グラフのデカルト積における木幅の下界

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あらまし 本論文では，2つのグラフ G_1 と G_2 のデカルト積として表わされるグラフ $G_1 \square G_2$ の tree-width に対するある下界を与える．より具体的には，G_1 の Hadwiger 数と G_2 の PI 数の積が $G_1 \square G_2$ のブランブル数の下界となることを示す．また本下界を用いた応用例も示す．
キーワード デカルト積，木幅，ブランブル，Hadwiger 数

A lower bound for tree-width of Cartesian product graphs

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Abstract In this paper, we give a lower bound for tree-width of Cartesian product graphs. To be more precise, we show that the bramble number of Cartesian product of graphs G_1 and G_2 is at least Hadwiger number of G_1 times PI number of G_2. We also demonstrate applications of the lower bound.
Key words Cartesian product, Tree-width, Bramble, Hadwiger number

1. Introduction

The concept of *tree-width* have been making great contributions on pure and algorithmic graph theory for recent twenty years. Roughly speaking, the tree-width of a graph G, denoted by $tw(G)$, is a graph parameter to measure how close G is to a tree. In this paper, we give a lower bound for tree-width of Cartesian product graphs. To this end, we use another graph parameter *bramble number* which is essentially the same as tree-width: Bramble number of a graph G, denoted by $bn(G)$, is the maximum order of a bramble of G. A *bramble* $\mathsf{B} = \{B_1, \dots, B_{|\mathsf{B}|}\}$ of G is a collection of connected subgraphs of G such that any B_i and B_j in B, B_i and B_j intersect or are joined by an edge. The order of B is the least number of vertices to cover every B_i in B, namely the size of minimum hitting set of B. Seymour and Thomas showed that the bramble number of a graph is precisely the tree-width of the graph plus one [10]. A merit to use bramble for showing lower bound on tree-width, is that a lower bound can be found *constructively*.

1.1 Motivation

Our study was motivated by the following natural question which arises from a study of inapproximability of bramble number: Is there a suitable graph product operation under which the tree-width of a resulting product graph can be determined only by tree-width of its factor graphs? A famous example of this type of question is *clique number* under *strong product operation*. Clique number $\omega(G)$ of a graph G is the size of largest clique in G. Strong product of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \otimes G_2$, is the graph whose vertex set is $V_1 \times V_2$, and edge set is $\{((u_1, v_1), (u_2, v_2)) \mid [u_1 = u_2 \vee \{u_1, u_2\} \in E_1] \wedge [v_1 = v_2 \vee \{v_1, v_2\} \in E_2]\}$. It is known that for any graphs G_1 and G_2, $\omega(G_1 \otimes G_2)$ equals $\omega(G_1) \times \omega(G_2)$.

To consider the above question, let us observe the following. Let G be a graph and $\mathsf{B} = \{B_1, \dots, B_{|\mathsf{B}|}\}$ be an optimal bramble of G. Then it is not difficult to see that $\mathsf{B} \times \mathsf{B} = \{B_{1,1}, \dots, B_{|\mathsf{B}|,|\mathsf{B}|}\}$ is a bramble of $G \otimes G$, where $B_{i,j} = \{(u, v) \mid u \in B_i, v \in B_j\}$. And let H be a minimum hitting set of B, then $H^2 = \{(u, v) \mid u, v \in H\}$ is a hitting set of $G \otimes G$. So if H^2 is a minimum hitting set of $G \otimes G$ then we have a lower bound for tree-width of $G \otimes G$.

Unfortunately there is a drawback in the above observation, that is, H^2 is not a minimum hitting set of $\mathsf{B} \times \mathsf{B}$ in general, and it would be difficult to determine the order of $\mathsf{B} \times \mathsf{B}$ or obtain a lower bound for the order, unless we impose some restriction.

1.2 Related works

For a graph G and a complete graph K_n with n vertices, Bodlaender *et al.* showed implicitly that $tw(G \otimes K_n) = (tw(G) + 1) \cdot n - 1$ to demonstrate inapproximability with an absolute error guarantee for tree-width [1].

Lucena determined the exact tree-width of the Cartesian product graph of K_n and K_n [7], more precisely, $tw(K_n \square K_n) = n^2/2 + n/2 - 1$ for $n \geqq 3$, where $G_1 \square G_2$ denote the Cartesian product of two graphs G_1 and G_2. Our results can be considered as a formulation of the result of Lucena.

