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ハノイの塔問題に対する再帰方程式の一般化とその厳密解析

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Abstract

本稿では，ハノイの塔問題より一般化された再帰方程式 $T(n,\, \alpha,\, \beta) = \min_{1\leq t\leq n}\{\alpha\, T(n-t,\, \alpha,\, \beta) + \beta\, S(t,\, 3)\}$ ($S(t,3) = 2^t - 1$ は3本の塔を持つハノイの塔問題に対する最小解) に対する厳密な解析を行い，その一般解を導出する。すなわち，α と β が $\alpha \geq 2$ なる任意の自然数であるとき，$\{T(n,\alpha,\beta)\}$ の階差数列が $\beta 2^i \alpha^j$ $(i,\, j \geq 0)$ なる自然数が昇順に並んだものであることを示す。

Exact Analysis of the Recurrence Relations Generalized from the Tower of Hanoi

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Abstract

In this paper, we make exact analysis of the recurrence relations generalized from the Tower of Hanoi problem of the form $T(n,\alpha,\beta) = \min_{1\leq t\leq n}\{\alpha\, T(n-t,\alpha,\beta) + \beta\, S(t,3)\}$, where $S(t,3) = 2^t - 1$ is the optimal solution for the 3-peg Tower of Hanoi problem. It is shown that when α and β are natural numbers and $\alpha \geq 2$, the sequence of differences of $T(n,\alpha,\beta)$'s, i.e., $T(n,\alpha,\beta) - T(n-1,\alpha,\beta)$, consists of numbers of the form $\beta 2^i \alpha^j$ $(i,\, j \geq 0)$ lined in the increasing order.

1 Introduction

The Tower of Hanoi problem with 3 pegs was invented by E. Lucas in 1883 [9]. In 1907, it was generalized to the problem having 4 pegs by H.E. Dudeney [3]. Since then the original problem and its variants have not only been used as an introductory example of recursive algorithms, but have been also studied widely in computational research fields [2,5,7,8,11-14]. Stockmeyer's survey [13] lists more than 200 references, not included articles in psychological journals and textbooks in discrete mathematics. In the simplest case with 3 pegs and n disks, the algorithm of first moving the upper $n-1$ disks to the intermediate peg, then moving the bottom disk to the peg of destination, and finally moving the remaining $n-1$ disks to the destination, is the best possible and the total number of moves is $2^n - 1$. Somewhat surprisingly, for the general Tower of Hanoi problem with $k\,(\geq 4)$ pegs and n disks, the optimal solution is not known yet. The best upper bound is obtained by the algorithms by Frame [5] and Stewart [11]. Their algorithms are rediscovered many times ([12] lists them). Furthermore, in [8], Klavžar et al. have shown that seven different approaches to the multi-peg Tower of Hanoi problem, which include the ones by Frame and Stewart, are all equivalent. On the other hand, the subexponential lower bound was first proven by Szegedy [14] and it was improved by Chen et al.[2]. Since the upper bound is believed to be optimal, it is called the "presumed optimal" solution.

The Stewart's recursive algorithm for the 4-peg Tower of Hanoi is written as follows. For $1 \leq t \leq n$, consider the procedures of first moving the top $n-t$ disks to the intermediate peg using the 4 pegs, moving the remaining t disks to the destination using the available 3 pegs, and then moving the $n-t$ disks to the destination with the 4 pegs. The algorithm chooses the minimum one among them. When the total number of moves is denoted by $S(n,4)$, the recurrence relation is written as $S(n,\ 4) = \min_{1\leq t \leq n}\{2\,S(n-t,\ 4)+S(t,3)\}$. This is solved with the difference $S(n,4) - S(n-1,4) = 2^{i-1}$ for $t_{i-1} < n \leq t_i$, where t_i is the triangular number, i.e., $t_i = i(i+1)/2$. It is also possible to obtain the closed formula for $S(n,4)$.

To clarify the combinatorial structures latent in this type of recurrence relation, we investigate the general recurrence relation of the form $T(n,\alpha,\beta) = \min_{1\leq t\leq n}\{\alpha T(n-t,\alpha,\beta) + \beta S(t,3)\}$ $(n \geq 1), T(0,\alpha,\beta) = 0$, where α and β are arbitrary natural numbers. $S(n,4)$ is then written as $S(n,4) = T(n,2,1)$.

