# 局所辺連結度を保存するオイラーグラフ節点分離

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**概要**

グラフの節点分離とは，節点を複数の節点で置き換える操作のことであり，縮約の逆操作に対応する．本報告では，グラフが無向グラフである場合と有向グラフである場合の両方に対し，操作後のグラフがオイラーグラフであるという仮定のもと，節点分離が局所辺連結度を保存するための必要十分条件を明らかにする．また，操作後のグラフがさらにループを持たないための必要十分条件についても述べる．

# Eulerian Detachments with Local-Edge-Connectivity

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**abstract**

For a graph $G$, a detachment operation at a vertex transforms the graph into a new graph by splitting the vertex into several vertices in such a way that the original graph can be obtained by contracting all the split vertices into a single vertex. A graph obtained from a given graph $G$ by applying detachment operations at several vertices is called a detachment of graph $G$. We consider a detachment which preserves the local-edge-connectivity of the given graph $G$. In this report, we present necessary and sufficient conditions for a given graph/digraph to have an $r$-edge-connected Eulerian detachment. We also discuss conditions for a graph/digraph to admit a loopless $r$-edge-connected Eulerian detachment.

## 1 Introduction

For an undirected graph $G$, a *degree specification* $g = (\mathcal{V}, \rho)$ consists of a family $\mathcal{V} = \{V_v \mid v \in V\}$ of disjoint new vertex sets each of which corresponds to a vertex $v \in V$ and a function $\rho : V^* = \cup_{v \in V} V_v \to \mathbb{N}$ such that $\sum_{x \in V_v} \rho(x) = d(v; G)$ for each $v \in V$, where $d(v; G)$ denotes the degree of a vertex $v$ in $G$. A *g-detachment* $G^*$ of $G$ is a graph obtained from $G$ by replacing each $v \in V$ with vertices in $V_v$ changing end vertices of each edge $uv \in E$ from $u$ to some $x \in V_u$ (resp., from $v$ to some $y \in V_v$) so that $d(z; G^*) = \rho(z)$ holds for each $z \in V^*$. This is a reverse operation of contraction; $G$ is obtained from $G^*$ by contracting each $V_v$ into a single vertex $v$. Degree specification $g$ is called *even* if $\rho : V^* \to \mathbb{N}_{\text{even}}$. Moreover if $|V_v| = 1$ for $v \in V - s$ (i.e., only $s \in V$ is split into several vertices), $g$ may be denoted by $g(s)$.

Historically detachments are introduced by

Nash-Williams [10]. He showed a necessary and sufficient condition for the existence of $k$-edge-connected $g$-detachments. This result can be regarded as a generalization of the famous Euler's theorem, which shows the existence of Euler tours in Eulerian graphs; Euler's theorem tells the existence of 2-edge-connected $g$-detachments for Eulerian graphs, where $\rho(x) = 2$ for all $x \in V^*$. A counterpart of his result for digraphs was afterwards given by Berg, Jackson and Jordán [1]. Fleiner [2] showed a necessary and sufficient condition for the existence of a $g(s)$-detachment that is $k$-edge-connected in $V-s$. His result was generalized by Jordán and Szigeti [7] for the existence of $g(s)$-detachments that are $r$-edge-connected in $V-s$, which is formally stated as follows.

**Theorem 1** ([7]) *Let $G = (V, E)$ be an undirected graph, $s \in V$ be a specified vertex to which no cut-edges are incident, and $g(s)$ be a degree specification consisting of $V_s$ and $\rho : V_s \to \mathbb{N}$. There exists a $g(s)$-detachment $G^* = (V^*, E^*)$ of $G$ which is $r$-edge-connected in $V^* - V_s$ if and only if $G$ is $r$-edge-connected in $V - s$ and $\lambda(u, v; G - s) \geq r(u, v) - \sum_{x \in V_s} \lfloor \rho(x)/2 \rfloor$ holds for every pair $u, v \in V-s$.* □

Recently Nagamochi [9] considered the existence of loopless connected $g$-detachments and applied it to the graph inference problem.

In spite of the above efforts, characterizing conditions for the existence of $r$-edge-connected $g$-detachments remains open. Our main contribution is to present a necessary and sufficient condition for the existence of $r$-edge-connected $g$-detachments of undirected graphs $G$ and digraphs $D$ with even degree specifications $g$. Trivially $G$ and $D$ must be Eulerian to have such $g$-detachments. We also discuss conditions for such detachments to have no loops.