Djelloul provided upper bounds for tree-width and path-width of Cartesian product graphs [4], [5]. We use the upper bounds to evaluate our lower bounds.

1.3 Our results

Our original motivation is to find a product operation $\circledcirc$ for which there exists a "tractable" function f such that $tw(G_1 \circledcirc G_2) = f(tw(G_1), tw(G_2))$, or to find a "reasonable" lower bound function f such that $tw(G_1 \circledcirc G_2) \geqq f(tw(G_1), tw(G_2))$.

In this paper, we give a lower bound function for tree-width of Cartesian product graphs in terms of two graph parameters related to tree-width. To be more precise, we show that the bramble number of Cartesian product of graphs G_1 and G_2 is at least *Hadwiger number* of G_1 times *PI number* of G_2, that is, in our terminology $tw(G_1 \square G_2) \geqq \eta(G_1) \times \iota(G_2)$, where $\eta(G)$ and $\iota(G)$ denote Hadwiger number and PI number of G, respectively (See Section 2. for more detail on the terminologies).

We also demonstrate applications of the lower bound function. More precisely, by applying our lower bound function to Cartesian product graph of a complete graph and a grid, we practically determine the tree-width of the product graph. We also apply our lower bound function to Cartesian product graph of a complete graph and a complete multipartite graph. Unfortunately our lower bound function does not work for the case where both factor graphs have small tree-width. For example, the path P_n of n vertices has $tw(P_n) = 1$ and it is known that $tw(P_n \square P_n) = n$. On the other hand, our lower bound function gives $\eta(P_n) \times \iota(P_n) = 2 \times 1 = 2$. Fortunately, however, our lower bound function works well for the case where one of two factor graphs has large tree-width, like a complete graph.

1.4 Organization of this paper

The rest of the paper is organized as follows: Section 2. reviews basic definitions and notations, and introduce new terminology. Section 3. gives the lower bound function for tree-width of Cartesian product graphs. Section 4. demonstrates the application of the lower bound function. In Section 5. we make a simple observation of a relationship between Hadwiger number and bramble number.

2. Definitions and notations

Let G be a graph. $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. Let X and Y be subsets of $V(G)$. $G[X]$ denotes the subgraph of G induced by X. If $G[X]$ is connected, we say X is a *connected subset* or simply *connected*. *X and Y are joined* if X and Y do not intersect and there exists an edge such that one of its endpoints is in X and the other is in Y. *X and Y are touched* if they intersect or are joined. Note that in our definitions if X and Y are touched then they intersect iff they are not joined.

In this paper, P_n and K_n denote a path with n vertices and a complete graph with n vertices, respectively (Note that P_n is a path of length $n-1$). And $K_{n_1,n_2,\ldots,n_k}$ denotes a complete multipartite graph with k parts, where n_i is the number of vertices in the i-th part of the graph. For convenience, we assume that $n_1 \geqq n_2 \geqq \cdots \geqq n_k$.

2.1 Graph parameters

A *tree decomposition* of a graph G is a pair (X, T), where $\mathsf{X} = \{X_i \mid i \in V(T)\}$ is a collection of subsets of $V(G)$ and T is a tree, such that

- $\bigcup_{i \in V(T)} X_i = V(G)$,
- for each edge $\{v, w\} \in E(G)$, there is a *node* $i \in V(T)$ such that $v, w \in X_i$, and
- for each $v \in V(G)$ the set of nodes $\{i \mid v \in X_i\}$ forms a subtree of T.

The elements X_i's in X are called *bags*. The *width* of a tree decomposition (X, T) equals $\max_{i \in V(T)} |X_i| - 1$. The *tree-width* of G is the minimum width over all tree decompositions of G. A *path decomposition* of G is a tree decomposition (X, T) in which T is a path. The path-width of G is the minimum width over all path decompositions of G.

Let $\mathsf{S} = \{S_1, \ldots, S_k\}$ be a collection of connected subsets of $V(G)$. The *order* of S is the size of *hitting set* for S of minimum cardinality, that is the least number of vertices to cover every subset $S_i \in \mathsf{S}$. Generally speaking, it is difficult to obtain a good lower bound for the order of S. S is an *intersecting family*, a *joined family*, and a *touched family* (or a *bramble*) if any two subsets in S intersect, are joined, and are touched, respectively (Recall that all subsets in S are required to be connected). Clearly if S is a joined family then the order of S is $|\mathsf{S}|$. So one way to avoid the difficulty of evaluating the order is to take a joined family instead of an intersecting or a touched family. Actually we use this simple but useful idea in this paper.

