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線形相補性問題の符号可解性

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Abstract

線形相補性問題 (LCP) とは，線形計画や凸二次計画を特殊ケースとして含む数理計画問題である．LCP が符号可解であるとは，解の取りうる符号パターンの集合が係数要素の大きさによらずに定まることを言う．本研究では，係数行列の対角要素が全て非零である LCP に対して，その符号可解性に対する必要十分条件を与える．そして，符号可解 LCP の解の符号パターンを効率的に得るための組合せ的解法を提案する．その計算量は O(γ) (γ は係数の非零要素数の和) である．

Sign-Solvable Linear Complementarity Problems

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Abstract

This paper presents a connection between qualitative matrix theory and linear complementarity problems (LCPs). An LCP is said to be *sign-solvable* if the set of the sign patterns of the solutions is uniquely determined by the sign patterns of the given coefficients. We provide a characterization for sign-solvable LCPs such that the coefficient matrix has nonzero diagonals, which can be tested in polynomial time. This characterization leads to an efficient combinatorial algorithm to find the sign pattern of a solution for these LCPs. The algorithm runs in O(γ) time, where γ is the number of the nonzero coefficients.

1 Introduction

This paper deals with linear complementarity problems (LCPs) in the following form:

$$\begin{array}{lll} \mathrm{LCP}(A,b): & \text{find} & (w,z) \\ & \text{s.t.} & w = Az + b, \\ & & w^{\mathrm{T}} z = 0, \\ & & w \geq 0,\ z \geq 0, \end{array}$$

where A is a real square matrix, and b is a real vector. The LCP, introduced by Cottle [4], Cottle and Dantzig [5], and Lemke [17], is one of the most widely studied mathematical programming problems, which contains linear programming and convex quadratic programming. Solving LCP(A,b) for an arbitrary matrix A is NP-complete [3], while there are several classes of matrices A for which the associated LCPs can be solved efficiently. For details of the theory of LCPs, see the books of Cottle, Pang, and Stone [6] and Murty [21].

The *sign pattern* of a real matrix A is the $\{+,0,-\}$-matrix obtained from A by replacing each entry by its sign. When we develop an LCP model in practice, the entries of A and b are subject to many sources of uncertainty including errors of measurement and absence of information.

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On the other hand, the sign patterns of A and b are structural properties independent of such uncertainty. This motivates us to provide a combinatorial method that exploits the sign patterns before using numerical information.

Sign pattern analysis for matrices and linear systems, called *qualitative matrix theory*, was originated in economics by Samuelson [25]. Various results about qualitative matrix theory are compiled in the book of Brualdi and Shader [1]. For a matrix A, we denote by $\mathcal{Q}(A)$ the set of all matrices having the same sign pattern as A, called the *qualitative class* of A. The qualitative class of a vector is defined similarly. A square matrix A is said to be *sign-nonsingular* if $\tilde{A}$ is nonsingular for any $\tilde{A} \in \mathcal{Q}(A)$. The problem of recognizing sign-nonsingular matrices has many equivalent problems in combinatorics [18, 22, 26, 28], while its time complexity had been open for a long time. In 1999, Robertson, Seymour, and Thomas [23] presented a polynomial-time algorithm for solving this problem (cf. McCuaig [19, 20]).

For linear programming, Iwata and Kakimura [11] proposed sign-solvability in terms of qualitative matrix theory. A linear program $\max\{cx \mid Ax = b, x \geq 0\}$, denoted by $\mathrm{LP}(A, b, c)$, is *sign-solvable* if the set of the sign patterns of the optimal solutions of $\mathrm{LP}(\tilde{A}, \tilde{b}, \tilde{c})$ is the same as that of $\mathrm{LP}(A, b, c)$ for any $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$, and $\tilde{c} \in \mathcal{Q}(c)$. They showed that recognizing sign-solvability of a given LP is NP-hard, and gave a sufficient condition for sign-solvable linear programs, which can be tested in polynomial time. Moreover, they devised a polynomial-time algorithm to obtain the sign pattern of an optimal solution for linear programs satisfying this sufficient condition.