This report is organized as follows. Section 2 introduces notations. Sections 3 shows new results on edge-splittings in Eulerian digraphs and undirected graphs. Section 4 considers conditions for the existence of $g$-detachments for Eulerian digraphs and undirected graphs with even degree specifications $g$. Section 5 makes some concluding remarks.

## 2 Preliminaries

Let $\mathbb{N}$ (resp., $\mathbb{N}_{\text{even}}$) denote the set of positive integers, (resp., positive even integers). We may represent a set $\{x\}$ of a single element by $x$.

We denote by $G = (V, E)$ an undirected graph with a vertex set $V$ and an undirected edge set $E$, where $E$ may contain parallel edges and loops. For a vertex $v \in V$, we let $N(v)$ denote the set of neighbors of $v$. For nonempty sets $X, Y \subseteq V$, $c(X, Y; G)$ denotes the number of edges in $G$ such that one end vertex is in $X$ and the other is in $Y$. We may denote $c(X, V-X; G)$ by $c(X; G)$. Note that $c(v, v; G)$ means the number of loops incident to $v$. The *degree* $d(v; G)$ of a vertex $v$ is defined by $d(v; G) = 2c(v, v; G) + c(v, V-v; G)$.

Analogously to undirected graphs, we denote by $D = (V, A)$ a digraph of a vertex set $V$ and an arc set $A$, where $A$ also may contain parallel arcs and loops. For a vertex $v \in V$, let $N^+(v)$ (resp., $N^-(v)$) denote the set of heads (resp., tails) of arcs leaving (resp., entering) $v$. Let $c(X, Y; D)$ denote the number of arcs in $D$ whose tail is in $X$ and head is in $Y$. In addition, we let $c^+(X; D) = c(X, V-X; D)$ (i.e., the number of arcs leaving $X$) and $c^-(X; D) = c(V-X, X; D)$ (i.e., the number of arcs entering $X$) for each nonempty subset $X \subset V$. Note that $c(v, v; D)$ means the number of loops incident to $v$. We define the *in-* and *out-degree* of a vertex $v$ by $d^+(v; D) = c(v, V-v; D) + c(v, v; D)$ and $d^-(v; D) = c(V-v, v; D) + c(v, v; D)$, respectively. In this report, we mainly deal with Eulerian digraphs $D$, i.e., $d^+(v; D) = d^-(v; D)$ for all $v \in V$, where $c^+(X; D) = c^-(X; D)$ holds for all nonempty subsets $X \subset V$, and we may denote $c^+(X; D)$ by $c(X; D)$ for short.

Let $G - v$ (resp., $D - v$) denote the graph (resp., digraph) obtained from $G$ (resp., $D$) by removing a vertex $v$ and all edges (resp., arcs) incident to $v$.

The *local-edge-connectivity* $\lambda(u, v; G)$ between vertices $u$ and $v$ in $G$ is defined to be the maximum number of edge-disjoint paths between $u$ and $v$, which is equal to $\min\{c(X;G) \mid X \subset V, u \in X, v \in V-X\}$ by Menger's theorem. In a digraph $D$, the *local-edge-connectivity* $\lambda(u,v;D)$ from $u$ to $v$ is defined as the maximum number of arc-disjoint di-paths from $u$ to $v$, which equals to $\min\{c^+(X;G) \mid X \subset V, u \in X, v \in V-X\}$. Note that $\lambda(u,v;D) = \lambda(v,u;D)$ holds if $D$ is Eulerian. For a function $r : \binom{V}{2} \to \mathbb{N}$ (resp., $V \times V \to \mathbb{N}$), we similarly say that $G$ (resp., $D$) is *$r$-edge-connected in $X \subseteq V$* if $\lambda(u,v;G) \geq r(u,v)$ (resp., $\lambda(u,v;D) \geq r(u,v)$) for all $u,v \in X$. If $X = V$, $G$ (or $D$) is simply called *$r$-edge-connected.* Moreover, for an integer $k \in \mathbb{N}$, $k$-edge-connectivity means $r$-edge-connectivity with $r : \binom{V}{2} \to \{k\}$.

