

方形描画（フロアプラン）の個数について：厳密数え上げと下界と上界

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あらまし グラフを、各面が長方形であるように、かつ、辺が交差することなく平面に描画したものを方形描画と呼ぶ。本稿では、内面数 n の方形描画の個数 $R(n)$ に対して以下を示す。(i) $R(n)$ を $O(\text{poly}(n) \cdot 2^n)$ 時間で計算するアルゴリズムと $n = 30$ までの正確な個数。(ii) 極限 $c_R = \lim_{n \rightarrow \infty} R(n)^{1/n}$ が存在し、かつ、 $11.56 < c_R < 28.3$ 。

On the Number of Rectangular Drawings : Exact Counting and Lower and Upper Bounds

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Abstract A rectangular drawing is a plane drawing of a graph in which every inner face is a rectangle. In this paper, we consider the problem of counting the number of rectangular drawings with n faces, denoted by $R(n)$, and show the following : (i) There is an algorithm for counting $R(n)$ in time $O(\text{poly}(n) \cdot 2^n)$ which enables us to determine exact values of $R(n)$ for $n \leq 30$, and (ii) There is a limit $c_R = \lim_{n \rightarrow \infty} R(n)^{1/n}$ such that $11.56 < c_R < 28.3$.

1 Introduction

A rectangular drawing is a plane drawing in which every face is a rectangle. Since rectangular drawings play an important role in e.g., floorplanning, there are a number of researches on a good (i.e., compact and easy to manipulate) representation of a various types of rectangular drawings or floorplans in the VLSI community (see e.g., [1, 6, 9, 13]). In this paper, we focus on the problem of counting the number of rectangular drawings. For instance, there are six rectangular drawings with three faces as shown in right of Fig. 1.

Nakano [8] gave an algorithm to enumerate all rectangular drawings with n faces in $O(1)$ time per drawing. By implementing this algorithm, exact number of rectangular drawings with n faces, denoted by $R(n)$, for up to 13 have been reported [14]. In this paper, we give a faster algorithm for counting $R(n)$, which runs in time $O(\text{poly}(n) \cdot 2^n)$. Although the running time of our algorithm is still exponential in the number of faces, it is considerably faster than the known algorithm based on enumerating. This is because the running time of such an enumeration-based algorithm is inherently $\Omega(R(n))$ and we will show in the paper that $R(n) = \Omega(11.56^n) = \Omega(2^{3.53n})$. By implementing this algorithm, we compute the exact number of rectangular drawings with n faces for up to $n = 30$. It turns out that $R(30) \sim 6.68 \times 10^{25}$.

In this paper, we also consider an asymptotic behavior of the values of $R(n)$, since it is not known whether $R(n)$ has a closed form of expression. Yamanaka and Nakano [12, 13] gave two different methods for encoding rectangular drawings with n faces using $5n$ bits. This immediately implies that the number of rectangular drawings with n faces is at most $2^{5n} = 32^n$. To the best of our knowledge, this is the best known upper bound on $R(n)$. One of the main motivation of our work is to see whether their encoding is best possible in terms of its length.

In our problem, we treat two drawings in left of Fig. 1 as distinct drawings, since we consider not only partitions but also adjacency between faces. If we are only interested in partitions and hence we treat these two drawings as identical one, the objects we will count are known as *mosaic floorplans* (see e.g., [1] or [10] for a formal definition).



Figure 1: Left: Rectangular drawings with three faces. Right: Two rectangular drawings with four faces.