The following graph parameters play important roles in this paper, and there are several known results on Hadwiger number (See e.g. [3]) and bramble number (See [2]).

Definition

- The *PI number* of G, denoted by $\iota(G)$, is the maximum order of all possible intersecting families of G.
- The *Hadwiger number* of G, denoted by $\eta(G)$, is the maximum order of all possible joined families of G.
- The *bramble number* of G, denoted by $bn(G)$, is the maximum order of all possible touched families of G.

As mentioned in Introduction, Seymour and Thomas showed that for graph G, $bn(G) = tw(G) + 1$ [10]. A merit to use bramble for showing lower bound on tree-width is that a lower bound can be found constructively by taking suitable bramble and evaluating its order. Generally speaking evaluation for the order is a difficult task.

It should be mentioned that the following graph parameter *linked-*

ness is strongly related to PI number. Actually, in this paper, we will use an idea inherent in the concept of the graph parameter linkedness in order to obtain a lower bound for $\iota(G)$. The definition of linkedness is as follows.

Definition ([8], [9]) A subset S of $V(G)$ is *k-linked* if for any set $X \subseteq V(G)$ with $|X| < k$, there is a component of $G - X$ containing more than half of the vertices of S. The *linkedness* of G, denoted by $\lambda(G)$, is the largest k for which G has a k-linked set.

For each set $X \subseteq V(G)$ with $|X| < k$, we call the component containing more than half of the vertices of S by *big component of* $G - X$ *under* S. Typical example of linkedness is that the linkedness of a complete graph K_n with n vertices is $\lceil n/2 \rceil$ (Because $V(K_n)$ is a $\lceil n/2 \rceil$-linked set).

Remark 1. *The set of big components under S pairwise intersect since each of them has more than half of the vertices of S.*

Remark 2. *The order of the set of big components under a k-linked set S is at least k. Because if not, there is a hitting set X of size less than k. From the definition, $G - X$ should have a big component, and clearly X and the big component do not intersect.*

Since each big component is clearly connected and from Remark 1 they pairwise intersect, the set of big components is an intersect family. Thus from Remark 2, we have $\lambda(G) \leq \iota(G)$. In Section 4., we will demonstrate a lower bound for PI number of a complete multipartite graph by using an idea observed in Remark 2.

2.2 Cartesian product

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. *Cartesian product* $G_1 \square G_2$ is the graph whose vertex set is $V_1 \times V_2$, and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \square G_2$ iff either $[v_1 = v_2$ and $\{u_1, u_2\} \in E_1]$ or $[u_1 = u_2$ and $\{v_1, v_2\} \in E_2]$.

Remark 3. *Let* $\mathsf{B}_1 = \{B^1_1, B^1_2, \ldots, B^1_{|\mathsf{B}_1|}\}$ *be a touched family (i.e. a bramble) of a graph* G_1 *and* $\mathsf{B}_2 = \{B^2_1, B^2_2, \ldots, B^2_{|\mathsf{B}_2|}\}$ *be a touched family of a graph* G_2*, and let* $\mathsf{B}_1 \times \mathsf{B}_2 := \{B_{i,j} \mid 1 \leq i \leq |\mathsf{B}_1|, 1 \leq j \leq |\mathsf{B}_2|\}$*, where* $B_{i,j} = \{(u, v) \mid u \in B^1_i, v \in B^2_j\}$*. Then fortunately* $\mathsf{B}_1 \times \mathsf{B}_2$ *is a touched family (i.e. a bramble) of* $G_1 \otimes G_2$*. Unfortunately, however,* $\mathsf{B}_1 \times \mathsf{B}_2$ *is not a touched family of* $G_1 \square G_2$ *in general.*

3. Lower bound for tree-width of Cartesian product graphs

In this section, we give a lower bound function for tree-width of Cartesian product graphs. As mentioned in Subsection 1.2, Lucena determined the exact tree-width of the Cartesian product graph of two complete graphs with n vertices [7], more precisely, $tw(K_n \square K_n) = n^2/2 + n/2 - 1$ for $n \geqq 3$. Our lower bound function can be considered as a formulation of this result of Lucena. In fact, applying our lower bound function to $K_n \square K_n$ with odd n, the proof of our lower bound function is coincident with the proof of the Lucena's result. For even n, our lower bound function does not achieve the optimal value. In fact Lucena constructed a bramble in more sophisticated way for the case of even n.