For a digraph $D$, a *degree specification* $g = (\mathcal{V}, \rho^+, \rho^-)$ consists of $\mathcal{V} = \{V_v \mid v \in V\}$ and $\rho^+, \rho^- : V^* = \cup_{v \in V} V_v \to \mathbb{N}$ such that $\sum_{x \in V_v} \rho^+(x) = d^+(v;D)$ and $\sum_{x \in V_v} \rho^-(x) = d^-(v;D)$. A $g$-detachment $D^*$ of $D$ is a digraph obtained from $D$ by replacing each $v \in V$ with vertices in $V_v$ changing end vertices of each arc $uv \in A$ from $u$ to some $x \in V_u$ (resp., from $v$ to some $y \in V_v$) so that $d^+(z;G^*) = \rho^+(z)$ and $d^-(z;G^*) = \rho^-(z)$ hold for each $z \in V^*$. Degree specification $g$ is called *even* if $\rho^+(x) = \rho^-(x)$ for all $x \in V^*$, and we may denote $\rho^+$ and $\rho^-$ by $\rho$ in this case. Analogously to undirected graphs, we represent $g$ by $g(s)$ if $|V_v| = 1$ for $v \in V-s$.

Our main tool is *edge-splitting.* For an undirected graph $G = (V,E)$ and a vertex $s \in V$, *splitting* a pair $\{e = us, f = sv\}$ of edges incident to $s$ is an operation that replaces $e$ and $f$ by a new edge $uv$. We note that $e$ and $f$ are possibly self-loops incident to $s$. Let $G^{ef}$ denote the graph after splitting $\{e,f\}$. The edge-connectivity in $G^{ef}$ is equal to or smaller than that in $G$. Pair $\{e = us, f = sv\}$ is called *splittable* if $\lambda(u,v;G^{ef}) \geq \lambda(u,v;G)$ for any $u,v \in V-s$. In digraphs, the splittability of a pair of two arcs, one leaving $s$ and the other entering $s$ is defined analogously to undirected graphs. Edge-splitting is closely related to detachments since splitting $\{us, sv\}$ is equivalent to a $g(s)$-detachment with $g(s) = \{\{s,s'\},\rho\}$, $\rho(s) = d(s;G)-2$ and $\rho(s') = 2$ if we subdivide the split edge $uv$ into $us'$ and $s'v$.

The following condition for graphs to have splittable pairs is characterized by Mader [8] to answer an earlier conjecture by Lovász.

**Theorem 2** ([8]) *Let $G = (V,E)$ be an undirected connected graph and $s \in V$ be a vertex with $d(s) \neq 3$. If no cut-edge is incident to $s$, then there is at least one splittable pair of edges incident to $s$.* □

A simple proof of this theorem by Frank can be found in [4]. Frank [3] and Jackson [6] obtained a counterpart of this theorem in Eulerian digraphs.

In the following sections, we use a slightly stronger result, which we call *strong splittability,* in order to derive a characterization for graphs/digraphs to admit Eulerian $r$-edge-connected $g$-detachments. Let us first consider an undirected graph $G = (V,E)$ and a vertex $s \in V$. Let $r_G(x,y) = \lambda(x,y;G)$ if $x,y \in V-s$ and $r_G(x,y) = \min\{d(s;G)-2, \lambda(x,y;G)\}$ if $s \in \{x,y\}$. Obviously $G$ is $r_G$-edge-connected. We call a pair $\{e,f\}$ of edges incident to $s$ *strongly splittable* at $s$ if $G^{ef}$ is also $r_G$-edge-connected, i.e., splitting such a pair preserves the local-edge-connectivity between every two $x,y \in V-s$ and that between $s$ and the others up to $d(s;G)-2$. Obviously a strongly splittable pair is also splittable.

The condition for a graph to have a strongly splittable pair was presented by Fukunaga and Nagamochi [5] as follows.

**Theorem 3** ([5]) *Let $G = (V,E)$ be an undirected graph and $s$ be a vertex in $V$. If no cut-edge is incident to $s$ and $d(s;G) \neq 3$, then there is a strongly splittable pair at $s$.* □

We here review the following result on splittable pairs due to Frank [4].

**Theorem 4** ([4]) *Let $G = (V,E)$ be an undirected graph and $s$ be a vertex in $V$. If no* *cut-edge is incident to $s$ and $d(s;G)$ is even, then edges incident to $s$ can be partitioned into $d(s;G)/2$ disjoint splittable pairs.* □

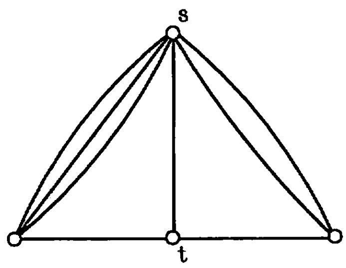


Figure 1: A graph that has no strongly splittable pair at $s$ containing edge $st$

This implies that there is a splittable pair containing an arbitrary edge incident to $s$. It is a natural question to ask whether there is a strongly splittable pair containing a specified edge if no cut-edge is incident to $s$ and $d(s;G)$ is even. Unfortunately there exists a counterexample to this, as shown in Figure 1. However, in this report, we prove that the answer to the question is affirmative for Eulerian undirected graphs and digraphs in Section 3.