There are many researches on the number of mosaic floorplans with n faces (see e.g., [1, 6, 9, 10]). The number is shown to be equal to the Baxter number $B(n)$, which is,

$$B(n) = \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \sum_{k=1}^n \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1} = \Theta\left(\frac{8^n}{n^4}\right).$$

Since it is obvious that $R(n) \geq B(n)$, the number of rectangular drawings with n faces satisfies $R(n) \geq \Omega(8^n/n^4)$. In this paper, we improve both upper and lower bounds on $R(n)$ to $\Omega(11.56^n) < R(n) < O(28.3^n)$. This implies that every binary encoding for rectangular drawings needs $\log_2 11.56^n > 3.53n$ bits. The proof for the lower bound includes an analysis of the first eigenvalue of a huge ($\sim 1.2 \cdot 10^7 \times 1.2 \cdot 10^7$) matrix with the aid of a computer, and that for the upper bound is based on analyzing the entropy of rectangular drawings. We also show that there is a one-to-one correspondence between rectangular drawings and “general” floorplans of a certain type [1, 9] in which an empty room is allowed to exist. Our bounds on $R(n)$ also improve the best known upper and lower bounds on the number of such “general” floorplans which has been discussed in e.g., [1, 9, 11].

The organization of the paper is as follows. In Section 2, we give some notations and definitions. In Section 3, we give an algorithm for counting the number of rectangular drawings. In Section 4, we show lower and upper bounds on the number of rectangular drawings and a one-to-one correspondence to general floorplans. Finally, in Section 5, we describe some open problems relating to this work.

2 Preliminaries

Below we give the formal definition of the objects that we will count. We follow the definition of Nakano [8] (see also [13]). A drawing of a graph is *plane* if it has no two edges intersect geometrically except at a vertex to which they are both incident. A *plane drawing* divides the plane into connected regions called *faces*. A *rectangular drawing* is a plane drawing that divides a rectangle into smaller ones such that each face is a rectangle. The number of faces of a rectangular drawing is defined to be the number of *inner* faces of it. In this paper, we only consider a rectangular drawing which has no vertex shared by four (or more) rectangles. A *based* rectangular drawing is a rectangular drawing with one designated base line on the boundary of the drawing, and we always draw the base as the lowermost horizontal line segment of the drawing. Two faces F_1 and F_2 are *ns-adjacent* if they share a horizontal line segment. Two faces F_1 and F_2 are *ew-adjacent* if they share a vertical line segment.

For two based rectangular drawings P_1 and P_2 , we say that P_1 and P_2 are isomorphic if P_1 and P_2 have a one-to-one correspondence between their faces preserving ns- and ew-adjacency, in which each base corresponding to the other. Here and hereafter, we usually drop the word “based” since we only consider based rectangular drawings. All rectangular drawings with at most four faces are shown in the left figure in Fig. 3. Note that two rectangular drawings in left of Fig. 1 are not isomorphic, since the adjacencies between the central vertical line are different.

If we don’t care the adjacencies between faces, i.e., consider two rectangular drawings in left of Fig. 1 as an identical one, the objects we will count are known as *mosaic floorplans* (e.g., [1, 10]). Note that the number of rectangular drawings with four faces is 24 whereas that of mosaic floorplans with four faces is 22.

3 Exact Counting

Let $R(n)$ denote the number of rectangular drawings with n faces. In this section, we give an algorithm to compute $R(n)$ in time $O(\text{poly}(n) \cdot 2^n)$.

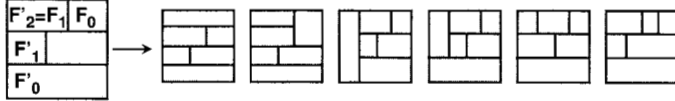


Figure 2: An example of a rectangular drawing with five faces of configuration $(1, 2, 0, 0, 1)$, and its childs.

The idea of our algorithm is as follows : We consider a *generating tree* \mathcal{T} (see the left figure in Fig. 3) for rectangular drawings introduced by Nakano [8] as a starting point of our algorithm. Each vertex in the tree \mathcal{T} at depth n is labeled by a rectangular drawing with n faces, and every rectangular drawing with n faces is labeled to some vertex at depth n . Here the depth of the root of the tree is considered to be 1. The tree \mathcal{T} is obtained by defining a certain relation between a rectangular drawing with $n - 1$ faces and that with n faces based on a *sweeping sequence*. See [8] for the details. Since the number of rectangular drawings with n faces $R(n)$ is given by the number of vertices at depth n in \mathcal{T} , we can count $R(n)$ by traversing the tree \mathcal{T} . However this algorithm apparently needs $\Omega(R(n))$ time, which seems to be intractable for e.g., $n \geq 20$. If we only have to count $R(n)$ (not have to enumerate all drawings) we can *merge* two vertices u and v in \mathcal{T} when we know that u and v have subtrees of an identical structure. This reduces the “width” of \mathcal{T} considerably, and hence reduces the running time of the algorithm for counting $R(n)$.