The main contribution of this paper is to exactly solve this relation for all natural numbers α and β. Especially, suppose that $\{a_n\}_{n\geq 1}$ is the integer sequence which consists of numbers of the form $2^i\alpha^j$ $(i,j \geq 0)$ lined in the increasing order. Then for $\alpha \geq 2$, the difference of $T(n,\alpha,\beta)$'s is written using this sequence as $T(n,\alpha,\beta) - T(n-1,\alpha,\beta) = \beta a_n$. $T(n,\alpha,\beta)$ is then computed by summing up the differences. We note that when $\alpha = 3$, $a_n = 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, \cdots$. These numbers are called "3-smooth numbers" and are explored in relation to the distribution of prime numbers [6] and new number representations [1,4,10].

The remaining of the paper is organized as follows: In Section 2, we state the main results. In Section 3, the proof of the main theorem is given. Some Tower of Hanoi variants are discussed in Section 4 and concluding remarks are given in Section 5. Finally, Appendix follows.

2 Main Results

2.1 Linearity of $T(n,\alpha,\beta)$ on β

We first show that $T(n,\alpha,\beta)$ is linear on β.

Theorem 1 *For any natural numbers α and β, $T(n,\alpha,\beta)$ is linear on β. Namely, $T(n,\alpha,\beta) = \beta\,T(n,\alpha,1)$ holds.*

Proof of Theorem 1: By induction on n. When $n = 0$, $T(0,\alpha,\beta) = 0 = \beta\,T(0,\alpha,1)$. Therefore, the equality holds. Next, suppose that for $n = k$, the equality holds. By the definition of $T(n,\alpha,\beta)$ and by the assumption of induction,

$$\begin{aligned} & T(k+1,\alpha,\beta) \\ = & \min_{1\leq t\leq k+1}\{\alpha T(k+1-t,\alpha,\beta) + \beta S(t,3)\} \\ = & \min_{1\leq t\leq k+1}\{\alpha\beta\,T(k+1-t,\alpha,1) + \beta S(t,3)\} \\ = & \beta \min_{1\leq t\leq k+1}\{\alpha T(k+1-t,\alpha,1) + S(t,3)\} \\ = & \beta T(k+1,\alpha,1). \end{aligned}$$

Therefore, the linearity of $T(n,\alpha,\beta)$ also holds for $n = k+1$.

This completes the proof of Theorem 1. □

We note that the linearity of $T(n,\alpha,\beta)$ also holds for any real number β.

2.2 Properties of $\Delta T(n,\alpha,1)$ and $T(n,\alpha,1)$

Owing to Theorem 1, it is enough to compute $T(n,\alpha,1)$ instead of $T(n,\alpha,\beta)$. We consider the following recurrence relation for $T(n,\alpha,1)$:

$$\begin{aligned} & T(n,\alpha,1) = \min_{1\leq t\leq n}\{\alpha T(n-t,\alpha,1)+S(t,3)\}\ (n\geq 1) \\ & T(0,\alpha,1) = 0 \qquad (1) \end{aligned}$$

Table 1: The values of $t_{\min}$, $T(n,3,1)$, and $\Delta T(n,3,1)$

n	1	2	3	4	5	6	7	8	9	10	11	12
$t_{\min}$	1	2	2	3	3	4	4	4	5	5	5	5
$T(n,3,1)$	1	3	6	10	16	24	33	45	61	79	103	130
$\Delta T(n,3,1)$	1	2	3	4	6	8	9	12	16	18	24	27

Table 2: The values of $t_{\min}$, $T(n,4,1)$, and $\Delta T(n,4,1)$

n	1	2	3	4	5	6	7	8	9	10	11	12
$t_{\min}$	1	2	2,3	3	3,4	4	4,5	4,5	5	5,6	5,6	5,6
$T(n,4,1)$	1	3	7	11	19	27	43	59	75	107	139	171
$\Delta T(n,4,1)$	1	2	4	4	8	8	16	16	16	32	32	32

Tables 1 and 2 show the values for $T(n,3,1)$ and $T(n,4,1)$ up to $n \leq 12$. In the tables, $t_{\min}$ is the value of the argument with which the right-hand side of the recurrence relation takes the minimum and $\Delta T(n,\alpha,1)$ is the differences of $T(n,\alpha,1)$'s.