## 3 Strongly Splittable Pair

We now consider edge-splitting in digraphs. Let $D = (V, A)$ be a digraph and $s \in V$ be a specified vertex. *Strong splittability* for digraph $D$ is defined in the same way with undirected graphs except for that $r_G$ is replaced by function $r_D$, where $r_D(x,y) = \lambda(x,y;D)$ if $x, y \in V-s$, $r_D(x,y) = \min\{d^+(s;D) - 1, \lambda(x,y;D)\}$ if $x = s$, and $r_D(x,y) = \min\{d^-(s;D) - 1, \lambda(x,y;D)\}$ if $y = s$. That is to say, splitting a strongly splittable pair preserves the local-edge-connectivity from $s$ to the other vertices up to $d^+(s;D)-1$, and from the other vertices to $s$ up to $d^-(s;D) - 1$, in addition to that between every two vertices in $V-s$. Note that $D$ is $r_D$-edge-connected.

In the rest of this report, we assume that $D$ is Eulerian. In this section, we also assume that $D$ has no loop incident to a designated vertex $s$ as we easily see that any pair containing a loop is strongly splittable.

Hence $r_D(x,y) = r_D(y,x)$ holds for every $x, y \in V$. It was proven by Frank [3] and Jackson [6] that there exists a splittable pair for Eulerian digraphs, although there are no results for any other classes of digraphs.

For a nonempty set $X \subseteq V-s$, let $R(X) = \max_{x\in X, y\in V-X} r_D(x,y)$ and $h(X) = c(X;D) - R(X)$. Since $D$ is $r_D$-edge-connected, it holds $c(X;D) \geq R(X)$ for all nonempty and proper subsets $X$ of $V$, and hence $h(X) \geq 0$, $\emptyset \neq X \subset V$. A subset $X$ of vertices is called *tight* if $h(X) = 0$ and $\emptyset \neq X \subseteq V-s$ (note that no tight subset $X$ contains $s$). Tight sets play an important role for the existence of strongly splittable pairs (We leave the proof to the reader).

**Lemma 1** *A pair $\{us, sv\}$ of arcs in an Eulerian digraph $D$ is strongly splittable if and only if no tight set contains both of $u$ and $v$.* □

Lemma 1 deals with Eulerian digraphs because we consider only those in this report. However, we remark that the statement remains valid for any digraphs if a tight set is redefined as a vertex set $X \subseteq V-s$ with $h^+(X) = 0$ or $h^-(X) = 0$, where $h^+(X) = \max_{x\in X, y\in V-X} r_D(x,y) - c^+(X;D)$ and $h^-(X) = \max_{x\in V-X, y\in X} r_D(x,y) - c^-(X;D)$.

We observe the following property of $h$ (the proof appears in [4] although the definition of $R$ is slightly different).

**Proposition 1** *For any $X, Y \subseteq V-s$, it holds either*

$$2h(X) + 2h(Y) \geq 2h(X\cap Y) + 2h(X\cup Y) + c(X-Y, Y-X; D) + c(Y-X, X-Y; D) \quad (1)$$

*or*

$$2h(X) + 2h(Y) \geq 2h(X-Y) + 2h(Y-X) + c(X\cap Y, V-X\cup Y; D) + c(V-X\cup Y, X\cap Y; D). \quad (2)$$

□

From the above facts, we have the next result on the existence of strongly splittable pairs in Eulerian digraphs, corresponding to Theorem 4.