In order to give a criterion to mergeability, we introduce the notion of a *configuration* of a rectangular drawing.

Definition 1 A *configuration* of a rectangular drawing P is a sequence of non-negative integers $(V, H, l_1, \dots, l_V, r_1, \dots, r_H)$ defined as follows: Let V be the number of inner vertical line segments touching to the upper outer face, and let H be the number of inner horizontal line segments touching to the left outer face. In other words, P has $V + 1$ inner faces sharing the uppermost horizontal line segment and $H + 1$ inner faces sharing the leftmost vertical line segment. For $i = 0, \dots, V$, define l_i as follows : Let F_0, F_1, \dots, F_V be the inner faces sharing the uppermost horizontal line, where they appear from right to left in this order. Let l_i be the number of (possibly outer) faces sharing the right boundary of F_i decreased by one (which equals to the number of junctions of the shape “ \lrcorner ” on the right boundary of F_i). Note that we always have $l_0 = 0$ and so we don’t include l_0 in a configuration. Similarly, let r_i ($i = 0, \dots, H$) be the number of (possibly outer) faces sharing the bottom boundary of F_i decreased by one (which equals to the number of junctions of the shape “ \top ” on the bottom boundary of F_i) where F'_0, F'_1, \dots, F'_H be the inner faces sharing the leftmost vertical line appearing from bottom to top. We don’t include r_0 in a configuration since r_0 is always 0. See Fig. 2 for an example. \square

The following theorem is implicitly in [8].

Theorem 1 Suppose that P is a rectangular drawing in the generating tree \mathcal{T} whose configuration is $(V, H, l_1, \dots, l_V, r_1, \dots, r_H)$. For convenience, we put $l_0 = r_0 = 0$. Then P has $\sum_{i=0}^V (l_i + 1) + \sum_{i=0}^H (r_i + 1)$ childs and their configurations are, for each $v \in \{0, \dots, V\}$ and for each $l'_v \in \{0, \dots, l_v\}$,

$$(v, H + 1, l_1, \dots, l_{v-1}, l'_v, r_1, \dots, r_H, V - v),$$

and, for each $h \in \{0, \dots, H\}$ and for each $r'_h \in \{0, \dots, r_h\}$

$$(V + 1, h, l_1, \dots, l_V, H - h, r_1, \dots, r_{h-1}, r'_h). \quad \square$$

For example, the configurations of the childs of a drawing of configuration $(1, 2, 0, 0, 1)$ (see Fig. 2) are

$$(0, 3, 0, 1, 1), (1, 3, 0, 0, 1, 0), (2, 0, 0, 2), (2, 1, 0, 1, 0), (2, 2, 0, 0, 0, 0), (2, 2, 0, 0, 0, 1).$$

The above theorem says that the structure of the tree \mathcal{T} is completely determined by configurations of drawings. If a configuration v is a child of u , we say that u *yields* v . The following fact shows that the width of our “generating graph” is $\Theta(n2^n)$.

Fact 1 The number of possible configurations of rectangular drawings with n faces is $(n + 1) \cdot 2^{n-2} - 2n + 3$.