When $\alpha = 3$, we observe that all the numbers of the sequence $\{2^i 3^j\}_{i,j\geq 0}$ appear in the increasing order as the differences of $T(n,3,1)$'s. When $\alpha = 4$, at some n's, $T(n,4,1)$ takes the minimum at two values of $t_{\min}$, which is essentially different from the case $\alpha = 3$. For clarifying the characteristics of $\Delta T(n,\alpha,1)$'s, we define a set of number sequences as follows. Let p and q be any natural numbers and let $\{a_n\}_{n\geq 1}$ be the sequence of numbers of the form $p^i q^j$ $(i,j \geq 0)$ which are lined in the increasing order. (Note that when $q = p^l$ for some integer l, $p^i q^j$'s such that $p^i q^j = p^{i'} q^{j'}$ and $(i,j) \neq (i',j')$ appear successively.) Then the sequence of differences of $T(n,\alpha,1)$'s is shown to be exactly of this form $\{p^i p^j\}$. Namely, we show the following theorem.

Theorem 2 *Let α be a natural number and let $\{a_n\}_{n\geq 1}$ be the number sequence which consists of numbers of the form $2^i \alpha^j$ $(i,j \geq 0)$ lined in the increasing order. Then for $n \geq 1$, the difference of $T(n,\alpha,1)$'s is written as follows.*

$$T(n,\alpha,1) - T(n-1,\alpha,1) = \begin{cases} 1 & (\alpha = 1) \\ a_n & (\alpha \geq 2) \end{cases}$$

Combining Theorems 1 and 2 leads to the following corollary.

Corollary 1 *Under the same condition with Theorem 2, $T(n,\alpha,\beta)$ is computed as follows.*

$$T(n,\alpha,\beta) = \begin{cases} \beta n & (\alpha = 1,\ n \geq 0) \\ \beta \sum_{i=1}^{n} a_i & (\alpha \geq 2,\ n \geq 1) \end{cases}$$

3 Proof of Theorem 2

When $\alpha = 1$, Equation (1) takes the minimum at $t = 1$ with $T(n,1,1) = T(n-1,1,1) + S(1,3) = T(n-1,1,1) + 1$. Therefore, $T(n,1,1) - T(n-1,1,1) = 1$ holds.

When $\alpha \geq 2$, the proof is divided into the following two cases: When α is not of the form 2^l for any integer $l \geq 1$ (Case 1); and otherwise (Case 2).

Case 1. We proceed by induction on n.

When $n = 0$, since $T(0,\alpha,1) = 0$ and $T(1,\alpha,1) = \alpha T(0,\alpha,1) + S(1,3) = 0 + (2^1 - 1) = 1$, $T(1,\alpha,1) - T(0,\alpha,1) = 1$. On the other hand, $a_1 = 2^0 \alpha^0 = 1$. Therefore, $T(1,\alpha,1) - T(0,\alpha,1) = a_1$ holds.

When $n \geq 1$, for $i \geq 0$, let k_i be the integer such that $a_{k_i} = 2^i$. We assume that the following equation holds up to k_i.

$$T(n,\alpha,1) - T(n-1,\alpha,1) = a_n \quad (1 \leq n \leq k_i) \tag{2}$$

We extend this equation for n's such that $k_i + 1 \leq n \leq k_{i+1}$. For brevity, define $T_{n,t} := \alpha T(n-t,\alpha,1) + S(t,3)$. Then $T(n,\alpha,1) = \min_{1\leq t\leq n}\{T_{n,t}\}$.

Now we clarify with which argument $T_{n,t}$ is minimized.

Lemma 1 *Under the assumption of the induction, the following statements hold.*

(i) When $k_i \leq n \leq k_{i+1} - 1$, $T(n, \alpha, 1) = \min_{1 \leq t \leq n}\{T_{n,t}\}$ takes the minimum at $t = i + 1$.

(ii) When $n = k_{i+1}$, $T(n, \alpha, 1) = \min_{1 \leq t \leq n}\{T_{n,t}\}$ takes the minimum at $t = i + 2$.

The next lemma on the sequence $\{a_n\} = \{p^i q^j\}$ plays the crucial role to prove Lemma 1 and Theorem 2.

Lemma 2 *Let p and q be any natural numbers such that $q \geq p \geq 2$ and let $\{a_n\}_{n \geq 1}$ be the sequence with numbers of the form $p^i q^j$ ($i, j \geq 0$) lined in the increasing order. Then the following statements hold.*

(i) When $q \neq p^l$ for any integer l, for any n such that $p^i < a_n < p^{i+1}$, $a_n = q a_{n-(i+1)}$.

(ii) When $q = p^l$ for some integer l, for any i and n such that $a_n = p^i$, $a_{n+1} = q a_{n-i}$.

A proof of Lemma 2 is given in Appendix. Throughout this section, Lemma 2 is used with $(p, q) = (2, \alpha)$.