In this paper, we introduce sign-solvability for linear complementarity problems. We say that $\mathrm{LCP}(A, b)$ is *sign-solvable* if the set of the sign patterns of the solutions of $\mathrm{LCP}(\tilde{A}, \tilde{b})$ coincides with that of $\mathrm{LCP}(A, b)$ for any $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. An $\mathrm{LCP}(A, b)$ such that all diagonal entries of A are nonzero is said to have *nonzero diagonals*. The class of LCPs with nonzero diagonals includes LCPs associated with positive definite matrices, P-matrices, and nondegenerate matrices, which are all of theoretical importance in the context of LCPs (e.g. [6, Chapter 3]). LCPs with P-matrices are related to a variety of applications such as circuit equations with piecewise linear resistances [8] and linear systems of interval linear equations [24]. We present a characterization for a sign-solvable $\mathrm{LCP}(A, b)$ with nonzero diagonals, and describe a polynomial-time algorithm to solve them from the sign patterns of A and b.

We first provide a sufficient condition for sign-solvable LCPs with nonzero diagonals. A square matrix A is *term-nonsingular* if the determinant of A contains at least one nonvanishing expansion term. A square matrix A is *term-singular* if it is not term-nonsingular. A matrix A is term-singular if and only if $\tilde{A}$ is singular for any $\tilde{A} \in \mathcal{Q}(A)$. An $m \times n$ matrix with $m \leq n$ is said to be *totally sign-nonsingular* if all submatrices of order m are either sign-nonsingular or term-singular, namely, if the nonsingularity of each submatrix of order m is determined uniquely by the sign pattern of the matrix. Totally sign-nonsingular matrices were investigated in the context of sign-solvability of linear systems [1, 13, 14, 27] (the terms "matrices with signed mth compound" and "matrices with signed null space" are used instead). Recognizing totally sign-nonsingular matrices can be done in polynomial time by testing sign-nonsingularity of related square matrices [11]. We show that, if the matrix $M = (A\ b)$ is totally sign-nonsingular and A has nonzero diagonals, then $\mathrm{LCP}(A, b)$ is sign-solvable.

We then present a characterization of sign-solvable LCPs with nonzero diagonals. A row of a matrix is called *mixed* if it has both positive and negative entries. A matrix is *row-mixed* if every row is mixed. For an $\mathrm{LCP}(A, b)$ with nonzero diagonals, we introduce the *residual row-mixed* matrix, which is the special submatrix of $M = (A\ b)$ defined in Section 3. Then $\mathrm{LCP}(A, b)$ with nonzero diagonals is sign-solvable if and only if its residual row-mixed matrix M' satisfies one of followings: M' does not contain the subvector of b, M' has no rows, or M' is totally sign-nonsingular. The residual row-mixed matrix can be obtained in polynomial time. Thus the sign-solvability of a given LCP(A, b) with nonzero diagonals can be recognized in polynomial time.

This characterization leads to an efficient combinatorial algorithm to solve a given LCP(A, b) with nonzero diagonals from the sign patterns of A and b. The algorithm tests the sign-solvability, and finds the sign pattern of a solution if it is a sign-solvable LCP with solutions. In this algorithm, we obtain a solution of LCP$(\tilde{A}, \tilde{b})$ for some $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. If LCP$(A, b)$ is sign-solvable, then LCP(A, b) has a solution with the same sign pattern as the obtained one. The time complexity is $\mathrm{O}(\gamma)$, where γ is the number of nonzero entries in A and b. We note that the obtained sign pattern easily derives a solution of the given LCP by Gaussian elimination. Thus a sign-solvable LCP with nonzero diagonals is a class of LCPs which can be solved in polynomial time.

Before closing this section, we give some notations and definitions used in the following sections. For a matrix A, the row and column sets are denoted by U and V. If A is a square matrix, suppose that U and V are both identical with N. We denote by a_{ij} the (i, j)-entry in A. Let $A[I, J]$ be the submatrix in A with row subset I and column subset J, where the orderings of the elements of I and J are compatible with those of U and V. The submatrix $A[J, J]$ is abbreviated as $A[J]$. The *support* of a row subset I, denoted by $\Gamma(I)$, is the set of columns having nonzero entries in the submatrix $A[I, V]$, that is, $\Gamma(I) = \{j \in V \mid \exists i \in I, a_{ij} \neq 0\}$. For a vector b, the jth entry of b is denoted by b_j. The vector $b[J]$ means the subvector with index subset J. The *support* of a vector b is the column index subset $\{j \mid b_j \neq 0\}$.