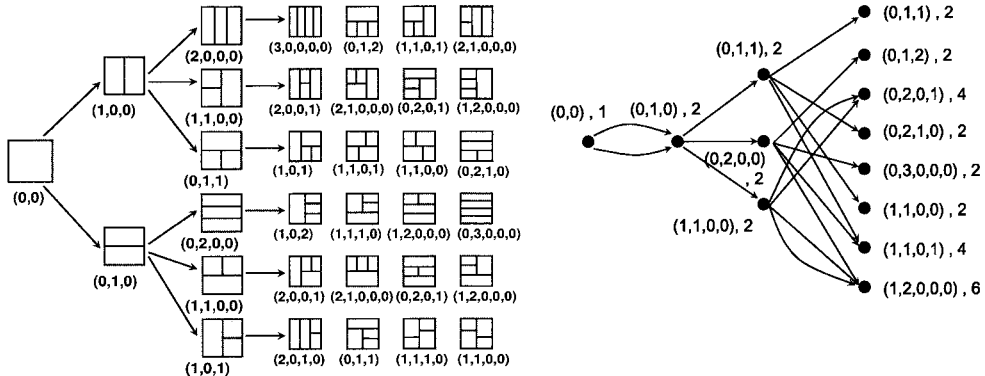


Figure 3: Left: A generating tree \mathcal{T} for rectangular drawings with their configurations. Right: A graph \mathcal{G}^R obtained from \mathcal{T} by merging vertices having same configuration (taking into account a mirror symmetry).

Proof It is not hard to verify that a sequence of non-negative integers $(V, H, l_1, \dots, l_V, r_1, \dots, r_H)$ is a configuration of some rectangular drawing with n faces if and only if it satisfies (i) $1 \leq V + H \leq n - 1$, and (ii) $l_1 + \dots + l_V + r_1 + \dots + r_H \leq n - 1 - (V + H)$, and doesn't satisfy (iii) $V = 0, 1 \leq H \leq n - 2$ and $r_1 = \dots = r_H = 0$, or (iv) $1 \leq V \leq n - 2, H = 0$ and $l_1 = \dots = l_V = 0$.

First we count the number of sequences that satisfies (i) and (ii). Suppose $V = v$ and $H = h$, the number of sequences that satisfies (ii) is $\binom{n-1}{v+h}$. Hence the number of sequences satisfying (i) and (ii) is

$$\sum_{\substack{v, h \geq 0 \\ 1 \leq v+h \leq n-1}} \binom{n-1}{v+h} = \sum_{k=1}^{n-1} \left((k+1) \binom{n-1}{k} \right) = (n+1) \cdot 2^{n-2} - 1.$$

Since the number of sequences satisfying (iii) or (iv) is $2(n-2)$, the number of possible configurations is given by

$$(n+1) \cdot 2^{n-2} - 1 - 2(n-2) = (n+1) \cdot 2^{n-2} - 2n + 3.$$

This completes the proof of the fact. \square

The following observation further reduces the number of configurations needed to consider by about a half : For a configuration $C = (V, H, l_1, \dots, l_V, r_1, \dots, r_H)$, the *mirror* of C , denoted by $\text{Mir}(C)$, is defined as $\text{Mir}(C) = (H, V, r_1, \dots, r_H, l_1, \dots, l_V)$. Since a subtree in \mathcal{T} whose root is C and that whose root is $\text{Mir}(C)$ have same structure, we can also merge C and $\text{Mir}(C)$ in \mathcal{T} .

Let \mathcal{C}_n be the set of all possible configurations of rectangular drawings with n faces. The *representative* for a configuration C is the lexicographically smaller element in $\{C, \text{Mir}(C)\}$. Let \mathcal{C}_n^R be the set of all representatives of $C \in \mathcal{C}_n$. The *multiplicity* of a configuration C for n is the number of rectangular drawings with n faces such that the representative of its configuration is C . If there is no fear of confusion, we simply call this the multiplicity of C .

The above argument naturally defines a directed multigraph \mathcal{G}^R each of whose vertices is labeled by a configuration of a rectangular drawing together with its multiplicity (Fig. 3, right). The graph \mathcal{G}^R has a layered structure, where the k -th layer consists of vertices corresponding to \mathcal{C}_k^R . Here and hereafter, we identify a vertex in \mathcal{G}^R with a configuration attaching to it. If a configuration $u \in \mathcal{C}_k^R$ yields a configuration $v \in \mathcal{C}_{k+1}^R$, then we place a directed edge from u to v . The right figure in Fig. 3 shows the first four layers of \mathcal{G}^R obtained from \mathcal{T} .

