点次数の増加上限制約を持つ2点連結グラフに対する線形時間3点連結化アルゴリズム

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[概要] 無向グラフ $G = (V, E)$ と各点 $v \in V$ の次数増加上限 $g(v) \in Z^+ \cup \{\infty\}$ が与えられたときに，次数増加上限制約を満たしつつ G に最小本数の辺を付加して 3 点連結グラフを構成する問題 (3VCA-DC) を考える．本稿では，G が 2 点連結である場合に本問題が線形時間で解けることを示す．

A Linear Time Algorithm for Tri-connectivity Augmentation of Bi-connected Graphs with Upper Bounds on Vertex-Degree Increase

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[Abstract] The 3-vertex-connectivity augmentation problem of a graph with degree constraints, 3VCA-DC, is defined as follows: "Given an undirected graph $G = (V, E)$, and an upper bound $g(v) \in Z^+ \cup \{\infty\}$ on vertex-degree increase for each $v \in V$, find a smallest set E' of edges such that $(V, E \cup E')$ is 3-vertex-connected and such that vertex-degree increase of each $v \in V$ by the addition of E' to G is at most $g(v)$, where Z^+ is the set of nonnegative integers." In this paper we show that checking the existence of a feasible solution and finding an optimum solution to 3VCA-DC for any bi-connected graph G can be done in $O(|V| + |E|)$ time.

1 Introduction

The connectivity augmentation problem of a graph asks for finding a smallest (or minimum-cost) set of edges whose addition to a given graph results in a graph satisfying some prescribed connectivity properties. This paper considers the 3-vertex-connectivity augmentation problem of a graph with degree constraints.

Given an undirected graph $G = (V, E)$ and a subset $S \subseteq V$, G is said to be *k-vertex-connected with respect to S* (or *k-connected with respect to S*) if and only if G has at least k internally-disjoint paths between any pair of vertices in S, where if $S = V$ then "with respect to S" is neglected. In particular, we say that G which is k-connected with respect to S is *bi-connected (tri-connected) with respect to S* if $k = 2$ ($k = 3$, respectively).

The k-vertex-connectivity augmentation problem for a specified set of vertices of a graph with degree constraints, kVCA-SV-DC, is defined as follows: "Given a positive integer k, an undirected graph $G = (V, E)$, a specified set $S \subseteq V$ and an upper bound $g(v) \in Z^+ \cup \{\infty\}$ on vertex-degree increase for each $v \in V$, find a smallest set E' of edges such that $G + E'$ is k-connected with respect to S and such that vertex-degree increase of each $v \in V$ by the addition of E' to G is at most $g(v)$, where $G + E' = (V, E \cup E')$ and Z^+ is the set of nonnegative integers." We call any set F of edges a *solution to* kVCA-SV-DC if $G + F$ is k-connected with respect to S, and we say that any set F' of edges is *feasible* (or more precisely *g-feasible*) if F' includes at most $g(v)$ edges incident to v for each $v \in V$. Any feasible solution of minimum cardinality is called an *optimum solution to* kVCA-SV-DC. kVCA-SV-DC has application to designing communication networks. kVCA-SV-DC with $S = V$ is denoted as kVCA-DC. kVCA-SV-DC with $g(v) = \infty$ for all $v \in V$ is denoted as kVCA-SV. kVCA-SV with $S = V$ is denoted as kVCA. Figure 1 shows an instance of 3VCA-DC.

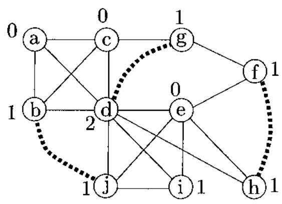

Figure 1: An instance, $G = (V, E)$ and $g : V \to Z^+ \cup \{\infty\}$, of 3VCA-DC, where G is bi-connected and the number beside each vertex $v \in V$ denotes $g(v)$. The edge set $\{(b, j), (d, g), (f, h)\}$ shown by bold dotted lines is an optimum solution. Notice that if $g(b) = 0$ then there is no feasible solution because $g(\{a, b\}) = g(a) + g(b) = 0$.

We summarize known results on kVCA-SV-DC. For 2VCA, a polynomial time algorithm was proposed by Eswaran and Tarjan [1], and a linear time algorithm was proposed Rosenthal and Goldner [11] and Hsu and Ramachandran [5]. For 3VCA, a polynomial time algorithm was devised by Watanabe and Nakamura [14], and a linear time algorithm was proposed by Hsu and Ramachandran [4]. For 4VCA, Hsu [3] devised an $O(|V| \log |V| + |E|)$ time algorithm. For any fixed k, Jackson and Jordán [6] first devised a polynomial time algorithm for kVCA. Concerning kVCA-SV, it was shown that kVCA-SV can be solved in linear time by reducing it to kVCA by Watanabe, Higashi, and Nakamura [13] if $k = 2$ and by Mashima and Watanabe [10] if $k = 3$. For 2VCA-SV-DC, we proposed a linear time algorithm [9].

In this paper we show that checking the existence of a feasible solution and finding an optimum solution to 3VCA-DC for any bi-connected graph G can be done in $O(|V| + |E|)$ time.

The paper is organized as follows. Section 2 provides some definitions and notations. Section 3 shows a necessary and sufficient condition for the existence of a feasible solution to 3VCA-DC. Section 4 presents a linear time algorithm for finding an optimum solution to 3VCA-DC. The concluding remarks are given in Sect. 5.

2 Preliminaries

2.1 Basic Definitions

A singleton set $\{x\}$ may be written as x. The notation "$\subseteq$" means inclusion, and "$\subset$" does proper inclusion. For a real number x, let $\lceil x \rceil$ ($\lfloor x \rfloor$, respectively) denote the smallest integer not less than x (the largest integer not more than x).

An *undirected graph* $G = (V(G), E(G))$ consists of a finite and nonempty set $V(G)$ of vertices and a finite set $E(G)$ of undirected edges. $V(G)$ and $E(G)$ are often denoted as V and E, respectively. We assume that graphs have neither multiple edges nor self-loops unless otherwise stated. An edge whose endvertices are u and v is denoted by (u, v). The *degree* of a vertex v in G, $d_G(v)$, is the number of edges incident to v in G. For a set E' of edges with $E' \cap E = \emptyset$, $G + E'$ denotes the graph $(V, E \cup E')$. For a set $X \subseteq V \cup E$, $G - X$ denotes the graph obtained from G by deleting X, where any edge incident to $v \in X$ is also removed. A *subgraph* of G is any graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V$, let $\Gamma(X; G) = \{v \in V - X \mid (u, v) \in E(G)$ for some $u \in X\}$ and $\xi(X; G) = V - (X \cup \Gamma(X; G))$. If G is clear from the context, $\Gamma(X; G)$ and $\xi(X; G)$ may be denoted as $\Gamma(X)$ and $\xi(X)$, respectively. Let $V(E') = \{u, v \in V \mid (u, v) \in E'\}$ for any set E' of edges. For any function $f : V \rightarrow Z^+ \cup \{\infty\}$ and any set $X \subseteq V$, we use the notation $f(X) = \sum_{v \in X} f(v)$.