This paper is organized as follows. In Section 2, we provide a sufficient condition using totally sign-nonsingular matrices. Section 3 gives a characterization for sign-solvable LCPs with nonzero diagonals. In Section 4, we describe a polynomial-time algorithm to solve sign-solvable LCPs with nonzero diagonals from the sign patterns of the given coefficients. In this paper, we omit proofs of all lemmas and theorems, due to the space limitation. They may be found in the technical report [12].

2 Totally Sign-Nonsingular Matrices

In this section, we give a sufficient condition for sign-solvable LCPs using totally sign-nonsingular matrices. For that purpose, we define *sign-nondegenerate* matrices. A square matrix A is *nondegenerate* if every principal minor is nonzero. A matrix A is nondegenerate if and only if LCP(A, b) has a finite number of solutions for any vector b [6]. Recognizing nondegenerate matrices is co-NP-complete [2, 21]. A square matrix A is said to be *sign-nondegenerate* if $\tilde{A}$ is nondegenerate for any $\tilde{A} \in \mathcal{Q}(A)$. Then the following lemma holds, which implies that sign-nondegeneracy can be tested in polynomial time.

Lemma 2.1. *A square matrix A is sign-nondegenerate if and only if A is a sign-nonsingular matrix with nonzero diagonals.*

We now obtain the following theorem. For LCP(A, b), let M be the matrix in the form of $M = (A\ b)$, where the column set is indexed by $N \cup \{g\}$.

Theorem 2.2. *For a linear complementarity problem* LCP(A, b) *with nonzero diagonals, if the matrix $M = (A\ b)$ is totally sign-nonsingular, then* LCP(A, b) *is sign-solvable.*

Sign-solvable LCPs do not necessarily satisfy this sufficient condition. Indeed, consider

LCP(A, b), where A and b are defined to be

$$A = \begin{pmatrix} -p_1 & -p_2 \\ +p_3 & +p_4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 \\ +p_5 \end{pmatrix}$$

for positive constants $p_1, \ldots, p_5 > 0$. Then LCP(A, b) has a unique solution $w = (0\ p_5)^{\mathrm{T}}$ and $z = 0$, and hence LCP(A, b) is sign-solvable. However, this does not satisfy the condition of Theorem 2.2, as A is not sign-nonsingular.

3 Sign-Solvable LCPs with Nonzero Diagonals

In this section, we describe a characterization for a sign-solvable LCP(A, b) with nonzero diagonals.

3.1 The Residual Row-Mixed Matrix

We first introduce the *residual row-mixed* matrix of LCP(A, b) with nonzero diagonals.

For each row index i, the ith equation of LCP(A, b) is represented by

$$w_i = \sum_{j \in \Gamma(\{i\})} a_{ij} z_j + b_i. \tag{1}$$

First assume that M has a nonpositive row i, that is, $b_i \leq 0$ and $a_{ij} \leq 0$ for all $j \in N$. Suppose that $b_i < 0$. Since any solution of LCP(A, b) is nonnegative, the ith row implies that LCP(A, b) has no solutions. Next suppose that $b_i = 0$. Then, if LCP(A, b) has a solution (w, z), the solution (w, z) must satisfy that $z_j = 0$ for any $j \in \Gamma(\{i\})$.

Next assume that M has a nonnegative row i, that is, $b_i \geq 0$ and $a_{ij} \geq 0$ for all $j \in N$. Let (w, z) be a solution of LCP(A, b). If $w_i > 0$, then the complementarity implies $z_i = 0$. Suppose that $w_i = 0$. Since any solution is nonnegative, (w, z) must satisfy $z_j = 0$ for any $j \in \Gamma(\{i\})$, and hence $z_i = 0$ by $a_{ii} \neq 0$. Thus, if LCP(A, b) has a solution and M has a nonnegative row i, any solution of LCP(A, b) must satisfy that $z_i = 0$. Note that there exists $j \in \Gamma(\{i\})$ with $z_j > 0$ if and only if the left-hand side of (1) is positive, i.e., $w_i > 0$.