The algorithm for counting $R(n)$ is now obvious. Let $\mathcal{C}_1^R = \{(0,0)\}$ and the multiplicity of the configuration $(0,0)$ is set to 1. Then, for $k = 2, \dots, n$ do the followings : Compute \mathcal{C}_k^R and the multiplicities of configurations therein from \mathcal{C}_{k-1}^R using Theorem 1. Finally, output the sum of the multiplicities over all configurations in \mathcal{C}_n^R . From Theorem 1, we have $|\mathcal{C}_k^R| = \Theta(k2^k)$. Since the outdegree of a vertex in the k -th layer of \mathcal{G}^R is at most $k+1$, the running time of the algorithm is obviously $O(\text{poly}(n) \cdot 2^n)$.

By implementing this algorithm, we compute exact number of rectangular drawings with n faces for up to $n = 30$.

n	$R(n)$	n	$R(n)$
1	1	16	533, 530, 004, 810
2	2	17	4, 970, 471, 875, 914
3	6	18	47, 169, 234, 466, 788
4	24	19	455, 170, 730, 152, 340
5	116	20	4, 459, 456, 443, 328, 824
6	642	21	44, 300, 299, 824, 885, 392
7	3, 938	22	445, 703, 524, 836, 260, 400
8	26, 194	23	4, 536, 891, 586, 511, 660, 256
9	186, 042	24	46, 682, 404, 846, 719, 083, 048
10	1, 395, 008	25	485, 158, 560, 873, 624, 409, 904
11	10, 948, 768	26	5, 089, 092, 437, 784, 870, 584, 576
12	89, 346, 128	27	53, 845, 049, 871, 942, 333, 501, 408
13	754, 062, 288	28	574, 315, 446, 827, 677, 760, 726, 480
14	6, 553, 942, 722	29	6, 172, 046, 042, 022, 742, 439, 905, 880
15	58, 457, 558, 394	30	66, 800, 176, 075, 969, 887, 041, 459, 650

A simple calculation verifies that $R(n)/R(n-1)$ is monotonically increasing with n for $n \leq 30$. We conjecture that this is true for every n , which immediately implies $R(n) = \Omega(10.82^n)$ since $R(30)/R(29) \sim 10.823$. We will show a better lower bound on $R(n)$ in the next section.

4 Upper and Lower Bounds

In this section, we analyze an asymptotic behavior of $R(n)$.

Let $m, n \geq 1$ be arbitrary fixed integers. For a rectangular drawing P with m faces and Q with n faces, let $P \mid Q$ be an arbitrary rectangular drawing with $m+n$ faces obtained by overlapping the rightmost line segment of P and the leftmost line segment of Q . For two drawings P_1 and P_2 with m faces and two drawings Q_1 and Q_2 with n faces, if $P_1 \neq P_2$ or $Q_1 \neq Q_2$ then $P_1 \mid Q_1 \neq P_2 \mid Q_2$. This implies that $R(n)$ is super-multiplicative, i.e., $R(n+m) \geq R(n) \cdot R(m)$, and hence the limit $\lim_{n \rightarrow \infty} R(n)^{1/n}$ exists. Let c_R be this limit. Below we give an upper and lower bound on c_R .

4.1 Lower Bound

Theorem 2 $c_R > 11.56$.

Proof Let \mathcal{G}^R be a graph defined in the last section. The number of rectangular drawings with n faces is equal to the number of paths of length $n-1$ in \mathcal{G}^R . The difficulties on estimating $R(n)$ come from the fact that the width of the graph \mathcal{G}^R is increasing exponentially with the depth of the graph. Below we consider a subgraph of \mathcal{G}^R with a limited width and estimate the number of paths in the subgraph to show a lower bound on the number of paths in the original graph.