**Theorem 5** *For an Eulerian digraph $D = (V, A)$, a vertex $s \in V$ and an arc $e$ entering (resp., leaving) $s$, there is another arc $f$ leaving $s$ (resp., entering $s$) such that $\{e, f\}$ is a strongly splittable pair at $s$.*

**Proof:** Let $e = us$ (i.e., an arc from $u$ to $s$) without loss of generality. Suppose that there is no strongly splittable pair at $s$ containing $e$. By Lemma 1, there is a tight set $X_v$ for each $v \in N^+(s)$ which contains both $u$ and $v$,

Let $v, w \in N^+(s)$. Then it holds $c(X_v \cap X_w, V - (X_v \cup X_w); D) \geq d(u, s; D) \geq 1$. We see that (2) does not hold for $X_v$ and $X_w$, since otherwise we would have

$$\begin{aligned} 0 + 0 &= h(X_v) + h(X_w) \\ &\geq h(X_v - X_w) + h(X_w - X_v) \\ &\quad + c(X_v \cap X_w, V - (X_v \cup X_w); D) \\ &\quad + c(V - (X_v \cup X_w), X_v \cap X_w; D) \\ &\geq 0 + 0 + 1 + 0, \end{aligned}$$

a contradiction. Therefore by Proposition 1, (1) holds as follows;

$$\begin{aligned} 0 + 0 &\geq h(X_v) + h(X_w) \\ &\geq h(X_v \cup X_w) + h(X_v \cap X_w) \\ &\quad + c(X_v - X_w, X_w - X_v) \\ &\quad + c(X_w - X_v, X_v - X_w), \end{aligned}$$

which implies that $X_v \cup X_w$ is a tight set in $D$. From this, we can see that a maximal tight set $X$ contains $N^+(s) \cup \{u\}$ and satisfies $c(X; D) \geq c(s; D)$.

Let $R(X) = r_D(x, y)$, where $x \in X$ and $y \in V - X$. If $y = s$, it holds

$$\begin{aligned} c(X; D) \geq c(s; D) = d(s; D) &\geq r_D(x, s) + 1 \\ &= r_D(x, y) + 1 = R(X) + 1. \end{aligned}$$

This implies $h(X) \geq 1$, contradicting tightness of $X$. Otherwise (i.e., $y \neq s$), it holds $\lambda(x, y; D) = \lambda(x, y; D - s)$ by $N^+(s) \subseteq X$. We also have $\lambda(x, y; D - s) \leq c(X; D) - c(s; D)$. Hence,

$$\begin{aligned} R(X) = r_D(x, y) &= \lambda(x, y; D) \\ &\leq c(X; D) - c(s; D) \leq c(X; D) - 1, \end{aligned}$$

which implies that $h(X) \geq 1$, a contradiction again. □

We use the following property in Section 4.

**Theorem 6** *For an Eulerian digraph $D$, a strongly splittable pair $\{e = us, f = sv\}$ can be chosen so that $u \neq v$ unless $|N^+(s) \cup N^-(s)| = 1$.*

**Proof:** By Theorem 5, $D$ has a strongly splittable pair. If such a pair consists of arcs $us$ and $su$, then there is no tight set containing $u$ by Lemma 1. Since $|N^+(s) \cup N^-(s)| \neq 1$, there is a vertex $v \neq u$ such that $v \in N^+(s) \cup N^-(s)$. Assume $v \in N^+(s)$ without loss of generality. Then $\{us, sv\}$ is strongly splittable in $D$. □

From Theorems 5 and 6, we can easily obtain a counterpart for undirected graphs as follows (We leave the proof to the reader).

**Theorem 7** *Let $G = (V, E)$ be an Eulerian undirected graph and $s$ be a specified vertex in $V$. For each edge $e = us \in E$, there is an edge $f = vs$ incident to $s$ such that $\{e, f\}$ is a strongly splittable pair.* □

**Theorem 8** *For an Eulerian undirected graph $G$, a strongly splittable pair $\{e = us, f = sv\}$ can be chosen so that $u \neq v$ unless $|N(s)| = 1$.* □

# 4 Eulerian Detachments

In this section, we consider Eulerian digraphs $D$ which may have loops, and show that there exists a $g$-detachment of $D$ for any even degree specification $g$.

For a digraph $D = (V, A)$ and a degree specification $g$ (possibly not even), let

$$r_g(x, y) = \min\{\rho^+(x), \rho^-(y), \lambda(u, v; D)\}$$

if $x \in V_u$ and $y \in V_v$ for some $u, v \in V$, where we define $\lambda(u, v; D) = +\infty$ if $u = v$. Note that it holds $\lambda(x, y; D^*) \leq r_g(x, y)$ for any $g$-detachments $D^*$ and $x, y \in V^*$. We call a $g$-detachment $D^*$ of $D$ *admissible* if $D^*$ is $r_g$-edge-connected, i.e., $\lambda(x, y; D^*) \geq r_g(x, y)$ for all $x, y \in V^*$. This means that admissible $g$-detachments preserve the local-edge-connectivity as much as possible. The admissibility is defined also for $g(s)$-detachments since $g(s)$-detachments form a subclass of $g$-detachments. By proving the existence of admissible $g$-detachments for even degree specification $g$, we show a necessary and sufficient condition for a digraph to have an $r$-edge-connected $g$-detachment.