Therefore, if M has a nonnegative or nonpositive row, then we know that some entries of any solution must be zero. We can repeat this process as follows. Set $M^{(1)} = M$. For a positive integer ν and a matrix $M^{(\nu)}$, let $I_-^{(\nu)}$ be the set of nonpositive rows in $M^{(\nu)}$, and $I_+^{(\nu)}$ be the set of nonnegative rows that have a nonzero entry in $M^{(\nu)}$. If $\Gamma(I_-^{(\nu)})$ contains the index g, then the LCP has no solutions. Define $I^{(\nu)} = I_+^{(\nu)} \cup I_-^{(\nu)}$ and $J^{(\nu)} = I_+^{(\nu)} \cup \Gamma(I_-^{(\nu)})$. Then any solution (w, z) of LCP(A, b) satisfies $z_j = 0$ for any $j \in J^{(\nu)}$. Let $M^{(\nu+1)}$ be the matrix obtained from $M^{(\nu)}$ by deleting the rows indexed by $I^{(\nu)}$ and the columns indexed by $J^{(\nu)}$. Repeat this for $\nu = 1, 2, \ldots$ until $I^{(\nu)} = J^{(\nu)} = \emptyset$, that is, until either $M^{(\nu)}$ is row-mixed or $M^{(\nu)}$ has no rows.

We call the remaining row-mixed submatrix M' the *residual row-mixed* matrix of LCP(A, b). Note that, if LCP(A, b) has solutions, the column index g is not deleted in each iteration.

Assume that the column set of M' contains the index g. Let M' be in the forms of $M' = (A'\ b')$, where b' is the subvector of b and A' is the submatrix of A with row set U' and column set V'. We denote $\bar{U}' = N \setminus U'$ and $\bar{V}' = N \setminus V'$. Since A has nonzero diagonals, $\bar{U}' \subseteq \bar{V}'$ holds, and hence we have $V' \subseteq U'$. Suppose that M' has no rows. Then $\bar{V}' = N$ holds, which means that any solution (w, z) of LCP(A, b) must satisfy $z = 0$. Since g is not deleted in each iteration, the vector b is nonnegative. Thus $(b, 0)$ is a unique solution of LCP(A, b). Next suppose that M' is row-mixed. Consider the following system:

$$\begin{aligned} &w = A'z + b', \\ &w_i^{\mathrm{T}} z_i = 0, \ \text{for any } i \in V', \\ &w \geq 0, \ z \geq 0. \end{aligned} \tag{2}$$

We claim that there exists a one-to-one correspondence between solutions of LCP(A, b) and (2). For a solution (w, z) of LCP(A, b), the pair $(w[U'], z[V'])$ is a solution of (2). Conversely, let (w', z') be a solution of (2). Define (w, z) to be $z[V'] = z'$, $z[\bar{V}'] = 0$, and $w = Az + b$. Then $w[U'] = A'z' + b' = w' \geq 0$ holds. Moreover, since each row in $A[\bar{U}', V']$ is nonnegative, we have $w[\bar{U}'] = A[\bar{U}', V']z' + b[\bar{U}'] \geq 0$. By $V' \subseteq U'$, the pair (w, z) satisfies the complementarity $w^{\mathrm{T}} z = 0$. Thus (w, z) is a solution of LCP(A, b).

3.2 Characterization

Using the residual row-mixed matrix M' of LCP(A, b), we have the following theorem.

Theorem 3.1. *Let* LCP(A, b) *be a linear complementarity problem with nonzero diagonals, and* M' *be the residual row-mixed matrix. Then* LCP(A, b) *is sign-solvable if and only if one of the followings holds:*

- *The column set of* M' *does not contain the index* g.
- *The residual row-mixed matrix* M' *has no rows.*
- *The residual row-mixed matrix* M' *is totally sign-nonsingular.*

In order to prove this theorem, we give some definitions. A linear system $Ax = b$ has *signed nonnegative solutions* if the set of the sign patterns of nonnegative solutions of $\tilde{A}x = \tilde{b}$ is the same as that of nonnegative solutions of $Ax = b$ for any $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. A matrix A is said to have *signed nonnegative null space* if $Ax = 0$ has signed nonnegative solutions. Matrices with signed nonnegative null space were examined by Fisher, Morris, and Shapiro [9]. They showed that a row-mixed matrix has signed nonnegative null space if and only if it is the matrix called *mixed dominating*, which is defined to be a row-mixed matrix which does not contain a square row-mixed submatrix. By the result of mixed dominating matrices, the following two lemmas hold.