Proof of Lemma 1: The difference $T_{n,t+1} - T_{n,t}$ is computed as follows.

$$\begin{aligned} & T_{n,t+1} - T_{n,t} \\ = & \{\alpha T(n-(t+1), \alpha, 1) + S(t+1, 3)\} \\ & - \{\alpha T(n-t, \alpha, 1) + S(t, 3)\} \\ = & -\alpha\{T(n-t, \alpha, 1) - T(n-t-1, \alpha, 1)\} \\ & + (2^{t+1} - 1) - (2^t - 1) \\ = & -\alpha a_{n-t} + 2^t \quad \text{(by Assumption (2)).} \quad (3) \end{aligned}$$

(i) When $k_i \leq n \leq k_{i+1} - 1$, we first show that for $t < i+1$, $T_{n,t}$ is monotonically decreasing. At Equation (3), when $t < i + 1$, both $-a_{n-t}$ and 2^t take the maximums at $t = i$. Therefore,

$$\begin{aligned} T_{n,t+1} - T_{n,t} & \leq -\alpha a_{n-i} + 2^i \\ & < -\alpha a_{k_i - i} + 2^i \text{ (since } k_i \leq n) \\ & = -a_{k_i+1} + a_{k_i} \text{ (by Lemma 2(i))} \\ & < 0. \end{aligned}$$

Thus, $T_{n,t}$ is monotonically decreasing when $t < i + 1$.

When $t \geq i + 1$, both a_{n-t} and 2^t take the minimums at $t = i + 1$. Therefore,

$$\begin{aligned} T_{n,t+1} - T_{n,t} & \geq -\alpha a_{n-(i+1)} + 2^{i+1} \\ & \geq -\alpha a_{k_{i+1}-1-(i+1)} + a_{k_{i+1}} \\ & \quad \text{(since } n \leq k_{i+1} - 1) \\ & = -a_{k_{i+1}-1} + a_{k_{i+1}} \text{ (by Lemma 2(i))} \\ & > 0. \end{aligned}$$

Thus, $T_{n,t}$ is monotonically increasing when $t \geq i + 1$. Consequently, when $k_i \leq n \leq k_{i+1} - 1$, $T_{n,t}$ takes the minimum at $t = i + 1$.

(ii) When $n = k_{i+1}$, the argument is exactly the same with the case $n = k_i$ in (i). So, $T_{k_{i+1},t}$ takes the minimum at $t = i + 2$.

This completes the proof of Lemma 1. □

Now we are ready to prove Case 1 of Theorem 2. It is further divided into two subcases: When $k_i + 1 \leq n \leq k_{i+1} - 1$ (Case 1-1); and when $n = k_{i+1}$ (Case 1-2).

Case 1-1. By Lemmas 1 and 2, $T(n, \alpha, 1) - T(n - 1, \alpha, 1)$ is computed for $k_i + 1 \leq n \leq k_{i+1} - 1$ as follows.

$$\begin{aligned} & T(n, \alpha, 1) - T(n-1, \alpha, 1) \\ = & T_{n,i+1} - T_{n-1,i+1} \\ = & \alpha\{T(n-(i+1), \alpha, 1) - T(n-1-(i+1), \\ & \alpha, 1)\} + S(i+1, 3) - S(i+1, 3) \\ = & \alpha a_{n-(i+1)} \quad \text{(by Assumption (2))} \\ = & a_n \quad \text{(by Lemma 2(i)).} \end{aligned}$$

Thus, Case 1-1 is shown.

Case 1-2. When $n = k_{i+1}$, we should prove $T(k_{i+1}, \alpha, 1) - T(k_{i+1} - 1, \alpha, 1) = a_{k_{i+1}}$ $(= 2^{i+1})$. By Lemma 1, $T(k_{i+1}, \alpha, 1)$ and $T(k_{i+1} - 1, \alpha, 1)$ take the minimums at $t = i + 2$ and $t = i + 1$, respectively. Therefore,

$$\begin{aligned} & T(k_{i+1}, \alpha, 1) - T(k_{i+1} - 1, \alpha, 1) \\ = & T_{k_{i+1},i+2} - T_{k_{i+1}-1,i+1} \\ = & \alpha\{T(k_{i+1} - (i+2), \alpha, 1) - T(k_{i+1} - 1 - \\ & (i+1), \alpha, 1)\} + S(i+2, 3) - S(i+1, 3) \\ = & (2^{i+2} - 1) - (2^{i+1} - 1) = 2^{i+1}. \end{aligned}$$

Thus, Case 1-2 is shown and the proof for Case 1 is completed.

