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再帰的経路順序停止性をもつ項書き換えシステムの直和の停止性

栗原正仁　　加地郁夫
北海道大学工学部

項書き換えシステムは、再帰的経路順序を用いてその停止性が証明できるとき、再帰的経路順序停止性をもつという。本論文では、項書き換えシステム R_1 と R_2 がともに再帰的経路順序停止性をもつときに限りその直和 $R_1 \oplus R_2$ も再帰的経路順序停止性をもつことを示す。この結果は、R_1 と R_2 の停止性がいかに証明されたかにのみ依存し、非分解性、非重複性、左線形性などの構文的性質に陽に依存しない点で新しい。

Termination of the Direct Sum of Rpo-Terminating Term Rewriting Systems

Masahito KURIHARA and Ikuo KAJI
Faculty of Engineering, Hokkaido University
Kita 13 Nishi 8, Kita-ku, Sapporo, 060 Japan

A term rewriting system is said to be *rpo-terminating* if it's termination is proved with the recursive path ordering method. We prove that the direct sum $R_1 \oplus R_2$ of term rewriting systems R_1 and R_2 is rpo-terminating iff both R_1 and R_2 are so. The result is novel in that it depends only upon *how we proved* both R_1 and R_2 terminating, rather than explicit syntactic properties of the terminating systems, such as non-collapsing, non-duplicant, and left-linear.

1 Introduction

A term rewriting system[3] R is a finite set of rewrite rules $M \to N$, where M and N are terms constructed from variables and function symbols. The *direct sum* $R_1 \oplus R_2$ is the union of two term rewriting systems with disjoint function symbols. A term rewriting system is *terminating* iff there is no infinite reduction sequence. Since establishing termination is in general a difficult task, it had been desired that we could construct terminating systems from smaller ones:

[Conjecture] *$R_1 \oplus R_2$ is terminating iff both R_1 and R_2 are so.*

Unfortunately, however, Toyama[6] recently discovered a counterexample in which R_1 and R_2 are terminating while $R_1 \oplus R_2$ is not. The conjecture was modified:

[Conjecture] (Toyama) *$R_1 \oplus R_2$ is terminating and confluent iff both R_1 and R_2 are so.*

However, it was also refuted by Klop and Barendregt.[6] Very recently, Rusinowitch[4] and Toyama, et. al.[7] presented positive results on this material:

[Theorem] (Rusinowitch) *$R_1 \oplus R_2$ is terminating and non-collapsing iff both R_1 and R_2 are so.*

[Theorem] (Rusinowitch) *$R_1 \oplus R_2$ is terminating and non-duplicant iff both R_1 and R_2 are so.*

[Theorem] (Toyama, et. al.) *$R_1 \oplus R_2$ is terminating, confluent, and left-linear iff both R_1 and R_2 are so.*

where a system is *collapsing* if it contains a rule whose right-hand side is a variable, and *duplicant* if it contains a rule whose right-hand side has strictly more occurrences of one variable than its left-hand side.

These results explicitly depend upon the syntactic properties of the systems such as non-collapsing, non-duplicant, and left-linear. In this letter, we present a new result discovered from another point of view:

[Theorem] *$R_1 \oplus R_2$ is rpo-terminating iff both R_1 and R_2 are so.*

where a system is *rpo-terminating* iff it is proved to be terminating with the recursive path ordering method.[2] The result is novel in that it depends only upon *how we proved* both R_1 and R_2 terminating, rather than explicit syntactic properties of the terminating systems.

2 Rpo-termination

Let V be a set of *variables*, denoted by $x, y, z, \ldots$, and F be a set of *function symbols*, denoted by $f, g, h, \ldots$. A *term*, denoted by $s, t, u, \ldots$, is defined as usual[5] in terms of variables and function symbols. $T(F)$ and $T(F,V)$ denote the set of terms on F and $F \cup V$, respectively. A *substitution*, denoted by $\theta, \sigma, \ldots$, is a mapping from V to $T(F,V)$. As usual,[5] it is naturally extended to a mapping from $T(F,V)$ to $T(F,V)$.

[Definition] The *depth* is the function from $T(F,V)$ to the set of natural numbers defined as follows:

$$depth(s) = \begin{cases} 1, & \text{if } s \text{ is a constant or a variable;} \\ 1 + \max_i\{depth(s_i)\}, & \text{if } s = f(s_1, \ldots, s_n). \end{cases}$$

[Definition] [2] Let $\succ$ be a partial ordering (i.e. irreflexive and transitive relation) on a set F of function symbols. The *recursive path ordering* induced by $\succ$ is the ordering $\succ^*$ on $T(F)$ defined recursively as follows:

$s = f(s_1, \ldots, s_m) \succ^* g(t_1, \ldots, t_n) = t$
iff
$s_i \succeq^* t$ for *some* i $(1 \leq i \leq m)$, or
$f \succ g$ and $s \succ^* t_j$ for *all* j $(1 \leq j \leq n)$, or
$f = g$ and $\{s_1, \ldots, s_m\} \succ\!\!\succ^* \{t_1, \ldots, t_n\}$

where $\succ\!\!\succ^*$ is the multiset ordering[1] induced by $\succ^*$, and $\succeq^*$ means $\succ^*$ or permutatively congruent (equivalent up to permutation of subterms).