Lemma 3.2 (Fischer and Shapiro [10]). *If a row-mixed matrix* A *has signed nonnegative null space, then the rows of* A *are linearly independent.*

A matrix A is said to have *row-full term-rank* if A has a term-nonsingular submatrix with row size. A matrix A has *column-full term-rank* if A^{T} has row-full term-rank.

Lemma 3.3 (Fischer, Morris, and Shapiro [9]). *An* $n \times (n + 1)$ *row-mixed matrix has signed nonnegative null space if and only if it is a totally sign-nonsingular matrix with row-full term-rank.*

Using the following lemmas, we obtain Theorem 3.1.

Lemma 3.4. *Suppose that the matrix* $(A\ b)$ *is row-mixed. If the linear system* $Ax + b = 0$ *has signed nonnegative solutions, then it has a solution all of whose entries are positive.*

Lemma 3.5. *Suppose that $M = (A\ b)$ is row-mixed. The linear system $Ax + b = 0$ has signed nonnegative solutions if and only if M has signed nonnegative null space.*

We close this section with an example of sign-solvable LCPs with nonzero diagonals. Consider LCP(A, b), where A and b have the sign patterns, respectively,

$$\begin{pmatrix} + & + & 0 & 0 & 0 \\ - & + & + & 0 & + \\ + & - & + & - & 0 \\ - & 0 & - & - & + \\ 0 & - & + & 0 & + \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ + \\ 0 \\ 0 \\ - \end{pmatrix}.$$

The residual row-mixed matrix is

$$\begin{pmatrix} + & - & 0 & 0 \\ - & - & + & 0 \\ + & 0 & + & - \end{pmatrix},$$

which is obtained from the matrix $(A\ b)$ by deleting the first two rows and the first two columns. This residual row-mixed matrix is totally sign-nonsingular, and hence LCP(A, b) is sign-solvable.

4 Algorithm for Sign-Solvable LCPs with Nonzero Diagonals

In this section, we describe an algorithm for a given LCP(A, b) with nonzero diagonals. The algorithm tests sign-solvability of LCP(A, b), and finds the sign pattern of a solution of LCP(A, b) if it is sign-solvable.

The algorithm starts with finding the residual row-mixed matrix M' as described in the previous section. If the column set of M' does not contain the index g, then LCP(A, b) is sign-solvable and has no solutions. Let M' be in the forms of $M' = (A'\ b')$, where b' is the subvector of b and A' is the submatrix of A with row set U' and column set V'. We denote $\bar{U}' = N \setminus U'$ and $\bar{V}' = N \setminus V'$. Note that $V' \subseteq U'$ holds. If M' has a row and M' is not totally sign-nonsingular, then return that LCP(A, b) is not sign-solvable by Theorem 3.1.

Assume that M' has no rows. Then LCP(A, b) is sign-solvable, and $(b, 0)$ is a unique solution of LCP(A, b). Next assume that M' has a row and $M' = (A'\ b')$ is totally sign-nonsingular. Then LCP(A, b) is sign-solvable by Theorem 3.1. Since M' is row-mixed, there exists $\tilde{M} = (\tilde{A}\ \tilde{b}) \in \mathcal{Q}(M)$ such that the sum of the columns of $\tilde{M}' \in \mathcal{Q}(M')$ is zero. Hence it follows from (2) that the pair (w, z), defined to be $z[\bar{V}'] = 0$, $z[V'] = \mathbf{+1}$, and $w = \tilde{A}z + \tilde{b}$, is a solution of LCP$(\tilde{A}, \tilde{b})$. This means that the vector w satisfies that $w_j > 0$ if $j \in \bar{U}'$ and $A[\{j\}, V']$ has nonzero entries, and $w_j = 0$ otherwise. Since LCP(A, b) is sign-solvable, (w, z) is the sign pattern of a solution of LCP(A, b).