Case 2. Now $\alpha = 2^l$ for some integer $l \geq 1$. Similarly to Case 1, we proceed by induction on n. For $i \geq 0$, let k_i be the largest index n such that $a_n = 2^i$.

When $n = 0$, the proof is exactly the same with Case 1.

When $n \geq 1$, we assume that the following equation holds up to k_i.

$$T(n, \alpha, 1) - T(n-1, \alpha, 1) = a_n \quad (1 \leq n \leq k_i)$$

We extend this equation up to $k_i + 1 \leq n \leq k_{i+1}$, i.e., for n's such that $a_n = 2^{i+1}$. Similarly to Lemma 1, we clarify with which argument $T_{n,t}$ is minimized.

Lemma 3 *Under the assumption of the induction, the following statements hold.*

(i) When $n = k_i$, $T(n, \alpha, 1) = \min_{1 \leq t \leq n}\{T_{n,t}\}$ *takes the minimum at* $t = i + 1$.

(ii) When $k_i + 1 \leq n \leq k_{i+1} - 1$, $T(n, \alpha, 1) = \min_{1 \leq t \leq n}\{T_{n,t}\}$ *takes the minimum at* $t = i + 1, i + 2$.

(iii) When $n = k_{i+1}$, $T(n, \alpha, 1) = \min_{1 \leq t \leq n}\{T_{n,t}\}$ *takes the minimum at* $t = i + 2$.

Proof of Lemma 3: Similarly to Lemma 1, we compute the difference $T_{n,t+1} - T_{n,t} = -\alpha a_{n-t} + 2^t$.

(i) When $n = k_i$, we first show that when $t < i + 1$, $T_{n,t}$ is monotonically decreasing. When $t < i+1$, again both $-a_{n-t}$ and 2^t take the maximums at $t = i$. Therefore,

$$\begin{aligned} T_{k_i,t+1} - T_{k_i,t} &\leq -\alpha\, a_{k_i-i} + 2^i \\ &= -a_{k_i+1} + a_{k_i} \text{ (by Lemma 2(ii))} \\ &< 0. \end{aligned}$$

Thus, $T_{k_i,t}$ is monotonically decreasing when $t < i + 1$.

When $t \geq i + 1$, both $-a_{n-t}$ and 2^t take the minimums at $t = i + 1$. Therefore,

$$\begin{aligned} T_{k_i,t+1} - T_{k_i,t} &\geq -\alpha a_{k_i-(i+1)} + 2^{i+1} \\ &> -\alpha a_{k_i-i} + 2^{i+1} \\ &= -a_{k_i} + 2^{i+1} \text{ (by Lemma 2(ii))} \\ &= -2^i + 2^{i+1} > 0. \end{aligned}$$

Thus, $T_{k_i,t}$ is monotonically increasing when $t \geq i + 1$. In all, when $n = k_i$, $T_{k_i,t}$ takes the minimum at $t = i + 1$.

(ii) When $k_i + 1 \leq n \leq k_{i+1} - 1$, we note that a_n is equal to 2^{i+1} constantly due to the definition of k_i. When $t < i + 1$, both a_{n-t} and 2^t take the maximums at $t = i$. Therefore,

$$\begin{aligned} T_{n,t+1} - T_{n,t} &\leq -\alpha\, a_{n-i} + 2^i \\ &= -a_{n+1} + 2^i \text{ (by Lemma 2(ii))} \\ &< -a_{k_i} + 2^i = 0. \end{aligned}$$

Thus, $T_{n,t}$ is monotonically decreasing when $t < i + 1$.

When $t = i + 1$, $T_{n,t+1} - T_{n,t}$ is computed as

$$\begin{aligned} T_{n,i+2} - T_{n,i+1} &= -\alpha a_{n-(i+1)} + 2^{i+1} \\ &= -\alpha a_{(n-1)-i} + 2^{i+1} \\ &= -a_n + 2^{i+1} \text{ (by Lemma 2(ii))} \\ &= 0. \end{aligned}$$

Therefore, $T_{n,i+2} = T_{n,i+1}$ holds.

When $t > i + 1$, both $-a_{n-t}$ and 2^t take the minimums at $t = i + 2$. Therefore,

$$\begin{aligned} T_{n,t+1} - T_{n,t} &\geq -\alpha a_{n-(i+2)} + 2^{i+2} \\ &> -\alpha a_{n-i} + 2^{i+2} \\ &= -a_{n+1} + 2^{i+2} \text{ (by Lemma 2(ii))} \\ &> 0. \end{aligned}$$

Thus, $T_{n,t}$ is monotonically increasing when $t > i + 1$. In all, when $k_i + 1 \leq n \leq k_{i+1} - 1$, $T_{k_i,t}$ takes the minimum at $t = i + 1,\ i + 2$.