**Lemma 2** *Let $D = (V, A)$ be an Eulerian digraph and $g$ be an even degree specification consisting of $\{V_v \mid v \in V\}$ and $\rho : V^* = \cup_{v \in V} V_v \to \mathbb{N}$. Then there exists an admissible $g$-detachment of $D$.*

**Proof:** In the following, we show how to construct an admissible $g$-detachment for an arbitrary $g$. For this, it suffices to consider constructing an admissible $g(s)$-detachment for $s \in V$ since splitting all vertices $v \in V$ into $V_v$ preserving admissibility finally gives an admissible $g$-detachment of $G$.

Suppose that $V_s = \{s_1, \ldots, s_n\}$ and that we have already obtained an admissible detachment $D_i = \{V \cup \{s_1, \ldots, s_i\}, A_i\}$ of $D$ such that

$$d(x; D_i) = \begin{cases} d(x; D) & \text{if } x \in V - s, \\ \rho(s_j) & \text{if } x = s_j,\ 1 \leq j \leq i, \\ \sum_{j=i+1}^{n} \rho(s_j) & \text{if } x = s. \end{cases}$$

Note that it holds $\lambda(x, y; D_i) = r_g(x, y)$ if $\{x, y\} \subseteq V \cup \{s_1, \ldots, s_i\} - s$, $\lambda(x, y; D_i) = \min\{d(s; D_i), \lambda(x, y; D)\}$ if $s \in \{x, y\} \subset V$, and $\lambda(x, y; D_i) = \min\{d(s; D_i), \rho(s_j)\}$ if $\{x, y\} = \{s, s_j\}$, $1 \leq j \leq i$, by admissibility. In the below, we show how to construct an admissible detachment $D_{i+1} = \{V \cup \{s_1, \ldots, s_{i+1}\}, A_{i+1}\}$ from $D_i$ such that $d(s_j; D_{i+1}) = \rho(s_j)$ for $j = 1, \ldots, i+1$ and $d(s; D_{i+1}) = \sum_{j=i+2}^{n} \rho(s_j)$. This inductively proves the lemma since $D_n - s$ is an admissible $g(s)$-detachment of $D$ (notice that $d(s; D_n) = 0$).