We now summarize the algorithm description.

Algorithm: An algorithm for LCPs with nonzero diagonals.

Input: A linear complementarity problem LCP(A, b) with nonzero diagonals.

Output: The sign pattern of a solution if LCP(A, b) is sign-solvable.

Step 1: Set $M^{(1)} = M$ and $\nu = 1$. Repeat the following until $I^{(\nu)} = J^{(\nu)} = \emptyset$.

- **1-1:** Find $I_-^{(\nu)}$ and $I_+^{(\nu)}$, where $I_-^{(\nu)}$ is the set of nonpositive rows in $M^{(\nu)}$, and $I_+^{(\nu)}$ is the set of nonnegative rows that have a nonzero entry in $M^{(\nu)}$.

1-2: If $g \in \Gamma(I_-^{(\nu)})$, then return that LCP$(A, b)$ is sign-solvable and has no solutions.

1-3: Let $I^{(\nu)} = I_+^{(\nu)} \cup I_-^{(\nu)}$ and $J^{(\nu)} = I_+^{(\nu)} \cup \Gamma(I_-^{(\nu)})$. Define $M^{(\nu+1)}$ to be the matrix obtained by deleting the rows indexed by $I^{(\nu)}$ and the columns indexed by $J^{(\nu)}$ from $M^{(\nu)}$.

1-4: Set $\nu = \nu + 1$ and go back to Step 1.

Step 2: Let $M' = (A' \ b')$ be the remaining submatrix, and U', V' be the row and column sets of A', respectively. If M' has a row and M' is not totally sign-nonsingular, then return that LCP(A, b) is not sign-solvable. Otherwise go to Step 3.

Step 3: Return that LCP(A, b) is sign-solvable and do the following.

3-1: If U' is empty, then return the sign pattern of a solution $(w, z) = (b, 0)$.

3-2: Otherwise, return the sign pattern of (w, z) defined to be

$$\operatorname{sgn} z_j = \begin{cases} +, & \text{if } j \in V' \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \operatorname{sgn} w_j = \begin{cases} +, & \text{if } j \in K \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where K is the set of rows which have nonzero entries in $A[\bar{U}', V']$, that is, $K = \{j \in \bar{U}' \mid \Gamma(\{j\}) \cap V' \neq \emptyset\}$.

Applying this algorithm to the example at the end of Section 3, we obtain the sign pattern of a solution, $w = (\ 0\ +\ \ 0\ \ 0\ \ 0)^{\mathrm{T}}$ and $z = (\ 0\ \ 0\ +\ +\ +)^{\mathrm{T}}$.

Based on this algorithm, we can compute a solution of a sign-solvable LCP as well as the sign pattern of a solution. Suppose that M' has a row. The solution (w, z) with the obtained sign pattern satisfies that $A'z[V'] + b' = 0$, $z[\bar{V}'] = 0$. Since A' is nonsingular by total sign-nonsingularity of M', we can compute a solution of LCP(A, b) by performing Gaussian elimination.

The running time bound of the algorithm is now given as follows. Note that an $n \times (n+1)$ row-mixed matrix A is a totally sign-nonsingular matrix with row-full term-rank if and only if all square submatrices of order n are sign-nonsingular [1, Theorem 5.3.3]. Such matrix is called an *S-matrix* in [1, 16], which can be recognized in $\mathrm{O}(n^2)$ time [15].

Theorem 4.1. *For a linear complementarity problem* LCP(A, b) *with nonzero diagonals, let n be the matrix size of A, and γ the number of nonzero entries in A and b. Then the algorithm tests sign-solvability in $\mathrm{O}(n^2)$ time, and, if* LCP(A, b) *is sign-solvable, the algorithm finds the sign pattern of a solution in $\mathrm{O}(\gamma)$ time.*

References

[1] R. A. Brualdi and B. L. Shader, *Matrices of Sign-solvable Linear Systems*, Cambridge University Press, Cambridge, 1995.

[2] R. Chandrasekaran, S. N. Kabadi, and K. G. Murty, Some NP-complete problems in linear programming, *Operations Research Letters*, 1 (1982), pp. 101–104.