(iii) When $n = k_{i+1}$, the proof is the same with the case (i).

This completes the proof of Lemma 3. □

Now we are ready to prove Case 2 of Theorem 2, i.e., $T(n, \alpha, 1) - T(n-1, \alpha, 1) = a_n$ for $k_i + 1 \leq n \leq k_{i+1}$. In this case, by Lemma 3(i), (ii), and (iii), we observe that $T_{n,t}$ takes the minimum at least at $t = i + 2$ and $T_{n-1,t}$ takes the minimum at least at $t = i + 1$. (For all of the three cases, we choose such common arguments to simplify the computation.) Therefore, for $k_i + 1 \leq n \leq k_{i+1}$,

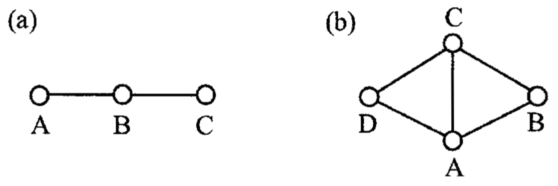

Figure 1: Variants of the Tower of Hanoi problem on graphs.

$$\begin{aligned} & T(n,\alpha,1) - T(n-1,\alpha,1) \\ = \; & T_{n,i+2} - T_{n-1,i+1} \\ = \; & \alpha\{T(n-(i+2),\alpha,1) - T(n-1-(i+1), \\ & \alpha,1)\} + (2^{i+2}-1) - (2^{i+1}-1) \\ = \; & 2^{i+1}. \end{aligned}$$

Therefore, the proof for Case 2 is shown.

This completes the proof of Theorem 2. □

4 On Tower of Hanoi Variants on Graphs

One of the motivation for considering the recurrence relations for $T(n,\alpha,\beta)$ is because they appear in some variants of the Tower of Hanoi problem. For example, we consider the Tower of Hanoi problem on the graphs in Fig. 1, where pegs are located on all of the vertices and disks are moved only through the edges. The objective for the graph in Fig. 1(a) (and (b), resp.) is to move all the n disks from A to C (and A to B, resp.). Then these problems admit algorithms with the following recurrence relations, respectively.

$$\begin{aligned} T_1(n,3,2) &= 3T_1(n-1,3,2)+2 \quad (n \geq 2) \\ T_1(0,3,2) &= 0, \quad T_1(1.3.2) = 1 \end{aligned}$$

$$T_2(n,3,1) = \min_{1 \leq t \leq n} \{3T_2(n-t,3,1) + S(t,3)\}$$

$$T_2(0,3,1) = 0$$

Therefore, the analysis of the recurrence relations for $T(n,\alpha,\beta)$ could be used for these types of Tower of Hanoi variants.

5 Concluding Remarks

We made exact analysis of the recurrence relations generalized from the Tower of Hanoi problem. The differences of $T(n,\alpha,\beta)$'s had unexpectedly simple form such as $\{2^i\alpha^j\}$. It has to be noted that the results of this paper are not the one to improve the bounds of the original multi-peg Tower of Hanoi problem, rather, the contribution should lie on clarifying the combinatorial structures in the set of recurrence relations generalized from the Stewart's algorithm. Relations with number theory, especially with smooth numbers and the properties of the sequence $\{p^iq^j\}$ should be further explored.

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A Proof of Lemma 2

In this appendix, we prove Lemma 2.

Lemma 2 *Let p and q be any natural numbers such that $q \geq p \geq 2$ and let $\{a_n\}_{n\geq 1}$ be the sequence with numbers of the form $p^i q^j$ $(i, j \geq 0)$ lined in the increasing order. Then the following statements hold.*

(i) When $q \neq p^l$ for any integer l, for any n such that $p^i < a_n < p^{i+1}$, $a_n = qa_{n-(i+1)}$.

(ii) When $q = p^l$ for some integer l, for any i and n such that $a_n = p^i$, $a_{n+1} = qa_{n-i}$.

In the proof of Lemma 2, we use the following lemma.