A *path between u and v*, or a *(u, v)-path*, is an alternating sequence of vertices and edges $u = v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n = v$ ($n \geq 0$) such that if $n \geq 1$ then $v_0, \ldots, v_n$ are all distinct and $e_i = (v_{i-1}, v_i)$ for each i, $1 \leq i \leq n$. If $n \geq 2$ then the vertices $v_1, v_2, \cdots, v_{n-1}$ are called the *inner vertices* of the path. A set of paths is said to be *internally-disjoint* if no two of them have an inner vertex in common. A *cycle* is a (v_0, v_n)-path, $n \geq 2$, together with the edge (v_0, v_n). An undirected graph G is *connected* if G has a path between any pair of vertices in G; otherwise G is *disconnected*. Any maximal connected subgraph of G is called a *connected component* or simply a *component* of G. G is *acyclic* if G contains no cycles. A *forest* is an acyclic undirected graph, and a *tree* is a connected forest. A *leaf* of a tree is a vertex with only one edge incident to it.

Let G be a connected graph. Any subset R of V is called a *separator of G* if and only if $G - R$ is disconnected. Any separator of minimum cardinality is called a *minimum separator of G*. The *vertex-connectivity $\kappa(G)$ of G* is the cardinality of any minimum separator of G, where if G is a complete graph then $\kappa(G)$ is defined to be $|V| - 1$. If $\kappa(G) \geq k$ for a positive integer k then G is said to be *k-vertex-connected* (or *k-connected*); in particular, if $\kappa(G) \geq 2$ ($\kappa(G) \geq 3$) then G is said to be *bi-connected* (*tri-connected*, respectively). Note that $\kappa(G) \geq k$ holds if and only if G has at least k internally-disjoint paths between any pair of vertices in V.

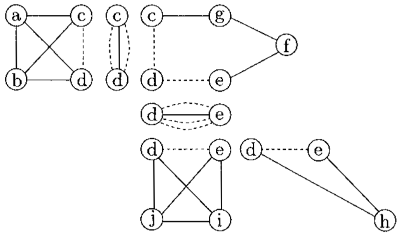

Figure 2: Tri-components of the graph G in Fig. 1. Edges represented as broken lines are virtual edges.

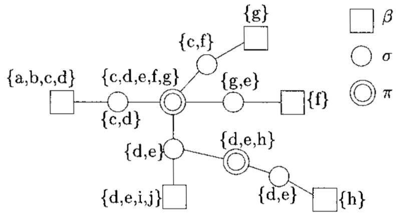

Figure 3: 3-blk(G) of the graph G in Fig. 1.

Let G be a bi-connected graph. For any pair of vertices $u, v \in V$, $\{u, v\}$ is called a *separation pair of G* if and only if $\{u, v\}$ is a separator of G. Let SP(G) denote the class of all separation pairs of G, where the part "(G)" is often omitted for simplicity unless any confusion arises. Let $K \in$ SP(G). Any component of $G - K$ is called a *K-component* of G. The *separating degree* of K in G is the number of K-components of G, and it is denoted by $d_G(K)$ or simply $d(K)$. In this paper, we often identify a K-component (or a component) with its vertex set.

2.2 Tri-Components and 3-Block Graph

Let G be a bi-connected graph. *Tri-connected components* (*tri-components* for short) [2, 12] of G are constructed from G by the two operations "split" and "merge". Tri-components can be partitions into three classes: tri-connected graphs, polygons and bonds, where a *bond* is a graph consisting of two vertices and at least three multiple edges between them. For the details of tri-components, refer to [2, 12]. Figure 2 shows tri-components of the graph G in Fig. 1. Any separation pair shared by at least two tri-components, each being either a tri-connected graph or a polygon, is called a separation pair of *adjacent type*, and any separation pair consisting of nonadjacent vertices in a polygon is called a separation pair of *nonadjacent type*. Note that any separation pair is either of adjacent type or of nonadjacent type. Let $SP_A(G) = \{K \in SP(G) \mid K$ is of adjacent type$\}$ and $SP_N(G) = SP(G) - SP_A(G)$, where the part "($G$)" is often omitted for simplicity unless any confusion arises.

Remark 2.1 $d_G(K) = 2$ *for any* $K \in SP_N(G)$. □

The 3-block graph [4], 3-blk(G), of a bi-connected graph G is constructed as follows. The vertex set of 3-blk(G) consists of three kinds of vertices called *σ-vertices*, *π-vertices* or *β-vertices*: we create a *σ-vertex* for each separation pair of adjacent type, a *π-vertex* for each polygon, and a *β-vertex* for each tri-component that is a tri-connected graph. Then distinct vertices u, v created above are connected by an edge if and only if either (1) or (2) holds: (1) $\{u, v\}$ is a pair of a σ-vertex and a β-vertex, and the separation pair represented by the σ-vertex is contained in the tri-connected graph represented by the β-vertex; (2) $\{u, v\}$ is a pair of a σ-vertex and a π-vertex, and the separation pair represented by the σ-vertex is contained in the polygon represented by the π-vertex. Moreover, for each each vertex $w \in V(G)$ with $d_G(w) = 2$, do the following: create a β-vertex u for $\{w\}$, and a σ-vertex v for $\Gamma(w; G)$; connect u and v by an edge; connect by an edge the vertex v and the π-vertex representing the polygon to which w belongs. The resulting graph is 3-blk(G). Figure 3 shows 3-blk(G) of the graph G in Fig. 1. 3-blk(G) of a bi-connected graph G is a tree and it can be constructed in $O(|V|+|E|)$ time by using the algorithm in [2] for finding tri-components of G.

Let $v \in V$(3-blk(G)). The degree of v in 3-blk(G) is denoted by $d_{3\text{-blk}(G)}(v)$ or simply $d(v)$ unless any confusion arises. If $d(v) = 1$ then v is called a *leaf* of 3-blk(G). Note that any leaf of 3-blk(G) is a β-vertex.

Let V_σ(3-blk(G)) denote the set of all σ-vertices of 3-blk(G). Any subset $K \subseteq V$ represented by a σ-vertex of 3-blk(G) is called a *σ-pair of* G. Let SI(G) or simply SI denote the class of all σ-pairs of G.