Theorem 1. *Let* G_1 *and* G_2 *be graphs. Then* $bn(G_1 \square G_2) \geqq \eta(G_1) \times \iota(G_2)$.

Proof. Let $\mathsf{J} = \{J_1, \ldots, J_{\eta(G_1)}\}$ be a joined family of G_1 and $\mathsf{I} = \{I_1, \ldots, I_{|\mathsf{I}|}\}$ be an intersecting family of G_2 with order of $\iota(G_2)$, and let $\mathsf{B} := \{B_{i,j} \mid 1 \leqq i \leqq \eta(G_1), 1 \leqq j \leqq |\mathsf{I}|\}$, where $B_{i,j} = \{(u, v) \mid u \in J_i, v \in I_j\}$.

First we show that B is a bramble of $G_1 \square G_2$. To show this, let us first verify that each $B_{i,j}$ is connected. It is known that the Cartesian product of two graphs is connected iff both factors are connected (See e.g. [6]). So each $B_{i,j}$ is connected.

Let us check that any $B_{i,j}$ and $B_{p,q}$ are touched. Since I is an intersecting family, there exists a vertex $u \in V(I_j) \cap V(I_q)$. If $i = p$, we have $(v, u) \in B_{i,j} \cap B_{p,q}$ for any vertex v in $J_i = J_p$, so $B_{i,j}$ and $B_{p,q}$ intersect. If $i \neq p$, there is an edge $\{v, w\}$ in G_1 such that $v \in J_i$ and $w \in J_p$, so $(v, u) \in B_{i,j}$ and $(w, u) \in B_{p,q}$ are adjacent in $G_1 \square G_2$. It is worth to note that it is an advantage here to take I (an intersecting family) not a bramble (a touched set) (Recall Remark 3).

Now we show that the order of B is $\eta(G_1) \times \iota(G_2)$. So let us consider the set $H := \{(u, v) \mid u \in H_{\mathsf{J}}, v \in H_{\mathsf{I}}\}$, where H_{J} and H_{I} are minimum hitting sets of J and I, respectively. Then clearly H is a hitting set of B (Note that $|H_{\mathsf{J}}| = \eta(G_1)$ and $|H_{\mathsf{I}}| = \iota(G_2)$). Furthermore H is minimum. that is, in order to cover every set in B at least $\eta(G_1) \times \iota(G_2)$ vertices are required. The reason is as follows. Partition H into $\{(J_1, \mathsf{I}), (J_2, \mathsf{I}), \ldots, (J_{|\mathsf{J}|}, \mathsf{I})\}$, where $(J_j, \mathsf{I}) = \{B_{j,i} \mid 1 \leqq i \leqq |\mathsf{I}|\}$. Then as $J_i \cap J_j = \emptyset$ for $i \neq j$, if (u, v) covers a set in (J_i, I) then (u, v) cannot cover any set in (J_j, I) for all $i \neq j$. Since the order of I is $\iota(G_2)$, we need at least $\iota(G_2)$ vertices to cover every set in (J_j, I) for each $1 \leqq j \leqq \eta(G_1)$. So we need a total of $\eta(G_1) \times \iota(G_2)$ vertices to cover every set in B.

It should be noted here that there is an advantage to taking J (a joined family) not a bramble (a touched family), because, in general, it is hard to estimate the order of B if we took a bramble instead of J.

As a result, we have that $bn(G_1 \square G_2) \geqq \eta(G_1) \times \iota(G_2)$. □

4. Applications

In this section, we practically determine the tree-width of Cartesian product of complete graphs and fundamental graphs such as grids and complete multipartite graph.

4.1 Complete graphs and grids

The following fact is a folklore result (See e.g. [5]).

Fact 1. *Let G be a connected graph with* $m \geqq 2$ *vertices. Then* $pw(P_n \square G) \leqq m$.