The following properties of $\succ^*$ are well-known:

- $s \succ^* t$ if t is a proper subterm of s.
- if s and t are constants, then $s \succ^* t$ *iff* $s \succ t$.

[Lemma 1] *Let* $\succ_1$ *and* $\succ$ *be partial orderings on the same domain* F. *Then* $\succ_1 \subseteq \succ$ *implies* $\succ_1^* \subseteq \succ^*$.

(Proof) Assume that $\succ_1 \subseteq \succ$ and $s \succ_1^* t$ $(s, t \in T(F))$. We show that $s \succ^* t$ by structural induction on $T(F)$.

When $depth(s) = depth(t) = 1$, both s and t are constants and we have from $s \succ_1^* t$ that $s \succ_1 t$. Hence, $s \succ t$. Therefore, $s \succ^* t$.

Assume as an inductive hypothesis that $\succ_1 \subseteq \succ$ and $s' \succ_1^* t'$ implies $s' \succ^* t'$ for all terms s' and t' such that $depth(s') \leq depth(s)$ and $depth(t') \leq depth(t)$ but $(depth(s'), depth(t')) \neq (depth(s), depth(t))$. Let $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$. From $s \succ_1^* t$, we have three cases:

(i) $s_i \succeq_1^* t$ for some i

(ii) $f \succ g$ and $s \succ_1^* t_j$ for all j

(iii) $f = g$ and $\{s_1, \ldots, s_m\} \succ\!\!\succ_1^* \{t_1, \ldots, t_n\}$

In Case (i), by the inductive hypothesis, $s_i \succeq^* t$. Hence, $s \succ^* t$. In Case (ii), we have $s \succ^* t$ again in a similar way. In Case (iii), we have $s \succ^* t$ by using the inductive hypothesis and the definition of multiset ordering $\succ\!\!\succ_1^*$:

$\exists X, Y : \emptyset \neq X \subseteq \{s_1, \ldots, s_m\}$,
$\{t_1, \ldots, t_n\} = (\{s_1, \ldots, s_m\} - X) \cup Y$,
$(\forall y \in Y)(\exists x \in X)\ x \succ_1^* y$.

Therefore, in all cases, we have that $s \succ^* t$. $\square$

[Definition] Let F_1 and F be sets of function symbols such that $F_1 \subseteq F$, and $\succ_1$ be a partial ordering on F_1. The *extension* of $\succ_1$ from F_1 to F is the partial ordering $\succ$ on F defined below:

$f \succ g$ iff $f, g \in F_1 \wedge f \succ_1 g$.

[Lemma 2] *Let F_1, F, $\succ_1$, and $\succ$ be the same as those in the above definition. Suppose s and t be two terms in $T(F_1, V)$ such that $s\theta \succ_1^* t\theta$ for any substitution $\theta : V \to T(F_1)$. Then $s\sigma \succ^* t\sigma$ for any substitution $\sigma : V \to T(F)$.*

(Proof) First, note that the term s cannot be a variable; otherwise, we would have $t\theta \succeq_1^* s\theta$ for some θ, which contradicts $s\theta \succ_1^* t\theta$.

By structural induction. When $depth(s) = depth(t) = 1$, both s and t are constants. (If t were a variable, we would obtain $s\theta = t\theta$ for $\theta = \{t \leftarrow s\}$.) Hence $s \succ_1 t$, so $s \succ t$. Therefore, $s\sigma \succ^* t\sigma$ for any substitution σ.

Assume that the claim holds for all terms s' and t' such that $depth(s') \le depth(s)$ and $depth(t') \le depth(t)$ but $(depth(s'), depth(t')) \ne (depth(s), depth(t))$.

(Case 1) When t is a variable, s must contain t as its proper subterm. Therefore, from the property of recursive path orderings, $s\sigma \succ^* t\sigma$ for any substitution σ.

(Case 2) When t is not a variable, let $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$. From $s\theta \succ_1^* t\theta$ for all θ, we have three cases:

(i) $s_i\theta \succeq_1^* t\theta$ for some i for all θ

(ii) $f \succ g$ and $s\theta \succ_1^* t_j\theta$ for all j and θ

(iii) $f = g$ and $\{s_1\theta, \ldots, s_m\theta\} \succ\succ_1^* \{t_1\theta, \ldots, t_n\theta\}$ for all θ.