Remark 2.2 *Let G be a bi-connected graph with $|V| \geq 4$.*
(1) The following conditions are equivalent: (a) $\kappa(G) \geq 3$; (b) $\mathrm{SP}(G) = \emptyset$*; (c)* $\mathrm{SI}(G) = \emptyset$*; (d) V_σ(3-blk(G)) $= \emptyset$.*
(2) $\mathrm{SI}(G) \subseteq \mathrm{SP}(G)$.
(3) Let $K \in$ $\mathrm{SI}(G)$ *and $V_\sigma(K) = \{v \in V_\sigma(\text{3-blk}(G)) \mid v$ represents $K\}$. Then, there is a σ-vertex $v \in V_\sigma(K)$ with $d(v) = d_G(K)$, and any $v' \in V_\sigma(K) - \{v\}$ satisfies $d(v') = 2$.*
(4) Let $K' \in$ $\mathrm{SP}(G) - \mathrm{SI}(G)$*. Then, $K' \in \mathrm{SP}_{\mathrm{N}}(G)$, and, therefore, $d_G(K') = 2$ by Remark 2.1.* □

Any graph represented by a β-vertex of 3-blk(G) is called a *3-block of G*. Any 3-block B of G is called a *singleton 3-block* if B consists of a single vertex of G; otherwise B is a *tri-connected 3-block*. Any singleton 3-block consists of one vertex whose degree in G is two, and there is exactly one polygon containing it. For any 3-block B of G, let $d_G(B)$ or simply $d(B)$ denote the number of separation pairs $K \in \mathrm{SP}(G)$ with $K \subseteq V(B)$, where we set $d_G(B) = 1$ for any singleton 3-block B. For any polygon P of G, let $d_G(P)$ or simply $d(P)$ denote the sum of the number of separation pairs $K \in \mathrm{SP}_{\mathrm{A}}(G)$ with $K \subseteq V(P)$ and the number of singleton 3-blocks included in P. For any $v \in V$(3-blk(G)), let K_v, B_v or P_v denote the σ-pair, the 3-block or the polygon represented by v, respectively. Note that $d(v) = d_G(B_v)$ for any β-vertex v and that $d(u) = d_G(P_u)$ for any π-vertex u.

Let u and v be distinct two leaves of 3-blk(G), and assume that the (u, v)-path in 3-blk(G) passes through a π-vertex w. Let $Q_1, Q_2, \ldots, Q_{d(w)}$ be a sequence of separation pairs of adjacent type and singleton 3-blocks, each included in P_w, appearing in this order on P_w, where distinct separation pairs may share one vertex. Let Q_i and Q_j, $1 \leq i < j \leq d(w)$, be two members of $\{Q_1, \ldots, Q_{d(w)}\}$, each represented by some vertex in the (u, v)-path. Then the (u, v)-path is said to be *adjacent on w* if $j - i = 1$ or $j - i = d(w) - 1$; otherwise *nonadjacent on w*.

Any separation pair K of G is said to be a *primary separation pair* if either $d(K) \geq 3$ or $K \subseteq V(B)$ for some 3-block B with $d(B) \geq 3$. Note that any primary separation pair is of adjacent type. For any $v \in V$ with $d(v) = 2$, let $P_G(v)$ or simply $P(v)$ denote the polygon of G containing v. Let T_G or simply T denote the set $\{v \in V \mid d(v) = 2, d(P(v)) \geq 3\}$.

2.3 Lower Bounds

Let G be a bi-connected graph with $|V| \geq 4$. Let v be any leaf of 3-blk(G) and let u be the σ-vertex adjacent to v in 3-blk(G). Then the subset $V(B_v) - K_u \subsetneq V$ is called a *leaf of G* or *the leaf of G represented by v*. Any singleton 3-block is a leaf of G. For any leaf v of 3-blk(G), let X_v denote the leaf of G represented by v. Let $\mathcal{L}(G)$ denote the class of all leaves of G, and let $l(G)$ denote the number of leaves of G (or equivalently the number of leaves of 3-blk(G)). Note that $X \cap X' = \emptyset$ for any distinct $X, X' \in \mathcal{L}(G)$. Let $d(G)$ denote the maximum separating degree of all separation pairs of G (or equivalently the maximum degree of all σ-vertices in 3-blk(G) by Remark 2.2).

Example 2.3 *For G in Fig. 1, $\mathcal{L}(G) = \{\{a, b\}, \{f\}, \{g\}, \{h\}, \{i, j\}\}$, $l(G) = 5$, and $d(G) = 3$.* □

Let F be any feasible solution to 3VCA-DC for G and g. For any $X \in \mathcal{L}(G)$, F contains at least one edge connecting a vertex in X with a vertex in $\xi(X)$. Since $\mathcal{L}(G)$ is pairwise disjoint, we have $|F| \geq \lceil l(G)/2 \rceil$, meaning that $g(V) \geq 2\lceil l(G)/2 \rceil$ and $g(X) \geq 1$ for any $X \in \mathcal{L}(G)$. Also, for any $K \in \mathrm{SP}$, F contains at least $d(K) - 1$ edges so that adding them to $G - K$ results in a connected graph. Therefore $|F| \geq d(G) - 1$ and $g(V - K) \geq 2(d(K) - 1)$ for any $K \in \mathrm{SP}$. Hence we have the following lemmas.

Lemma 2.4 *For any feasible solution F to 3VCA-DC for G and g, $|F| \geq \max\{d(G) - 1, \lceil l(G)/2 \rceil\}$.* □

Lemma 2.5 *If there is a feasible solution to 3VCA-DC for G and g then the following (1) and (2) hold.*
(1) $g(X) \geq 1$ for any $X \in \mathcal{L}(G)$, and if $l(G)$ is odd then $g(V) \geq l(G) + 1$.
(2) $g(V - K) \geq 2(d(K) - 1)$ for any $K \in$ SP. □

Next we investigate the situations where there is a σ-vertex $v \in V_\sigma$(3-blk(G)) with $d(v) - 1 \geq \lceil l(G)/2 \rceil$.

Lemma 2.6 *Let $u, v \in V_\sigma$(3-blk(G)) be distinct σ-vertices with $d(u) - 1 \geq \lceil l(G)/2 \rceil \leq d(v) - 1$. Then the following (1)–(3) hold:*
(1) $d(u) - 1 = d(v) - 1 = \lceil l(G)/2 \rceil$;
(2) $l(G)$ is even;
(3) $d(x) \in \{1, 2\}$ for any $x \in V$(3-blk(G)) $- \{u, v\}$.

Proof: Since $l(G)$ is the number of leaves of 3-blk(G),

$$l(G) \geq (d(u) - 1) + (d(v) - 1) \geq 2\lceil l(G)/2 \rceil \geq l(G).$$

Since the formula above must hold with equality, we have the lemma. □

Lemma 2.7 *For any $v \in V_\sigma$(3-blk(G)), the following (1) and (2) hold.*
(1) If $d(v) - 1 > \lceil l(G)/2 \rceil$ then $d(u) - 1 < \lceil l(G)/2 \rceil$ for any $u \in V_\sigma$(3-blk(G)) $- \{v\}$ and $d(v) = d(G)$.
(2) Suppose that $d(v) - 1 = \lceil l(G)/2 \rceil$. Then $d(v) = d(G)$, and the following (i) and (ii) hold.
(i) If $d(v) \geq 3$ then there are at most two σ-vertices $u \in V_\sigma$(3-blk(G)) with $d(u) = d(G)$, and moreover if there are two such σ-vertices then $l(G)$ is even.
(ii) If $d(v) = 2$ then $l(G) = 2$ and $d(u) = 2$ for any $u \in V_\sigma$(3-blk(G)) $- \{v\}$.

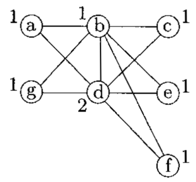

Figure 4: An instance for which there is no feasible solution to 3VCA-DC, where $l(G) = d(G) = 5$ and the condition (2) does not hold. $\{b, d\}$ is the separation pair K with $d(K) = d(G)$. Since $g(V - K) = 5$, there is no feasible solution.