Recall that \mathcal{C}_k^R is the set of all possible representative configurations of rectangular drawings with k faces. Let $\tilde{\mathcal{C}}_k$ be a subset of \mathcal{C}_k^R defined as $\tilde{\mathcal{C}}_k = \mathcal{C}_k^R \setminus \{(i, k-1-i, 0, \dots, 0) \mid 0 \leq i \leq \lfloor k/2 \rfloor\}$. In other words, $\tilde{\mathcal{C}}_k$ is a set of configurations obtained from \mathcal{C}_k^R by removing $(V, H, 0, \dots, 0)$ with $V+H = k-1$. The reason why we define $\tilde{\mathcal{C}}_k$ as above is to guarantee the irreducibility of a matrix A_k which we will define later. Remark that, for every $m > k$, $\tilde{\mathcal{C}}_k \subseteq \mathcal{C}_m^R$ holds.

Let \mathcal{H}_k be a subgraph of \mathcal{G}^R obtained by removing all the vertices in the first $k-1$ layers, and removing all the vertices whose configuration is not in $\tilde{\mathcal{C}}_k$ for the k' -th layer for each $k' \geq k$. Note that each two adjacent layers in \mathcal{H}_k is an isomorphic bipartite graph on the vertex set $\tilde{\mathcal{C}}_k \cup \tilde{\mathcal{C}}_k$. We denote this bipartite graph by \mathcal{H}_k^2 . The number of paths of length $n-1$ in \mathcal{G}^R , which is equal to $R(n)$, is lower bounded by the number of paths of length $n-k$ in the graph \mathcal{H}_k .

Table 1: The values of λ_k .

k	λ_k	k	λ_k	k	λ_k	k	λ_k
3	1	8	6.5792	13	9.4936	18	10.9136
4	2.1304	9	7.3569	14	9.8575	19	11.1059
5	3.4134	10	8.0202	15	10.1747	20	11.2775
6	4.6116	11	8.5873	16	10.4525	21	11.4312
7	5.6686	12	9.0739	17	10.6972	22	11.5695

Define $|\tilde{\mathcal{C}}_k| \times |\tilde{\mathcal{C}}_k|$ matrix A_k as follows : Each row and column of A_k is indexed by a configuration in $\tilde{\mathcal{C}}_k$. The (u, v) -entry of A_k is the number of paths from v to u in the graph \mathcal{H}_k^2 . Let $\mathbf{1}$ be the $|\tilde{\mathcal{C}}_k|$ -dimensional column vector whose entries are all 1. Then the number of paths of length n in \mathcal{H}_k is given by $\mathbf{1}^T A_k^n \mathbf{1}$.

Since we can show that A_k is irreducible in a sense that for some integer i , all entries of A_k^i is strictly positive¹ we can apply Perron-Frobenius theorem to show that A_k has a dominant eigenvalue λ_k of multiplicity one, and all entries of corresponding eigenvector is positive. This implies

$$\lim_{n \rightarrow \infty} (\mathbf{1}^T A_k^n \mathbf{1})^{1/n} = \lambda_k,$$

which intuitively says that the growth rate of the number of paths in \mathcal{H}_k is given by the first eigenvalue of the matrix A_k . Similar technique is recently used e.g., to obtain a lower bound on the number of permutations having a certain property [2], or the number of cycles in a planar graph [4].

Let $k = 22$. The matrix A_k has $|\tilde{\mathcal{C}}_{22}| = 12059104$ rows and columns. Since A_k is very sparse, i.e., the number of non-zero entries of each columns is at most $k + 1 = 23$, it is not so hard to compute the first eigenvalues of A_k with the aid of a computer. In fact, the number of edges in \mathcal{H}_k^2 is about 240 million, and hence A_k can be stored using $2.4 \times 10^8 \times 8$ bytes ~ 2 Gbytes of memory. Since the multiplicity of the first eigenvalue λ_{22} is shown to be one, this can easily be calculated by applying the power method to A_k (see e.g., [5, p.149]).

Let v be a column vector such that the first entry of v is 1 and all other entries are 0. Then compute $v := A_k v / \|A_k v\|$ iteratively until the vector v sufficiently converges. An approximation of the first eigenvalue is then given by $\|A_k v\|$. It should be noted that the irreducibility of A_k is used only for guaranteeing the convergence of this procedure in this proof and is not needed for the lower bound.