Lemma 4 *Let p and q be natural numbers such that $q \geq p \geq 2$ and let $\{a_n\}_{n\geq 1}$ be the sequence with numbers of the form $p^i q^j$ $(i, j \geq 0)$ lined in the increasing order. Then the following statements hold.*

(i) When $q \neq p^l$ for any integer l, for any integer $j \geq 0$, $|\{a_n \,|\, q^j < a_n < q^{j+1}\}| = \max\{i \,|\, i \in \mathbf{N},\ p^i < q^{j+1}\}$.

(ii) When $q = p^l$ for some integer l, for any integers j and k such that $j \geq 0$ and $0 \leq k \leq l-1$,

$$|\{a_n \,|\, a_n = p^k q^j = p^{jl+k}\}| = j + 1.$$

Proof of Lemma 4: (i) By induction on j. We define $I_j = \{a_n \,|\, q^j < a_n < q^{j+1}\}$ and $i_j = \max\{i \,|\, i \in \mathbf{N},\ p^i < q^{j+1}\}$.

When $j = 0$, $I_j = \{a_n \,|\, 1 < a_n < q\} = \{i \,|\, 1 < p^i < q\}$. Therefore, $|I_0| = i_0$.

Next, assume that $|I_j| = i_j$. Since $I_{j+1} = \{a_n \,|\, q^{j+1} < a_n < q^{j+2}\}$ is the union of the two sets $\{qa_n \,|\, a_n \in I_j\}$ and $\{p^i \,|\, i \in \mathbf{N},\ q^{j+1} < p^i < q^{j+2}\}$,

$$\begin{aligned}
|I_{j+1}| &= |\{qa_n \,|\, a_n \in I_j\}| + |\{p^i \,|\, q^{j+1} < a_k < q^{j+2}\}| \\
&= |I_j| + |\{p^i \,|\, i \in \mathbf{N},\ q^{j+1} < p^i < q^{j+2}\}| \\
&= \max\{i \,|\, i \in \mathbf{N},\ p^i < q^{j+1}\} + \\
&\quad\ |\{p^i \,|\, q^{j+1} < p^i < q^{j+2}\}| \\
&= |\{p^i \,|\, i \in \mathbf{N},\ p^1 < p^i < q^{j+2}\}| \\
&= i_{j+1}.
\end{aligned}$$

Therefore, (i) is proven.

(ii) In this case, the numbers in the sequence $\{p^i q^j\}$ are written as follows.

$$\begin{aligned}
p^i q^j &= 1,\ p,\ p^2,\ \cdots,\ p^{l-1}, \\
&\quad\ p^l,\ p^l,\ p^{l+1},\ p^{l+1},\ \cdots,\ p^{2l-1},\ p^{2l-1}, \\
&\quad\ \cdots \\
&\quad\ p^{jl},\ p^{jl},\ \cdots,\ p^{jl+k},\ \cdots,\ p^{(j+1)l-1}, \\
&\quad\ \cdots.
\end{aligned}$$

For any k such that $0 \leq k \leq l-1$, since there are $j+1$ ways to compute p^{jl+k} using p and $q = p^l$, p^{jl+k} appears $j+1$ times. Namely, $|\{a_n \,|\, a_n = p^k q^j = p^{jl+k}\}| = j + 1$ holds.

This completes the proof of Lemma 4. □

Proof of Lemma 2: (i) By induction on i.

When $i = 1$, since $q \geq p$, there is no a_n such that $p^0 = 1 < a_n < p^1 = p$. Therefore, the equality in (i) holds.

Now assume that for all n such that $p^{s-1} < a_n < p^s$, $1 \leq s \leq i$, $a_n = qa_{n-s}$ holds. We show that for any N such that $p^i < a_N < p^{i+1}$, $a_N = qa_{N-(i+1)}$ holds. We divide into two cases: When $a_N = q^j$ for some integer j (Case 1); and otherwise (Case 2).

Case 1. When $a_N = q^j$, there exist i a_n's between q^{j-1} and $a_N = q^j$ by Lemma 4(i). So, $a_{N-(i+1)}$ is equal to q^{j-1}. Therefore, $a_N = qa_{N-(i+1)}$ holds.

Case 2. When $a_N \neq q^j$ for any integer $j \geq 0$, a_N is divisible by p. So, there exists M such that $p^{i-1} < a_M < p^i$ and $a_N = pa_M$. Then by the assumption of the induction, $a_M = qa_{M-i}$. Therefore, $a_N = pa_M = p(qa_{M-i}) = q(pa_{M-i})$. To prove $a_N = qa_{N-(i+1)}$, it is enough to show that $pa_{M-i} = a_{N-(i+1)}$.