Proof: We prove (1). If $d(u) - 1 \geq \lceil l(G)/2 \rceil$ for some $u \in V_\sigma(\text{3-blk}(G)) - \{v\}$, then we have a contradiction $d(v) - 1 = \lceil l(G)/2 \rceil$ by Lemma 2.6 (1). Therefore $d(u) - 1 < \lceil l(G)/2 \rceil$ for any $u \in V_\sigma(\text{3-blk}(G)) - \{v\}$. Since $d(v)$ is the maximum degree of all σ-vertices in 3-blk(G), $d(v) = d(G)$ holds.

We prove (2). By Lemma 2.6 (1) (or (1) of this lemma), $d(u) - 1 \leq \lceil l(G)/2 \rceil$ for any $u \in V_\sigma(\text{3-blk}(G)) - \{v\}$. Therefore $d(v) = d(G)$. Let $V_\sigma^d = \{u \in V_\sigma(\text{3-blk}(G)) \mid d(u) = d(G)\}$. Suppose by contradiction that $d(v) \geq 3$ and $|V_\sigma^d| \geq 3$. Then, it follows from Lemma 2.6 (3) that $d(v) = 2$, a contradiction. Hence if $d(v) \geq 3$ then $|V_\sigma^d| \leq 2$, and if $|V_\sigma^d| = 2$ then $l(G)$ is even by Lemma 2.6 (2), showing (2)(i). Suppose that $d(v) = 2$. Then $l(G) = 2$ follows from $\lceil l(G)/2 \rceil = 1$. And, for any $u \in V_\sigma(\text{3-blk}(G)) - \{v\}$, we have $d(u) = 2$ since $d(v) \geq d(u) \geq 2$, showing (2)(ii). □

By Lemma 2.7, we have the next corollary.

Corollary 2.8 *If either $(d(G) - 1 > \lceil l(G)/2 \rceil)$ or $(d(G) - 1 = \lceil l(G)/2 \rceil$ and $l(G)$ is odd) then there is the unique σ-vertex $v \in V_\sigma(\text{3-blk}(G))$ with $d(v) = d(G)$; furthermore, K_v is the unique separation pair $K \in \text{SP}(G)$ with $d(K) = d(G)$.*

Proof: Lemma 2.7 shows the first part. Since $d(v) \geq 3$, K_v is a separation pair of adjacent type, and, therefore, $d(K_v) = d(v) = d(G)$. Hence the second part follows. □

3 The Existence Condition for Feasible Solutions

From this section, we assume that G is bi-connected with $|V| \geq 4$ unless otherwise stated. In this section we show a necessary and sufficient condition for the existence of a feasible solution to 3VCA-DC.

Theorem 3.1 (The existence condition for feasible solutions) *There is a feasible solution to 3VCA-DC for a bi-connected graph G with degree constraints by g if and only if the following (1)–(3) hold.*
(1) $g(X) \geq 1$ for any leaf $X \in \mathcal{L}(G)$.
(2) If either $(d(G) - 1 > \lceil l(G)/2 \rceil)$ or $(d(G) - 1 = \lceil l(G)/2 \rceil$ and $l(G)$ is odd) then $g(V - K) \geq 2(d(G) - 1)$ for the separation pair K with $d(K) = d(G)$. (See Fig. 4.)
(3) If $(d(G) - 1 < \lceil l(G)/2 \rceil$ and $l(G)$ is odd) then the following (a)–(c) hold.
(a) $g(V) \geq l(G) + 1$.
(b) If no polygon P with $d(P) \geq 3$ exists and one vertex $u \in V$ is shared by all primary separation pairs of G, then $g(V - u) \geq l(G) + 1$. (See Fig. 5.)
(c) If $l(G) = 3$, there is a polygon P with $d(P) = 3$, and P includes at least one singleton 3-block, then $g(V - T) \geq l(G) + 1 - |T|$. (See Fig. 6.) □

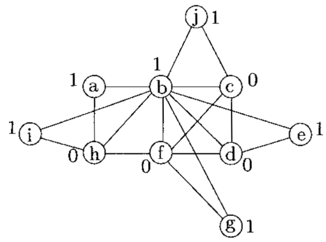

Figure 5: An instance for which there is no feasible solution to 3VCA-DC, where $l(G) = 5$, $d(G) = 3$, and the condition (3)(b) does not hold. Vertex b is shared by all members of $\{K \in \text{SP}(G) \mid d(K) \geq 3\} = \{\{b, f\}, \{b, h\}\}$ and $\{K \in \text{SP}(G) \mid K \subseteq V(B)$ for some 3-block B with $d(B) \geq 3\} = \{\{b, c\}, \{b, d\}, \{b, f\}\}$. Since $g(V - b) = 5$, there is no feasible solution.

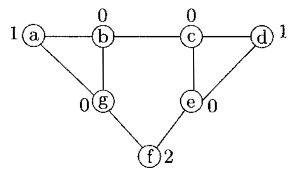

Figure 6: An instance for which there is no feasible solution to 3VCA-DC, where $l(G) = 3$, $d(G) = 2$, and the condition (3)(c) does not hold. Polygon P with $V(P) = \{b, c, e, f, g\}$ has $d(P) = 3$, and $T = \{v \in V(P) \mid d_G(v) = 2\} = \{f\}$. Since $g(V - T) = 2 < 4 - |T|$, there is no feasible solution.

We call the set of the conditions (1)–(3) given in Theorem 3.1 *the existence condition for feasible solutions.* Note that, in the condition (2), K is unique by Corollary 2.8. We call any graph satisfying all the assumptions of the condition (3)(b) or (3)(c) *a graph of type 1* or *of type 2*, respectively, and the vertex u in the condition (3)(b) is called *the center* of a graph of type 1, where we usually denote the center by u unless any confusion arises.

Lemma 3.2 *Let G be a graph of type 1 with center u. If $g(V - u) < l(G) + 1$ then there is no feasible solution.*

Proof: Suppose that there is a feasible solution F. It suffices to consider the case where $g(X) = 1$ for each $X \in \mathcal{L}(G)$, $g(V - u) = l(G)$, and $g(u) = \infty$. Let X be any leaf of G. Since there is no polygon P with $d(P) \geq 3$, there is a primary separation pair K of G such that some K-component H of G satisfies $X \subseteq H$ and $X' \cap H = \emptyset$ for any $X' \in \mathcal{L}(G) - \{X\}$. (Such K can be found easily by searching 3-blk(G) from the leaf which represents X. The first visited σ-vertex s satisfying $d(s) \geq 3$ or being adjacent to a β-vertex b with $d(b) \geq 3$ represents the desired separation pair K.) Since $u \in K$ and $K \notin \text{SP}(G + F)$, F must contain an edge connecting a vertex in X with a vertex in some $X' \in \mathcal{L}(G) - \{X\}$. Therefore, every leaf of G would be connected by an edge in F to another leaf of G, implying $g(V - u) \geq l(G) + 1$, a contradiction. □

Lemma 3.3 *Let G be a graph of type 2. If $g(V - T) < l(G) + 1 - |T|$ then there is no feasible solution.*

Proof: Suppose that there is a feasible solution F. It suffices to consider the case where $g(X) = 1$ for each $X \in \mathcal{L}(G) - \{\{v\} \mid$

$v \in T\}$, $g(V-T) = l(G) - |T|$, and $g(x) = \infty$ for each $x \in T$. F contains at least one edge connecting a vertex v in T with a vertex v' in some $X \in \mathcal{L}(G) - \{\{v\} \mid v \in T\}$. Suppose that such an edge (v, v') is added to G. Then, a new leaf X' containing X but not containing v appears in the resulting graph, and no more edge can be added to X' because $g(X') = g(X) = 1$, a contradiction. □

The necessity of the conditions (1)–(3) follows from Lemmas 2.5, 3.2 and 3.3, and the sufficiency is shown by the algorithm *Solve_3VCA-DC_aug2to3* which will be presented in the next section.