First, prepare $D' = (V \cup \{s_1, \ldots, s_{i+1}\}, A_i \cup A')$ from $D_i$ by adding a new vertex $s_{i+1}$ and an arc set $A'$ consisting of $\rho(s_{i+1})$ arcs $ss_{i+1}$ and $\rho(s_{i+1})$ arcs $s_{i+1}s$. Then it holds $d(x; D') = d(x; D_i) = d(x, D)$ if $x \in V - s$, $d(x; D') = d(s_j; D_i) = \rho(s_j)$ if $x = s_j$ with $1 \leq j \leq i$, $d(x; D') = \rho(s_{i+1})$ if $x = s_{i+1}$, $d(x; D') = d(s; D_i) + \rho(s_{i+1}) = 2\rho(s_{i+1}) + \sum_{j=i+2}^{n} \rho(s_j)$ if $x = s$. Moreover, $\lambda(x, y; D') = \lambda(x, y; D_i)$ is obvious if $s_{i+1} \notin \{x, y\}$. If $s_{i+1} \in \{x, y\}$, it holds $\lambda(x, y; D') = \min\{\rho(s_{i+1}), \lambda(s, z; D_i)\}$, where $z = \{x, y\} - s_{i+1}$ and $\lambda(s, s; D_i) = +\infty$. Hence for such $\{x, y\}$ (i.e., $s_{i+1} \in \{x, y\}$), it holds $\lambda(x, y; D') = \min\{\rho(s_{i+1}), \lambda(s, z; D)\} = r_g(s_{i+1}, z)$ if $\{x, y\} - s_{i+1} = z \in V - s$, $\lambda(x, y; D') = \min\{\rho(s_{i+1}), \rho(s_j)\} = r_g(s_j, s_{i+1})$ if $\{x, y\} = \{s_j, s_{i+1}\}$ with $1 \leq j \leq i$, $\lambda(x, y; D') = \min\{\rho(s_{i+1}), +\infty\} = \rho(s_{i+1})$ if $\{x, y\} = \{s, s_{i+1}\}$, where we used $\rho(s_{i+1}) \leq d(s; D_i)$ here. For each new arc $ss_{i+1}$, there is an arc $zs$ such that $\{ss_{i+1}, zs\}$ is strongly splittable and $z \neq s_{i+1}$ by Theorems 5 and 6, while $z$ is possibly $s$ if exists. Splitting such a pair decreases the in- and out-degree of $s$ by 1 respectively while preserving the local-edge-connectivity between any pair of vertices in $V \cup \{s_1, \ldots, s_{i+1}\} - s$, and between $s$ and the other vertices up to degree of $s$ after splitting. Analogously for each new arc $s_{i+1}s$, there is an arc $sz$ such that $\{s_{i+1}s, sz\}$ is strongly splittable and $z \neq s_{i+1}$. Let $D_{i+1}$ be the graph obtained by splitting such pairs successively. Then $D_{i+1}$ is a detachment of $D$. Moreover it holds $d(x; D_{i+1}) = d(x; D') = d(x, D)$ if $x \in V - s$, $d(x; D_{i+1}) = d(s_j; D') = \rho(s_j)$ if $x = s_j$ with $1 \leq j \leq i+1$, $d(x; D_{i+1}) = d(s; D') - 2\rho(s_{i+1}) = \sum_{j=i+2}^{n} \rho(s_j)$ if $x = s$. Furthermore, it also hold $\lambda(x, y; D_{i+1}) = \lambda(x, y; D')$ if $s \notin \{x, y\}$, and $\lambda(x, y; D_{i+1}) = \min\{d(s; D_{i+1}), \lambda(x, y; D')\}$ otherwise. This means $\lambda(x, y; D_{i+1}) = r_g(x, y)$ if $\{x, y\} \subseteq V \cup \{s_1, \ldots, s_{i+1}\} - s$,

$\lambda(x, y; D_{i+1}) = \min\{d(s; D_{i+1}), \lambda(x, y; D)\}$ if $s \in \{x, y\}$ and $\{x, y\} \subseteq V$, $\lambda(x, y; D_{i+1}) = \min\{d(s; D_{i+1}), \rho(s_j)\}$ if $\{x, y\} = \{s, s_j\}$ with $1 \leq j \leq i+1$. Hence $D_{i+1}$ is admissible, as required. □

If an original digraph has some loops, its detachments may have loops as well. As mentioned in Section 1, Nagamochi [9] showed a sufficient condition for an undirected graph to have a loopless connected $g$-detachment. Moreover we can see that there exists loopless $k$-edge-connected $g$-detachments if $k$ is even and $g$ satisfies a simple necessary condition by considering the proof of the theorem by Nash-Williams [10] (although we will not state the detail here). In the following, we extend our result in the above to loopless Eulerian $g$-detachments.

**Lemma 3** *Let $D = (V, A)$ be an Eulerian digraph and $g$ be an even degree specification consisting of $\{V_v \mid v \in V\}$ and $\rho : V^* = \cup_{v \in V} V_v \to \mathbb{N}$. Then $D$ has a loopless admissible $g$-detachment if and only if $2\rho(x) \leq c(v, v; D) + 2c(v, V - v; D)$ for all $v \in V$ and $x \in V_v$.*

**Proof:** First, we show necessity. Let us suppose that there exists a loopless admissible $g$-detachment $D^*$ of $D$. Consider a new vertex $x \in V_v$ for a vertex $v \in V$. Trivially it hold $c(x, V^* - V_v; D^*) \leq c(v, V - v; D)$ and $c(V^* - V_v, x; D^*) \leq c(V - v, v; D)$. Since every arc between $x$ and $V_v - x$ in $D^*$ is originally a loop in $D$ incident to $v$, it holds $c(x, V_v; D^*) + c(V_v, x; D^*) \leq c(v, v; D)$. By $c^+(x; D^*) = c(x, V_v; D^*) + c(x, V^* - V_v; D^*)$ and $c^-(x; D^*) = c(V_v, x; D^*) + c(V^* - V_v, x; D^*)$, it holds

$$\begin{aligned} 2\rho(x) &= c^+(x; D^*) + c^-(x; D^*) \\ &= c(x, V_v; D^*) + c(x, V^* - V_v; D^*) \\ &\quad + c(V_v, x; D^*) + c(V^* - V_v, x; D^*) \\ &\leq c(v, v; D) + 2c(v, V - v; D), \end{aligned}$$

implying the necessity.