By the inductive hypothesis and the definition of multiset orderings, it is easy to verify that in all cases we have that $s\sigma \succ^* t\sigma$ for any substitution $\sigma : V \to T(F)$. □

It is well known that recursive path orderings can be used to establish the termination of term rewriting systems:

[Lemma 3] (2) *A term rewriting system R over a set of terms $T(F)$ is terminating if there exists a partial ordering $\succ$ on F such that $l\theta \succ^* r\theta$ for each rule $l \to r$ in R and for any substitution $\theta : V \to T(F)$.*

The existence of a partial ordering $\succ$ in this lemma may be checked mechanically. If such an ordering exists, then we may conclude that a given system is terminating. Note that such an ordering may not exist even when the system is terminating. If it exists, we say that the system is *rpo-terminating*.

[Theorem] *$R_1 \oplus R_2$ is rpo-terminating iff both R_1 and R_2 are so.*

(Proof) The *only-if* part is trivial. We prove the *if* part. Let F_1 and F_2 be the disjoint set of function symbols contained in R_1 and R_2, respectively, and let $F = F_1 \cup F_2$. Since R_1 and R_2 are rpo-terminating, there exists a partial ordering $\succ_1$ on F_1 such that $l_1\theta \succ_1^* r_1\theta$ for each rule $l_1 \to r_1$ in R_1 and for any substitution $\theta : V \to T(F_1)$. Let $\succ_1'$ be the extension of $\succ_1$ from F_1 to F. Then by Lemma 2, $l_1\sigma \succ_1'^* r_1\sigma$ for each rule $l_1 \to r_1$ in R_1 and for any substitution

$\sigma : V \to T(F)$. Similarly, there exists a partial ordering $\succ_2$ on F_2 and its extension $\succ'_2$ from F_2 to F such that $l_2\sigma \succ'^*_2 r_2\sigma$ for each rule $l_2 \to r_2$ in R_2 and for any substitution $\sigma : V \to T(F)$. Let $\succ$ be the union of $\succ'_1$ and $\succ'_2$. Obviously, $\succ$ is a partial ordering on F. Since $\succ'_1 \subseteq \succ$ and $\succ'_2 \subseteq \succ$, we have $\succ'^*_1 \subseteq \succ^*$ and $\succ'^*_2 \subseteq \succ^*$ by Lemma 1. Hence we have that $l\sigma \succ^* r\sigma$ for each rule in $R_1 \oplus R_2$ and for any substitution $\sigma : V \to T(F)$. Therefore, $R_1 \oplus R_2$ is rpo-terminating. □

[Example] Let

$R_1 = \{x \cdot x \to x, \quad x \cdot (y + z) \to x \cdot y + x \cdot z\}$ and

$R_2 = \{(x^{-1})^{-1} \to x\}$.

The first rewrite rule in R_1 is collapsing and non-left-linear, and the second is duplicant. Hence, the three theorems by Rusinowitch and Toyama, et. al. described in the introduction cannot be applied. By the way, R_1 is shown to be rpo-terminating by defining $\succ_1$ as $\cdot \succ_1 +$. R_2 is also rpo-terminating. Therefore, by our theorem, $R_1 \oplus R_2$ is rpo-terminating.

3 Conclusion

We have presented a novel result on the termination of the direct sum of term rewriting systems. The authors claim that not only the result itself is novel but also the *kind* of the result is novel in that it focuses on the *termination proof method* (recursive path ordering), rather than explicit syntactic properties (e.g., being linear, non-collapsing, etc.). Also, the result is independent of the confluence. Proof with recursive path ordering is one of the most powerful methods that are suitable for semi-mechanical termination proof. Therefore, our result is very useful for applications which require semi-mechanical termination proof procedures. (Induction-less induction theorem proving based on the Knuth-Bendix completion procedure(5) is an example.) You can load and merge several disjoint, rpo-terminating systems together without losing termination. We believe that similar results will be obtained for many other termination proof methods, and it is left as future work.

References

(1) N. Dershowitz and Z. Manna: "Proving termination with multiset orderings", Commun. ACM, **22**, pp.465-476 (1979).

(2) N. Dershowitz: "Termination", Lecture Notes in Computer Science, **220**, pp.180-224 (1985).

(3) K. Futatsugi and Y.Toyama: "Term rewriting systems and their applications: a survey", Information Processing, **24**, pp.133-146 (1983).

(4) M. Rusinowitch: "On termination of the direct sum of term-rewriting systems", Inf. Process. Lett., **26**, pp.65-70 (1987).

(5) K. Sakai: "Knuth-Bendix algorithm and it's applications", Computer Software, **4**, pp.2-22 (1987).

(6) Y. Toyama: "Counterexamples to termination for the direct sum of term rewriting systems", Inf. Process. Lett., **25**, pp.141-143 (1987).

(7) Y. Toyama, J. W. Klop and H. P. Barendregt: "Termination for the direct sum of left-linear term rewriting systems", IEICE technical report, **COMP88-30**, pp.47-55 (1988).