Given G and g, we can check whether or not the existence condition for feasible solutions holds in $O(|V|+|E|)$ time. Hence we have the following theorem.

Theorem 3.4 *Checking the existence of a feasible solution to 3VCA-DC for any bi-connected graph G can be done in* $O(|V| + |E|)$ *time.* □

4 A Linear Time Algorithm

In this section, we assume that a bi-connected graph G with degree constraints by g satisfies the existence condition for feasible solutions, and we present a linear time algorithm *Solve_3VCA-DC_aug2to3* for finding an optimum solution to 3VCA-DC for G and g.

We partition the problem into the following three cases:

Case 1: $(d(G) - 1 > \lceil l(G)/2 \rceil)$ or
$(d(G) - 1 = \lceil l(G)/2 \rceil$ and $l(G)$ is odd);

Case 2: $(d(G) - 1 \le \lceil l(G)/2 \rceil$ and $l(G)$ is even);

Case 3: $(d(G) - 1 < \lceil l(G)/2 \rceil$ and $l(G)$ is odd).

And we present three algorithms, Algorithm 1 for Case 1, Algorithm 2 for Case 2, Algorithm 3 for Case 3, for finding an optimum solution. *Solve_3VCA-DC_aug2to3* uses these algorithms and is formally described as follows.

Algorithm *Solve_3VCA-DC_aug2to3*
Input: A bi-connected graph $G = (V, E)$ with degree constraints by $g : V \to Z^+ \cup \{\infty\}$, satisfying the existence condition for feasible solutions.
Output: An optimum solution E^* to 3VCA-DC for G and g.

1. Find an optimum solution E^* by using Algorithm 1, Algorithm 2 or Algorithm 3 according to the case to which the problem belongs, and output E^*.

Before describing three algorithms, we summary the known results on 3VCA for bi-connected graphs G. Any vertex $v \in V$ is said to be a *leaf-vertex of G* if $v \in X$ for some $X \in \mathcal{L}(G)$; otherwise *non-leaf-vertex of G*.

Lemma 4.1 ([4, 7, 14]) *Any optimum solution F to 3VCA for a bi-connected graph G satisfies (1)–(3):*
(1) If $(d(G) - 1 > \lceil l(G)/2 \rceil)$ *or* $(d(G) - 1 = \lceil l(G)/2 \rceil$ *and* $l(G)$ *is odd) then F consists of* $d(G) - 1$ *edges whose addition connects* $d(G)$ *components of* $G - K$*, where K is the separation pair with* $d(K) = d(G)$*, and every leaf of G contains at least one vertex incident to some edge in F.*
(2) If $(d(G) - 1 \le \lceil l(G)/2 \rceil$ *and* $l(G)$ *is even) then F consists of* $l(G)/2$ *edges each of which connects two leaf-vertices belonging to distinct leaves of G.*
(3) If $(d(G) - 1 < \lceil l(G)/2 \rceil$ *and* $l(G)$ *is odd) then F consists of* $\lfloor l(G)/2 \rfloor$ *edges each of which connects two leaf-vertices belonging to distinct leaves of G plus one edge connecting a leaf-vertex of G with a vertex which is not necessarily a leaf-vertex of G.* □

4.1 Algorithm 1 and Algorithm 2

In Cases 1 and 2, we can directly use the results shown in Lemma 4.1 to find an optimum solution. The difference is that we have to take into account g-feasibility. The following Algorithms 1 and 2 solve Cases 1 and 2, respectively. Algorithm 1 is based on the algorithm in [7], and Algorithm 2 uses an algorithm for 3VCA.

Algorithm 1 /* for Case 1 */

1. Find the separation pair K of G with $d(K) = d(G)$.
2. $k \leftarrow d(G)$. Let $C_1, \ldots, C_k$ be the k components of $G - K$, and let ℓ_i $(i = 1, \ldots, k)$ be the number of leaves of G contained in C_i.
3. Let $a_1, \ldots, a_k$ be k integers satisfying $\sum_{i=1}^{k} a_i = 2(k-1)$ and $\ell_i \le a_i \le g(C_i)$ for each $i = 1, \ldots, k$.
4. Construct a tree T with $V(T) = \{v_i \mid 1 \le i \le k\}$ satisfying $d_T(v_i) = a_i$ for each $v_i \in V(T)$.
5. $E^* \leftarrow \emptyset$. For each $(v_i, v_j) \in E(T)$, add an edge which connects a vertex in C_i with a vertex in C_j into E^* so that E^* finally satisfies that every leaf of G contains a vertex in $V(E^*)$ as well as g-feasibility of E^*. Output E^*.

Lemma 4.2 *Algorithm 1 finds an optimum solution to 3VCA-DC in Case 1 and can run in* $O(|V| + |E|)$ *time.*

Proof: Since the conditions (1) and (2) hold, the algorithm is executable. We assume by contradiction that $G + E^*$ is not tri-connected. Then $\mathrm{SP}(G + E^*) \ne \emptyset$, and let K' be any separation pair of $G + E^*$. We have $K' \in \mathrm{SP}(G)$ and $K' \ne K$ for the separation pair K with $d_G(K) = d(G)$. Therefore, $K' \subseteq K \cup H$ for some K-component H of G, and $K \subseteq K' \cup H'$ for some K'-component H' of G. Furthermore, H' includes all K-components except H, while H includes all K'-components except H'. Every K'-component C' with $C' \ne H'$ includes a leaf X of G, and, therefore, C' and H' is connected by an edge $e \in E^*$ which connects a vertex in X with a vertex in some K-component in H'. This means $G + E^* - K'$ is connected, contradicting $K' \in \mathrm{SP}(G + E^*)$. Hence $G + E^*$ is tri-connected. Since $|E^*| = d(G) - 1$, E^* is an optimum solution by Lemma 2.4 (or Lemma 4.1 (1)).