By implementing the above procedure on a computer, we have $\lambda_{22} \sim 11.5695$ together with an approximate eigenvector \vec{v} such that $\|\vec{v}\| = 1$. A direct computation shows that $A_k \vec{v} \geq (11.56)\vec{v}$ and every entries of \vec{v} is non-negative. Hence we have

$$\mathbf{1}^T A_k^n \mathbf{1} \geq \mathbf{1}^T A_k^n \vec{v} \geq \mathbf{1}^T (11.56^n) \vec{v} = \Omega(11.56^n),$$

which completes the proof of the theorem. \square

Apparently, it is possible to obtain a better lower bound by considering A_k with $k \geq 23$. However, we should note that the dimension of A_{22} is about $1.2 \cdot 10^7$ and it will roughly double for each additional k . We present the table of the values of the dominant eigenvalues λ_k for $k \leq 22$ (Table 1). It is almost surely that the limit $\lambda_\infty := \lim_{k \rightarrow \infty} \lambda_k \leq c_R$ exceeds 12. We *guess* that this limit lies between 13 and 15.

4.2 Upper Bound

Theorem 3 $c_R < 28.3$.

Due to the space constraint, we can only give a rough sketch of the proof for the upper bound. For a detailed proof, see [3].

It is well-known that the number of mosaic floorplans with n faces is equal to the number of Baxter permutations on $[n]$ (see e.g., [1]).

¹The proof is omitted due to the space restriction.

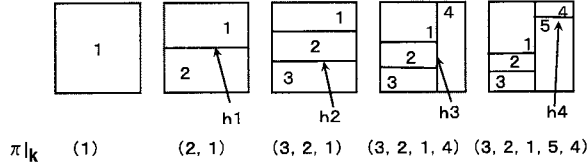


Figure 4: A generation of $\text{BP2FP}(\pi|_k)$ for $\pi = (3, 2, 1, 5, 4)$.

Definition 2 For positive integer n , let $[n]$ denote the set $\{1, \dots, n\}$. A permutation $\pi : [n] \rightarrow [n]$ is a Baxter permutation if there are no four indices $1 \leq i < j < k < l \leq n$ such that (i) $\pi(k) < \pi(i) + 1 = \pi(l) < \pi(j)$, or (ii) $\pi(j) < \pi(l) + 1 = \pi(i) < \pi(k)$. \square

Several bijections between mosaic floorplans and Baxter permutations have been proposed (see the introduction of [1] for a short history). Among them, we use the mapping BP2FP presented by Ackerman et al. [1] to show the upper bound. The mapping BP2FP is defined by an algorithm, which takes a Baxter permutation on $[n]$ as an input and outputs a mosaic floorplan with n faces [1].

A single mosaic floorplan is usually corresponding to a number of rectangular drawings.

Definition 3 For a permutation π on $[n]$ and $k \leq n$, let $\pi|_k$ denote the permutation on $[k]$ such that $\pi|_k = (\pi(i_1), \pi(i_2), \dots, \pi(i_k))$ where $i_1 < i_2 < \dots < i_k$ and $\pi(i_j) \leq k$ for every j .

Let $\pi = (\sigma_1, \dots, \sigma_n)$ be a Baxter permutation on $[n]$. Define the multiplicity of π , denoted by $\text{Mul}(\pi)$, as follows: Let $P(\pi)$ be a mosaic floorplan obtained by algorithm BP2FP with input π . Let h_1, \dots, h_{n-1} be line segments of $P(\pi)$ where h_k is a line segment introduced in generating from $\text{BP2FP}(\pi|_k)$ to $\text{BP2FP}(\pi|_{k+1})$ (see Fig 4). If h_i is a vertical line segment, let l_i be the number of the junctions of the shape ‘ \lrcorner ’ on h_i (which equals to the number of horizontal lines whose right end is on h_i) and r_i be the number of junctions of the shape ‘ \llcorner ’ on h_i (which equals to the number of horizontal lines whose left end is on h_i). Similarly, if h_i is a horizontal line segment, let l_i be the number of the junctions of the shape ‘ \perp ’ on h_i (which equals to the number of vertical lines whose bottom end is on h_i) and r_i be the number of junctions of the shape ‘ \top ’ on h_i (which equals to the number of vertical lines whose top end is on h_i). Then,

$$\text{Mul}(\pi) := \prod_{i=1}^{n-1} \binom{l_i + r_i}{l_i}.$$

We define the parameter of h_i by (l_i, r_i) . \square

For example, in the rightmost drawing in Fig 4, the parameter for h_3 is $(2, 1)$ and is $(0, 0)$ for h_1, h_2 and h_4 , and the multiplicity of the drawing is 3.