By combining Theorem 1 and Fact 1, we have Theorem 2 given below. As seen from the proof, Theorem 2 also holds if we replace tree-width tw with path-width pw in the statement.

Theorem 2. *Let $grid_{m,n}$ be a $m \times n$ grid graph with $m \geqq n$. Then, $\ell n - 1 \leqq tw(K_\ell \,\square\, grid_{m,n}) \leqq \ell n$.*

Proof. Firstly it is easy to see that $\iota(grid_{m,n}) \geqq \min\{m, n\}$ by using a similar argument to show that $bn(grid_{m,n}) = \min\{m, n\} + 1$ (See [2]).

Since $K_\ell \,\square\, grid_{m,n}$ is isomorphic to $(K_\ell \,\square\, P_n) \,\square\, P_m$, from Fact 1 we have $tw(K_\ell \,\square\, grid_{n,m}) \leqq pw(K_\ell \,\square\, grid_{n,m}) = pw((K_\ell \,\square\, P_n) \,\square\, P_m) \leqq |V(K_\ell \,\square\, P_n)| = \ell \cdot n$. On the other hand, from Theorem 1, we have $tw(K_\ell \,\square\, grid_{m,n}) = bn(K_\ell \,\square\, grid_{m,n}) - 1 \geqq \eta(K_\ell) \times \iota(grid_{m,n}) - 1 \geqq \ell \cdot n - 1$. □

Our lower bound function is tight for tree-width of Cartesian product of complete graphs and grids. For example, applying Theorem 2 to $P_2 \,\square\, grid_{2,2}$, we have $3 \leqq tw(P_2 \,\square\, grid_{2,2}) \leqq 4$. It is easy to verify that $tw(P_2 \,\square\, grid_{2,2}) = 3$.

4.2 Complete graph and complete multipartite graph

The following upper bound lemma can be proved by applying Theorem 3.4 in [5].

Lemma 1. *$tw(K_\ell \,\square\, K_{n_1,n_2,\ldots,n_k}) \leqq \ell \cdot (n - n_1 + 1) - 1$, where $n_1 \geqq n_2 \geqq \cdots \geqq n_k$ and $n = \sum_{i=1}^{k} n_i$.*

To estimate lower bound for tree-width of Cartesian product of complete graphs and complete multipartite graphs, we will consider the two cases: $n_1 \leqq \lfloor n/2 \rfloor$ and otherwise.

Lemma 2. *If $n_1 \leqq \lfloor n/2 \rfloor$, then $\ell \cdot \lceil n/2 \rceil - 1 \leqq tw(K_\ell \,\square\, K_{n_1,n_2,\ldots,n_k}) \leqq \ell \cdot n - n_1$, where $n_1 \geqq n_2 \geqq \cdots \geqq n_k$ and $n = \sum_{i=1}^{k} n_i$.*

Proof. The upper bound follows from Lemma 1. To derive a lower bound, we will estimate $\eta(K_\ell) \times \iota(K_{n_1,n_2,\ldots,n_k})$ (The other side $\eta(K_{n_1,n_2,\ldots,n_k}) \times \iota(K_\ell)$ gives a weaker bound).

To estimate $\iota(K_{n_1,n_2,\ldots,n_k})$, consider a set $\mathsf{S} := \{S \subseteq V(K_{n_1,n_2,\ldots,n_k}) \mid |S| = \lfloor n/2 \rfloor + 1\}$. Firstly it is easy to see that S is an intersecting family. In fact, as $n_1 \leqq \lfloor n/2 \rfloor$, any set S in S is connected and clearly any two sets in S intersect pairwise. Now we show that the order of S is at least $\lceil n/2 \rceil$. Suppose if not, there is a hitting set H of size at most $\lceil n/2 \rceil - 1$. Then $V(G) - H$ has at least $\lfloor n/2 \rfloor + 1$ vertices. Thus $V(G) - H$ has a set in S which cannot be covered by H. Therefore we have $\iota(K_{n_1,n_2,\ldots,n_k}) \geqq \lceil n/2 \rceil$.