Concerning time-complexity, Steps 1, 2 and 3 can be executed in $O(|V| + |E|)$ time using 3-blk(G) which can be computed in $O(|V| + |E|)$ time. In Step 4 we use the algorithm [8, Problem 7.47] for constructing a tree from a prescribed degree sequence, which runs in $O(|V|)$ time. Step 5 can be done in $O(|V|)$ time. The total time-complexity of Algorithm 1 is $O(|V| + |E|)$. □

Algorithm 2 /* for Case 2 */

1. Find an optimum solution F for 3VCA for G.
   /* $|F| = l(G)/2$, and each edge of F connects two leaf-vertices belonging to distinct leaves of G. */
2. $E^* \leftarrow \emptyset$. For each $(u, v) \in F$, do the following: find two leaves $X(u), X(v) \in \mathcal{L}(G)$ containing u, v, respectively; select a vertex $u' \in X(u)$ with $g(u') \ge 1$ and a vertex $v' \in X(v)$ with $g(v') \ge 1$; and add an edge (u', v') into E^*. Output E^*.

Lemma 4.3 *Algorithm 2 finds an optimum solution to 3VCA-DC in Case 2 and can run in* $O(|V| + |E|)$ *time.*

Proof: The correctness is obvious. Using a linear time algorithm [4] for 3VCA, Step 1 can be done in $O(|V| + |E|)$ time. Step 2 takes $O(|V|)$ time. Hence the lemma follows. □

4.2 Algorithm 3

In Case 3, we use the strategy of reducing the problem to Case 1 or Case 2 by adding several number of edges which can be a subset of an optimum solution. In Algorithm 3, at most two edges are added before the reduction. In the following we introduce two types of edge addition, called *Edge-Addition A* and *Edge-Addition B*, and show some properties of them. Then we present Algorithm 3.

4.2.1 Edge-Addition A

Edge-Addition A is to add an edge connecting two leaf-vertices of G so that the number of leaves is decreased by two if possible. Suppose that $l(G) \geq 5$. Let u be any leaf of 3-blk(G) with $g(X_u) \geq 2$, where X_u is the leaf of G represented by u. Select a leaf v of 3-blk(G) satisfying one of the following (a)–(c) [4]: (a) The (u, v)-path contains at least one β-vertex with degree at least four; (b) The (u, v)-path contains at least two vertices with degree at least three; (c) The (u, v)-path is nonadjacent on some π-vertex. Select a vertex $u' \in X_u$ with $g(u') \geq 1$ and a vertex $v' \in X_v$ with $g(v') \geq 1$, and add an edge (u', v') to G. If $l(G) = 3$ then let u be any leaf of 3-blk(G) with $g(X_u) \geq 2$ and $B_u \notin \{\{v\} \mid v \in T_G\}$ (i.e. B_u is not a singleton 3-block in some polygon P with $d(P) \geq 3$), let v be any leaf of 3-blk(G) with $v \neq u$, and add an edge connecting $u' \in X_u$ with $v' \in X_v$ to G, where $g(u') \geq 1$ and $g(v') \geq 1$.

Let G_A denote the graph obtained from G by executing Edge-Addition A (i.e. $G_A = G + \{(u', v')\}$) and let v^* be a vertex in $X_u \in \mathcal{L}(G)$ satisfying $g(v^*) \geq 1$ and, moreover, if $v^* = u'$ then $g(v^*) \geq 2$. Note that if $l(G) = 3$ then $v^* \notin T_G$. We update g by $g(u') \leftarrow g(u') - 1$ and $g(v') \leftarrow g(v') - 1$.

Remark 4.4 *$g(v^*) \geq 1$ for updated g (even if $v^* = u'$).* □

Lemma 4.5 ([4]) *(1) If $l(G) \geq 5$ then $l(G_A) = l(G) - 2$; (2) If $l(G) = 3$ then $l(G_A) = 2$.* □

Corollary 4.6 *(1) If $l(G) \geq 5$ then $\mathcal{L}(G_A) = \mathcal{L}(G) - \{X_u, X_v\}$, v^* is a non-leaf-vertex of G_A, $d(G_A) - 1 \leq \lceil l(G_A)/2 \rceil$ and $l(G_A)$ is odd.*
(2) If $l(G) = 3$ then G_A has a new leaf $X' \in \mathcal{L}(G_A) - \mathcal{L}(G)$, and $d(G_A) - 1 = \lceil l(G_A)/2 \rceil = 1$. □

Remark 4.7 *$g(X) \geq 1$ for any $X \in \mathcal{L}(G_A) \cap \mathcal{L}(G)$.* □

For the case $v^* \notin T_G$ we have the next lemma.

Lemma 4.8 *If $v^* \notin T_G$ then G_A has a tri-connected 3-block B with $v^* \in V(B)$ and has no $K \in$ SP(G_A) with $v^* \in K$.*

Proof: B_u is a tri-connected 3-block or a singleton 3-block containing v^*. If B_u is a singleton 3-block then there is the triangle $P(v^*)$ with $d(P(v^*)) = 2$. By adding (u', v'), B_u (or $P(v^*)$) and v' are included in a tri-connected 3-block of G_A. Also G has no $K \in$ SP(G) with $v^* \in K$ and so does G_A. □

The next lemma shows that if $l(G) = 3$ then Case 3 can be reduced to Case 2 after Edge-Addition A.

Lemma 4.9 *If $l(G) = 3$ then v^* is in the new leaf of G_A.*

Proof: By Lemmas 4.5 (2) and 4.8, we have $d_{G_A}(B) = 1$ for the tri-connected 3-block B with $v^* \in V(B)$. Since $v^* \notin K$ for any $K \in$ SP(G_A), v^* is in the new leaf of G_A. □

For the case $v^* \in T_G$ we have the next lemma.

Lemma 4.10 *Suppose that $v^* \in T_G$. Then any $K \in$ SP(G_A) with $v^* \in K$ has $d_{G_A}(K) = 2$. If G_A has a tri-connected 3-block B with $d_{G_A}(B) \geq 3$ and $v^* \in V(B)$ then B includes a separation pair $K \in$ SP(G_A) with $v^* \notin K$.*

Proof: In this case, $v^* = u'$. Any $K \in$ SP(G) with $v^* \in K$ consists of v^* and any vertex $x \in V(P(v^*))$ not adjacent to v^* in $P(v^*)$, that is, $K \in$ SP$_N(G)$. Therefore, for any $K \in$ SP(G_A) with $v^* \in K$, we have $K \in$ SP$_N(G)$ and hence $d_{G_A}(K) = 2$. By adding (u', v'), at most two separation pairs $K \in$ SP$_N(G)$ with $v^* \in K$ are included in B. Since $d_{G_A}(B) \geq 3$, B includes some $K \in$ SP(G_A) with $v^* \notin K$. □

The next lemma shows that if ($d(G_A) - 1 = \lceil l(G_A)/2 \rceil$ and $l(G_A)$ is odd) then Case 3 can be reduce to Case 1 after Edge-Addition A.

Lemma 4.11 *If ($d(G_A) - 1 = \lceil l(G_A)/2 \rceil$ and $l(G_A)$ is odd) then $g(V - K) \geq 2(d(G_A) - 1)$ for the separation pair $K \in$ SP(G_A) with $d_{G_A}(K) = d(G_A)$.*

Proof: Since $d_{G_A}(K) \geq 3$, we have $v^* \notin K$ by Lemmas 4.8 and 4.10. Hence $g(V - K) \geq \sum_{X \in \mathcal{L}(G_A)} g(X) + g(v^*) \geq l(G_A) + 1 = 2(\lceil l(G_A)/2 \rceil) = 2(d_{G_A}(K) - 1)$. □

Moreover, the next lemma concerning graphs of type 1 or of type 2 follows from Lemmas 4.8 and 4.10.