Based on the above definition, the number of rectangular drawings with n faces is given by the sum of $\text{Mul}(\pi)$ ’s over all Baxter permutations π on $[n]$. Since the number of Baxter permutations on $[n]$ is known to be at most 2^{3n} and $\text{Mul}(\pi) \leq 2^{2n}$ for every π (this can easily be proved using the fact $\sum_i (l_i + r_i) \leq 2n$), we have $R(n) \leq 2^{5n}$. This equals to the bound obtained from the fact that there is a method for encoding rectangular drawings with n faces using $5n$ bits [12, 13].

It seems to be natural to expect that this bound can be improved by a deeper analysis of $\text{Mul}(\pi)$. In fact, we can show the following lemma which says that the fraction of Baxter permutations π such that $\text{Mul}(\pi)$ is large is small.

Lemma 1 The number of Baxter permutations π on $[n]$ such that $\text{Mul}(\pi) = \Omega(2^{\beta n})$ is at most $O(2^{(1+\beta)n})$ for $\beta = 1.822$. \square

The proof of Lemma 1 is done by a careful analysis of $\text{Mul}(\pi)$ including some numerical examinations. See [3] for the proof. If we can establish Lemma 1, then the number of rectangular drawings with n faces is upper bounded by

$$O(2^{2.822n}) \cdot 2^{2n} + 2^{3n} \cdot O(2^{1.822n}) = O(2^{4.822n}) = O(28.3^n),$$

which is as desired.

4.3 Correspondence between Rectangular Drawings and General Floorplans

When we consider a problem to find a floorplan that minimizes criteria such as area or wire-length, it is well known that an optimal floorplan might contain empty rooms. Young et al. [11] showed that when searching for an optimal floorplan, it is enough to consider floorplans in which every empty room is at the center of pin-wheel structure and has no room-to-room neighbor. They call such floorplans as “general” floorplans (although Ackerman et al. [1] call these as “potentially optimal floorplans”). Shen and Chu [9] then showed that the number of such “general” floorplans is equal to the sum of $\text{Mul}(\pi)$'s over all Baxter permutations π on $[n]$ ([9, Sect. V]). By combining this with the fact that $\text{Mul}(\pi) = O(2^{2n}/\sqrt{n})$, they showed the number of “general” floorplans is between $\Omega(2^{3n}/n^4)$ and $O(2^{5n}/n^{4.5})$.

In the argument of the proof of the upper bound on $R(n)$, we show that $R(n)$ is the sum of $\text{Mul}(\pi)$'s over all Baxter permutation π on $[n]$, which is identical to the quantity described above. As a byproduct, our bounds of $\Omega(11.56^n) \sim \Omega(2^{3.53n})$ and $O(28.3^n) \sim O(2^{4.83n})$ give the current best upper and lower bounds on the number of “general” floorplans which has been investigated in e.g., [1, 9, 11].

5 Open Problems

In this paper, we give an algorithm that computes the number of rectangular drawings with n faces in time $O(\text{poly}(n) \cdot 2^n)$, and show that the growth rate of $R(n)$ is between 11.56 and 28.3. To narrow the gap between the upper and lower bounds is apparently an interesting open problem. We believe that the lower bound is closer to the truth; we *guess* that the value is around 14.

Some other open problems are listed below.

- Is there a procedure that computes $R(n)$ whose running time is a polynomial in n ?
- What is the value of the limit $\lambda_\infty := \lim_{n \rightarrow \infty} \lambda_n$? Is $\lambda_\infty = c_R$?
- Is there a natural class of permutations that has a bijection to the set of rectangular drawings?

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