Thus from Theorem 1 we have $\ell \cdot \lceil n/2 \rceil \leqq bn(K_\ell \,\square\, K_{n_1,n_2,\ldots,n_k})$. □

Lemma 3. *If $n_1 > \lfloor n/2 \rfloor$, then $\ell \cdot (n - n_1) - 1 \leqq tw(K_\ell \,\square\, K_{n_1,n_2,\ldots,n_k})$, where $n_1 \geqq n_2 \geqq \cdots \geqq n_k$ and $n = \sum_{i=1}^{k} n_i$.*

Proof. To estimate $\iota(K_{n_1,n_2,\ldots,n_k})$, consider a set $\mathsf{S} := \{S \subseteq V(K_{n_1,n_2,\ldots,n_k}) \mid |S| = n_1 + 1\}$. Then it is easy to see that S is an intersecting family with the order $n - n_1$ by using an argument similar to that in the proof of Lemma 2. Thus, from Theorem 1, we have $\ell \cdot (n - n_1) \leqq bn(K_\ell \,\square\, K_{n_1,n_2,\ldots,n_k})$. □

By combining Lemmas 2, 3 and Lemma 1, we have the following theorem.

Theorem 3. *Let $n_1 \geqq n_2 \geqq \cdots \geqq n_k$ and $n = \sum_{i=1}^{k} n_i$. Then*

$$\ell \cdot (n - n_1 + 1) - 1 \geqq tw(K_\ell \,\square\, K_{n_1,n_2,\ldots,n_k}) \geqq \begin{cases} \ell \cdot \lceil n/2 \rceil - 1 & n_1 \leqq \lfloor n/2 \rfloor, \\ \ell \cdot (n - n_1) - 1 & n_1 > \lfloor n/2 \rfloor. \end{cases}$$

4.3 Lower bound function in terms of Hadwiger number

From the definition, clearly $\eta(G) \leqq bn(G)$ holds, and it is known that for any graph G, $bn(G) \leqq 2\lambda(G)$ [8], hence $\eta(G)/2 \leqq \lambda(G)$. Hence, from Theorem 1, we have that $bn(G \,\square\, G) \geqq \eta(G)^2/2$ (Recall that $\lambda(G) \leqq \iota(G)$).

5. Hadwiger number and bramble number

In this section, we consider graphs G with $bn(G) = \eta(G)$. This is motivated from Hadwiger conjecture.

The conjecture "$\eta(G) \geqq \chi(G)$ for any graph G" is known as Hadwiger conjecture ($\chi(G)$ denotes the chromatic number of G). As we can see from the definitions, bramble number is somewhat similar to Hadwiger number. So, it is natural to ask whether or not $bn(G) \geqq \chi(G)$ holds for any graph G. Actually the inequality holds for any graph. To see this, consider a chordal graph G' such that G' is a supergraph of G and $tw(G') = tw(G)$. Then, since G' is a chordal graph, $\chi(G') = \omega(G')$ holds, where $\omega(G')$ denotes the size of largest complete graph in G'. So we have $tw(G) + 1 = tw(G') + 1 = \omega(G') = \chi(G') \geqq \chi(G)$.

Next let us observe a relationship between bramble number and Hadwiger number from the viewpoint of "covering and packing problem on hypergraphs." To this end, we need some definitions. Let B be a bramble of a graph G, and let $\mathsf{H_B}$ denote the hypergraph whose vertex set is $V(G)$ and edge set is B. Let P be a property on hypergraphs. For example, "Balanced," "Arboreal," and "Normal" can be considered as P. Then B *has P (or P-bramble)* if $\mathsf{H_B}$ has P. Let H be a hypergraph. A *covering* of H is a set of vertices of H intersecting each edge of H. The *covering number* of H, denoted by $\tau(\mathsf{H})$, is the size of a minimum covering of H. So, for any bramble B, $\tau(\mathsf{H_H})$ equals the order of B. A *packing* of H is a set of pairwise disjoint edges in H. The *packing number* of H, denoted by $\nu(\mathsf{H})$, is the size of a maximum packing set of H. So, for any bramble B, $\nu(\mathsf{H_H})$ equals the size of a maximum joint set contained in B. H has *Konig property* if $\tau(\mathsf{H}) = \nu(\mathsf{H})$ holds.

We are now ready to explain the relationship between bramble number and Hadwiger number. Let consider a graph G for which there exists an optimal bramble B having Konig property (i.e., $bn(G) = bn(\mathsf{B})$ and B has Konig property). For such a graph G, we have $bn(G) = \eta(G)$, so Hadwiger conjecture holds for such a graph.

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