Lemma 4.12 *If $l(G_A) \geq 3$ then (1) and (2) hold.*
(1) If G_A is a graph of type 1 then v^* is not the center of G_A.
(2) If G_A is a graph of type 2 then $v^* \notin T_{G_A}$.

Proof: We prove (1). Suppose that v^* is the center of G_A. If $v^* \notin T_G$ then $v^* \notin K$ for any $K \in$ SP(G_A) by Lemma 4.8, contradicting that v^* is the center of G_A. Therefore we have $v^* \in T_G$. By Lemma 4.10, any $K \in$ SP(G_A) with $v^* \in K$ has $d_{G_A}(K) = 2$. This shows that G_A has a tri-connected 3-block B with $d_{G_A}(B) \geq 3$ and $v^* \in V(B)$. Again by Lemma 4.10, B includes a separation pair $K \in$ SP(G_A) with $v^* \notin K$, contradicting that G_A is of type 1. Hence (1) follows.

Since v^* is a non-leaf-vertex of G_A, (2) holds. □

From the results mentioned above, we obtain the following corollary.

Corollary 4.13 *(1) The existence condition for feasible solutions holds for G_A and updated g.*
(2) If $l(G) \geq 5$ ($l(G_A) \geq 3$) then G_A has a non-leaf-vertex v^* with $g(v^*) \geq 1$ and $v^* \neq u$ if G_A is a graph of type 1. □

If ($d(G_A) - 1 < \lceil l(G_A)/2 \rceil$ and $l(G_A)$ is odd) then we can execute Edge-Addition B by using v^* in Corollary 4.13 (2) as a non-leaf-vertex to which a new edge is added.

4.2.2 Edge-Addition B

Edge-Addition B is to add an edge connecting a non-leaf-vertex with a leaf-vertex of G so that the number of leaves is decreased by one. Let v^* be a non-leaf-vertex of G with $g(v^*) \geq 1$, where we assume that v^* is not the center of G if G is a graph of type 1. We consider two cases.

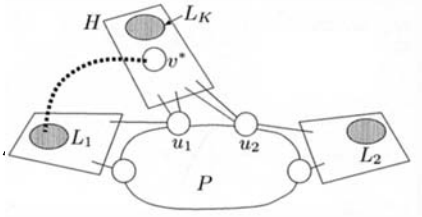

Figure 7: Edge-Addition B in Case B1, showing the case $Q_1, Q_2 \in \mathrm{SP_A}(G)$ and $v^* \in H$.

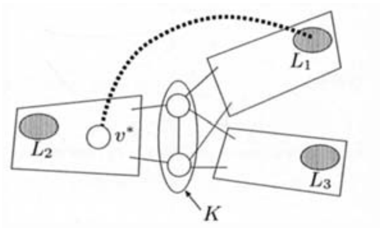

Figure 8: Edge-Addition B in Case B2, where $d(K) \geq 3$.

(Case B1) There is a polygon P with $d(P) \geq 3$. In this case, P includes a separation pair $K = \{u_1, u_2\} \in \mathrm{SP_A}(G)$ with $v^* \in H \cup K$, where H is a K-component not containing $V(P) - K$. See Fig. 7. Let Q_1, Q_2 be a separation pair of adjacent type or a singleton 3-block included in P such that Q_1, K, Q_2 appear in this order consecutively on P. If $Q_1 \in \mathrm{SP_A}(G)$ then let $L_1 \in \mathcal{L}(G)$ be any leaf in a Q_1-component not containing $V(P) - Q_1$; if Q_1 is a singleton 3-block (which is a leaf) then let $L_1 = Q_1$. For Q_2, let L_2 be a leaf of G defined similarly. If $v^* \in H \cup \{u_2\}$ then select a vertex $v'' \in L_1$ with $g(v'') \geq 1$; if $v^* = u_1$ then select a vertex $v'' \in L_2$ with $g(v'') \geq 1$. And add an edge (v^*, v'') to G.

Example 4.14 *For G in Fig. 1, $d(G) - 1 < \lceil l(G)/2 \rceil$ and $l(G)$ is odd. G has a non-leaf-vertex $v^* =$ d with $g(v^*) \geq 1$ and a polygon P with $V(P) = \{$c, d, e, f, g$\}$ and $d(P) \geq 3$. For $K = \{$c, d$\} \in \mathrm{SP_A}(G)$ and a K-component $H = \{$a, b$\}$, $v^* \in H \cup K$ holds. Then an edge* (d, g) *is added to G.* □

(Case B2) There is no polygon P with $d(P) \geq 3$. Then there is a primary separation pair K of G with $v^* \notin K$ (even if G is a graph of type 1). If $d(K) \geq 3$ then let $L_1 \in \mathcal{L}(G)$ be any leaf in a K-component not containing v^*, select a vertex $v'' \in L_1$ with $g(v'') \geq 1$ and add an edge (v^*, v'') to G (Fig. 8). If $d(K) = 2$ then do the following. Let B be a 3-block with $K \subseteq V(B)$ and $d(B) \geq 3$. Let $L_K \in \mathcal{L}(G)$ be any leaf in the K-component not containing $V(B) - K$. Let $K_1 \in \mathrm{SP}(G) - \{K\}$ be any separation pair in $V(B)$, and let $L_1 \in \mathcal{L}(G)$ be any leaf in a K_1-component not containing $V(B) - K_1$. If v^* and $V(B) - K$ are in a K-component then select a vertex $v'' \in L_K$ with $g(v'') \geq 1$ (Fig. 9); otherwise select a vertex $v'' \in L_1$ with $g(v'') \geq 1$. And add an edge (v^*, v'') to G.

Let G_B denote the graph obtained from G by executing Edge-Addition B (i.e. $G_B = G + \{(v^*, v'')\}$). For any 3-block or any leaf Y of G (or G_B), let $\beta(Y)$ denote the β-vertex representing Y in 3-blk(G) (or in 3-blk(G_B)).

Lemma 4.15 $l(G_B) = l(G) - 1$.

Proof: (Case B1) Suppose that $v^* \in H$ (Fig. 7). Then G_B has a 3-block B_1 containing $\{v^*, u_1, u_2\} \cup L_1$. Let $L_K \in \mathcal{L}(G)$ be any leaf in H. Then $L_K, L_2 \in \mathcal{L}(G_B)$, and there are two paths in 3-blk(G_B), one from $\beta(B_1)$ to $\beta(L_K)$ and one from $\beta(B_1)$ to $\beta(L_2)$. Hence $\beta(B_1)$ is not a leaf of 3-blk(G_B), and $l(G_B) = l(G) - 1$.