By the definition of $\{a_n\}$ and since $p^i \in \{a_n \mid a_M < a_n < a_N\}$,

$$\begin{aligned}
& |\{a_n \mid a_M/q < a_n < a_N/q\}| \\
= & |\{a_n \mid a_M < a_n < a_N\}| - 1 \\
= & (N - M - 1) - 1 \\
= & N - M - 2. \qquad (4)
\end{aligned}$$

Using (4) and $a_M/q = a_{M-i}$, a_N/q is computed as $a_N/q = a_{(M-i)+(N-M-2)+1} = a_{N-(i+1)}$. Therefore, $a_N = qa_{N-(i+1)}$ holds. This completes the proof of Lemma 2(i).

(ii) Suppose that $q = p^l$ for some integer $l \geq 1$. For $j \geq 0$ and $0 \leq k \leq l-1$, let $G_{j,k}$ be the subsequence of $\{a_n\}$ which consists of p^{jl+k}'s. By Lemma 4(ii), note that $|G_{j,k}| = j + 1$. Furthermore, for $j \geq 0$, let G_j be the union of $G_{j,k}$'s for $0 \leq k \leq l-1$, i.e.,

$$\begin{aligned}
G_j &= \bigcup_{k=0}^{l-1} G_{j,k} \\
&= \{p^{jl}, \cdots, p^{jl}, \cdots, p^{jl+k}, \cdots, p^{jl+k}, \\
&\quad \cdots, p^{(j+1)l-1}, \cdots, p^{(j+1)l-1}\}.
\end{aligned}$$

We note that in this notation, same numbers are not identified as opposed to the usual definition of a set.

Now, it is enough to show that for any $a_n = p^{jl+k}$ in G_j, $a_{n+1} = qa_{n-(jl+k)}$.

Let $a_n = p^{jl+k}$ be any elements in G_j with $j \geq 0$ and $0 \leq k \leq l-1$. Suppose that a_n is the tth element (p^{jl+k}) in $G_{j,k}$, where $1 \leq t \leq j+1$. Then n and $n - (jl+k)$ are explicitly written as follows.

$$\begin{aligned}
n &= \sum_{m=1}^{j} |G_m| + (j+1)k + t \\
&= \sum_{m=1}^{j} ml + (j+1)k + t \\
&= \frac{lj(j+1)}{2} + (j+1)k + t.
\end{aligned}$$

$$\begin{aligned}
n - (jl+k) &= \frac{lj(j+1)}{2} + (j+1)k + t \\
&\quad - (jl+k) \\
&= \frac{l(j-1)j}{2} + jk + t \\
&= \sum_{m=1}^{j-1} |G_m| + jk + t.
\end{aligned}$$

Since t is within $1 \leq t \leq j+1$ and $|G_{j-1,k}| = j$, the place $a_{n-(jl+k)}$ is located in $\{G_j\}$ differs in the following two cases: When $1 \leq t \leq j$ (Case 1); and when $t = j+1$ (Case 2). We consider each of these cases.

Case 1. When $1 \leq t \leq j$, $a_{n-(jl+k)}$ is the tth element in $G_{j-1,k}$. So, $a_{n-(jl+k)} = p^{(j-1)l+k}$. Therefore, we obtain $qa_{n-(jl+k)} = p^l p^{(j-1)l+k} = p^{jl+k} = a_{n+1}$.

Case 2. When $t = j+1$, a_n is the last element in $G_{j,k}$. Then, $a_{n-(jl+k)}$ is the first element in $G_{j-1,k+1}$, i.e., $p^{(j-1)l+k+1}$ except for the case $a_n = p^{(j+1)l-1}$ is the last element in G_j. We consider this exceptional case later. Now since $a_{n-(jl+k)} = p^{(j-1)l+k+1}$, $qa_{n-(jl+k)} = p^l p^{(j-1)l+k+1} = p^{jl+k+1} = a_{n+1}$. Note that the last equality holds because a_n is the last p^{jl+k} in $G_{j,k}$.

We finally consider the exceptional case, that is, when $a_n = p^{(j+1)l-1}$ is the last element in G_j. In this case, $k = l-1$, so $a_{n-(jl+k)} = a_{n+1-(j+1)l}$. Since $|G_j| = (j+1)l$ and a_{n+1} is the first element in G_{j+1}, $a_{n+1-(j+1)l}$ is the first element in G_j, i.e., p^{jl}. Therefore, $qa_{n-(jl+k)} = p^l p^{jl} = p^{(j+1)l} = a_{n+1}$.

This completes the proof of Lemma 2. □