[3] S. J. Chung, NP-completeness of the linear complementarity problem, *Journal of Optimization Theory and Applications*, 60 (1989), pp. 393–399.

[4] R. W. Cottle, *The Principal Pivoting Method of Quadratic Programming*, in Mathematics of Decision Sciences, Part 1, G. B. Dantzig and A. F. Veinott, eds., American Mathematical Society, Providence R. I., 1968, pp. 142–162.

[5] R. W. Cottle and G. B. Dantzig, Complementary pivot theory of mathematical programming, *Linear Algebra and Its Applications*, 1 (1968), pp. 103–125.

[6] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*, Academic Press, 1992.

[7] G. E. Coxson, The P-matrix problem is co-NP-complete, *Mathematical Programming*, 64 (1994), pp. 173–178.

[8] J. T. J. V. Eijndhoven, Solving the linear complementarity problem in circuit simulation, *SIAM Journal on Control and Optimization*, 24 (1986), pp. 1050–1062.

[9] K. G. Fischer, W. Morris, and J. Shapiro, Mixed dominating matrices, *Linear Algebra and Its Applications*, 270 (1998), pp. 191–214.

[10] K. G. Fischer and J. Shapiro, Mixed matrices and binomial ideals, *Journal of Pure and Applied Algebra*, 113 (1996), pp. 39–54.

[11] S. Iwata and N. Kakimura, Solving linear programs from sign patterns, *Mathematical Programming*, (to appear).

[12] N. Kakimura, Sign-solvable linear complementarity problems, in Lecture Notes in Computer Science, M. Fischetti and D. P. Williamson, eds., vol. 4513, Springer-Verlag, 2007, pp. 397–409.

[13] S.-J. Kim and B. L. Shader, Linear systems with signed solutions, *Linear Algebra and Its Applications*, 313 (2000), pp. 21–40.

[14] ——, On matrices which have signed null-spaces, *Linear Algebra and Its Applications*, 353 (2002), pp. 245–255.

[15] V. Klee, Recursive structure of S-matrices and $O(m^2)$ algorithm for recognizing strong sign-solvability, *Linear Algebra and Its Applications*, 96 (1987), pp. 233–247.

[16] V. Klee, R. Ladner, and R. Manber, Sign-solvability revisited, *Linear Algebra and Its Applications*, 59 (1984), pp. 131–158.

[17] C. E. Lemke, Bimatrix equilibrium points and mathematical programming, *Management Science*, 11 (1965), pp. 681–689.

[18] L. Lovász and M. D. Plummer, *Matching Theory*, vol. 29 of Annals of Discrete Mathematics, North-Holland, Amsterdam, 1986.

[19] W. McCuaig, Brace decomposition, *Journal of Graph Theory*, 38 (2001), pp. 124–169.

[20] ——, Pólya's permanent problem, *The Electronic Journal of Combinatorics*, 11 (2004), p. R79.

[21] K. G. Murty, *Linear Complementarity, Linear and Nonlinear Programming*, Internet Edition, 1997.

[22] G. Pólya, Aufgabe 424, *Archiv der Mathematik und Physik*, 20 (1913), p. 271.

[23] N. Robertson, P. D. Seymour, and R. Thomas, Permanents, Pfaffian orientations, and even directed circuits, *Annals of Mathematics*, 150 (1999), pp. 929–975.

[24] J. Rohn, Systems of linear interval equations, *Linear Algebra and Its Applications*, 126 (1989), pp. 39–78.

[25] P. A. Samuelson, *Foundations of Economics Analysis*, Harvard University Press, 1947; Atheneum, New York, 1971.

[26] P. Seymour and C. Thomassen, Characterization of even directed graphs, *Journal of Combinatorial Theory, Series B*, 42 (1987), pp. 36–45.

[27] J.-Y. Shao and L.-Z. Ren, Some properties of matrices with signed null spaces, *Discrete Mathematics*, 279 (2004), pp. 423–435.

[28] V. V. Vazirani and M. Yannakakis, Pfaffian orientations, 0-1 permanents, and even cycles in directed graphs, *Discrete Applied Mathematics*, 25 (1989), pp. 179–190.