Suppose that $v^* = u_2$. If $Q_1 \in \mathrm{SP_A}(G)$ then G_B has a 3-block B'_1 containing $\{v^*\} \cup Q_1 \cup L_1$, and, in 3-blk($G_B$), the path between $\beta(L_K)$ and $\beta(L_2)$ passes through $\beta(B'_1)$, showing $l(G_B) = l(G) - 1$. If Q_1 is a singleton 3-block then $L_1 = Q_1 = \{v''\}$, and $\{v^*, v''\} \in \mathrm{SP_A}(G_B)$. Since v'' is no longer a leaf-vertex in G_B, $l(G_B) = l(G) - 1$. In the case $v^* = u_1$, a similar discussion shows $l(G_B) = l(G) - 1$.

(Case B2) Suppose that $d(K) \geq 3$ (Fig. 8). G_B has a 3-block B_2 containing $K \cup \{v^*, v''\}$. Let $L_2 \in \mathcal{L}(G)$ be any leaf in the K-component containing v^*. Let $L_3 \in \mathcal{L}(G)$ be any leaf in a K-component containing neither v'' nor v^*. Then, in 3-blk(G_B), the path between $\beta(L_2)$ and $\beta(L_3)$ passes through $\beta(B_2)$, showing $l(G_B) = l(G) - 1$.

Suppose that $d(K) = 2$ and $K \subseteq V(B)$ for a 3-block B with $d(B) \geq 3$. G_B has a 3-block B'_2 containing $B \cup \{v^*, v''\}$. Let $K_2 \in \mathrm{SP}(G) - \{K, K_1\}$ be any separation pair in $V(B)$, and let $L_2 \in \mathcal{L}(G)$ be any leaf in a K_2-component not containing $V(B) - K_2$. If v^* and $V(B) - K$ are contained in a K-component then, in 3-blk(G_B), the path between $\beta(L_1)$ and $\beta(L_2)$) passes through $\beta(B'_2)$. Otherwise, in 3-blk(G_B), the path between $\beta(L_K)$ and $\beta(L_2)$) passes through $\beta(B'_2)$. $\beta(B'_2)$ is not a leaf of 3-blk(G_B). Hence $l(G_B) = l(G) - 1$. □

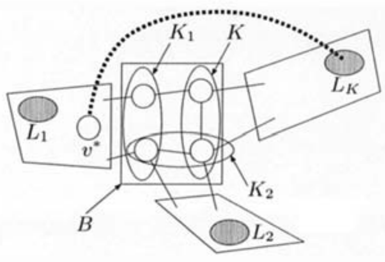

Figure 9: Edge-Addition B in Case B2, where $d(K) = 2$ and $K \subseteq V(B)$ for a 3-block B with $d(B) \geq 3$, showing the case where v^* and $V(B) - K$ are in a K-component.

4.2.3 Description of Algorithm 3

Now we describe Algorithm 3. Figure 10 shows overview of Algorithm 3.

Algorithm 3 /* for Case 3 */

1. (1.1) $E^* \leftarrow \emptyset$.
   (1.2) If there is a non-leaf-vertex v^* with $g(v^*) \geq 1$ such that if G is a graph of type 1 then v^* is not the center of G, select such a vertex v^* and go to Step 3.
   (1.3) Otherwise go to Step 2.
2. (2.1) Add an edge (u', v') to G by Edge-Addition A as described in Sec. 4.2.1; $E^* \leftarrow E^* \cup \{(u', v')\}$; $G \leftarrow G + \{(u', v')\}$; $g(u') \leftarrow g(u') - 1$ and $g(v') \leftarrow g(v') - 1$.
   (2.2) If the problem for new G and g is in Case 1 or 2 then find an optimum solution A by Algorithm 1 or 2, respectively; $E^* \leftarrow E^* \cup A$; and output E^* and halt.
   (2.3) Otherwise select a non-leaf-vertex v^* with $g(v^*) \geq 1$ such that if G is a graph of type 1 then v^* is not the center of G; and go to Step 3.
3. (3.1) Add an edge (v^*, v'') by Edge-Addition B as described in Sec. 4.2.2; $E^* \leftarrow E^* \cup \{(v^*, v'')\}$; $G \leftarrow G +$

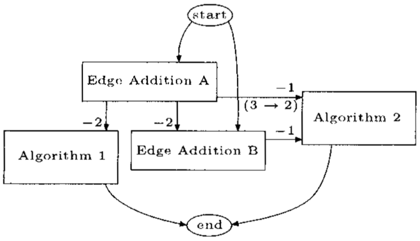

Figure 10: Overview of Algorithm 3, where the negative integer beside each arrow shows decrease of the number of leaves.

$\{(v^*, v'')\}$; $g(v^*) \leftarrow g(v^*) - 1$ and $g(v'') \leftarrow g(v'') - 1$.
(3.2) Find a optimum solution A for new G and g by Algorithm 2; $E^* \leftarrow E^* \cup A$; and output E^*.

Lemma 4.16 *Algorithm 3 finds an optimum solution to 3VCA-DC in Case 3 and can run in $O(|V| + |E|)$ time.*

Proof: We show the correctness of Algorithm 3. Suppose that a non-leaf-vertex v^* is selected in Step 1. By Lemma 4.15, $l(G_B) = l(G) - 1$, which is even. Therefore, the problem is reduced to Case 2 and we can apply Algorithm 2 for G_B. The optimum solution obtained by Algorithm 2 with the edge added by Edge-Addition B is an optimum solution whose number of edges is $\lceil l(G)/2 \rceil$.

Next we consider the case Step 2 is executed. After Edge-Addition A in Step 2, the existence condition for feasible solutions holds by Corollary 4.13 (1). Therefore, if the problem is reduced to Case 1 or Case 2 then we can execute algorithm 1 or Algorithm 2, respectively. If the problem is in Case 3, there is a desired non-leaf-vertex v^* by Corollary 4.13 (2), so we can execute Edge-Addition B in Step 3, and we find an optimum solution A to G_B by Algorithm 2. In each case we have $|E^*| = \lceil l(G)/2 \rceil$ for E^* output by Algorithm 3. Hence it is an optimum solution.

Since all of Edge-Addition A, Edge-Addition B, Algorithm 1, Algorithm 2 can be executed in $O(|V| + |E|)$ time, Algorithm 3 can run in $O(|V| + |E|)$ time. □

From Lemmas 4.2, 4.3 and 4.16, *Solve_3VCA-DC_aug2to3* finds an optimum solution and runs in $O(|V| + |E|)$ time. We obtain the following theorem.

Theorem 4.17 *If a bi-connected graph G with degree constraints by g satisfies the existence condition for feasible solutions then finding an optimum solution to 3VCA-DC for G and g can be done in $O(|V| + |E|)$ time and the cardinality of the optimum solution equals* $\max\{d(G) - 1, \lceil l(G)/2 \rceil\}$. □

5 Concluding Remarks

We have shown that checking the existence of a feasible solution and finding an optimum solution to 3VCA-DC for any bi-connected graph $G = (V, E)$ can be done in $O(|V| + |E|)$ time, where if there is a feasible solution then the optimum value to 3VCA-DC is equal to the optimum one to 3VCA. Devising a polynomial time algorithm for 3VCA-DC for not bi-connected graph is left for future research.

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