In the next, we show sufficiency. We consider constructing an admissible $g(s)$-detachment of $D$. We have already shown that this can be done by an operation described in the proof of Lemma 2. Let us consider this again. If some loops are incident to $s$ in $D' = (V \cup \{s_1, \ldots, s_i, s_{i+1}\}, A_i \cup A')$, pairs $\{ss, ss_{i+1}\}$ and $\{ss, s_{i+1}s\}$ are strongly splittable because splitting such a pair is equivalent to deleting one loop incident to s. At splitting on $s$ in order to obtain $D_{i+1}$, we first continue choosing one of such pairs as long as some loops are incident to $s$. Then, no loops incident to $s$ remain in $D_{n-1}$ (and hence in $D_n$) by the following reason; It holds $\sum_{i=1}^{n} \rho(s_i) = d(s; D) = c(s, s; D) + c(s, V - s; D)$ by the hypothesis. Since $2\rho(s_n) \leq c(s, s; D) + 2c(s, V - s; D)$, it holds $\sum_{i=1}^{n-1} 2\rho(s_i) \geq c(s, s; D)$, which implies the above claim. If no loops are incident to $s$, we choose other strongly splittable pairs $\{xs, ss_{i+1}\}$ or $\{sx, s_{i+1}s\}$ such that $x \neq s_{i+1}$. This operation generates no loop obviously. Hence we can construct an admissible $g(s)$-detachment such that no loop is incident to a vertex in $V_s$, and therefore a loopless $g$-detachment. □

**Theorem 9** *Let $D = (V, A)$ be an Eulerian digraph and $g$ be an even degree specification consisting of $\{V_v \mid v \in V\}$ and $\rho : V^* = \cup_{v \in V} V_v \to \mathbb{N}$. Then there exists an $r$-edge-connected $g$-detachment of $D$ if and only if $\lambda(u, v; D) \geq r(x, y)$ for all $x \in V_u$ and $y \in V_v$ with $u \neq v$ and $\rho(x) \geq r(x, y)$ for all $x \in V^*$ and $y \in V^* - x$. Such a $g$-detachment can be constructed without generating any loop if and only if $2\rho(x) \leq c(v, v; D) + 2c(v, V - v; D)$ for all $v \in V$ and $x \in V_v$.*

**Proof:** First, let us consider the former part. Necessity is obvious. We can also derive the sufficiency from Lemma 2 since admissible detachments are $r$-edge-connected, i.e., $r_g(x, y) \geq r(x, y)$ for all $x, y \in V^* = \cup_{v \in V} V_v$, if $\lambda(u, v; D) \geq r(x, y)$ for $x \in V_u$ and $y \in V_v$ with $u \neq v$ and $\rho(x) \geq r(x, y)$ for $x \in V^*$ and $y \in V^* - x$.

Next, we consider the latter part. Necessity is proven as in the same way with

Lemma 3. Sufficiency is derived from the existence of loopless admissible detachments, proven in Lemma 3. □

For an undirected graph $G = (V, E)$, we can derive a counterpart from Theorem 9 although we leave the proof to the readers.

**Theorem 10** *Let $G = (V, E)$ be an Eulerian undirected graph and $g$ be an even degree specification consisting of $\{V_v \mid v \in V\}$ and $\rho : V^* = \cup_{v \in V} V_v \to \mathbb{N}_{\rm even}$. Then there exists an $r$-edge-connected $g$-detachment of $G$ if and only if $\lambda(u, v; G) \geq r(x, y)$ for all $x \in V_u$ and $y \in V_v$ with $u \neq v$ and $\rho(x) \geq r(x, y)$ for all $x \in V^*$ and $y \in V^* - x$. Such a $g$-detachment can be constructed without generating any loop if and only if $\rho(x) \leq c(v, v; G) + c(v, V - v; G)$ for all $v \in V$ and $x \in V_v$.*

## 5 Concluding remarks

We have proved the existence of strongly splittable pairs in Eulerian digraphs and undirected graphs. Based on this result, we have derived necessary and sufficient conditions for Eulerian digraphs and undirected graphs to admit $r$-edge-connected $g$-detachments. We have also presented necessary and sufficient conditions for such $g$-detachments to be loopless. Nevertheless, it remains open to characterize conditions for general graphs to have $r$-edge-connected $g$-detachments.

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