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ビンの容量を制限したキューブパッキング問題のNP完全性について

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あらまし

Look Up Table (LUT) ベースの FPGA のテクノロジーマッピングにおいて, 2段部分回路の LUT への割り当てはキューブパッキング問題として定式化できる. 共通信号線の効果を考慮しない場合にはビンパッキング問題として定式化でき, LUT の入力線数を現実的な値に限ると高速に最適解を求めることができる. 本論文では LUT の入力線数を固定した場合のキューブパッキング問題の計算複雑度を考察し, NP 完全であることを証明する.

和文キーワード　テクノロジーマッピング, ビンパッキング, キューブパッキング, NP 完全, FPGA

Cube-Packing Problem with Fixed Bin-Capacity(≥ 3) is NP-complete

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Abstract

In technology mapping of Look Up Table (LUT) based FPGA, the problem of mapping a two-level subcircuit into LUTs is formulated as the Cube-Packing problem. Without taking account of the advantage of a signal to multiple gates, it is formulated as the Bin-Packing problem. If the number of inputs of LUTs is limited to a practical value, the Bin-Packing problem can be solved optimally very fast. In this paper, we consider the computational complexity of the Cube-Packing in the case that the number of inputs of LUTs is fixed, and prove the problem to be NP-complete.

英文 key words　Technology Mapping, Bin-Packing, Cube-Packing, NP-complete, FPGA

1 Introduction

According to the development of electronics industry, it has become more vital to reach the market with new products in the shortest possible time. Products are repeatedly prototyped and checked. Semi-custom approaches such as standard cells and Mask Programmed Gate Arrays (MPGAs) require extensive time and high cost to manufacture. Field Programmable Gate Arrays (FPGAs) have emerged as the ultimate solution to these problems, because they provide instant manufacturing and very low-cost prototyping.

An FPGA is composed of logic blocks and routing resources. To implement a given circuit on an FPGA chip, the circuit is divided into subcircuits each of which is small enough to be implemented to a logic block. Such a procedure is called the technology mapping. Lookup Tables (LUTs) are often used for logic blocks for its advantage that a k-input LUT can implement any boolean function of k variables.

Minimizing the total number of LUTs used for a circuit allows implementation of larger circuits since the number of LUTs available in an FPGA chip is fixed. One approach is described as follows. Given a multistage logic circuit, traversing from primary inputs to primary outputs, a two-level subcircuit which consists of a gate and it's fanin gates is extracted and mapped into LUTs. [1][3][2][7]. The problem of mapping a two-level circuit can be formulated as the following Cube-Packing problem: Given a set of variables, a set of cubes each of which consists of a subset of variables and an integer called capacity, the set of cubes are divided into as few groups as possible in each of which the number of variables included does not exceed the capacity. Each variable corresponds to the input signal in the circuit, each cube to the gate, and the capacity to the number of inputs to LUTs. By treating such a variable included in multiple cubes in common as a different variable for each cube, the problem of mapping two-level circuits can be formulated as Bin-Packing problem: Given a set of boxes each of which has specific size and capacity, the set of boxes are divided into as few groups such that the sum of size of boxed in each group is not exceeding the capacity.

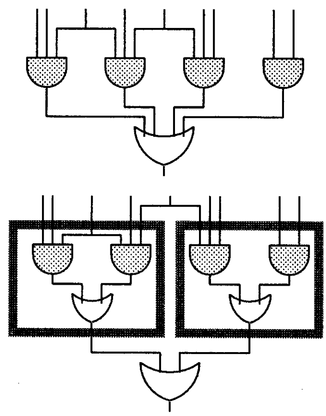

Figure 1: A two-level circuit and mapping into LUTs of $k = 5$

Bin Packing problem being NP-hard, it is known that it can be solved in polynomial time by exhaustive search when the capacity of bins is fixed [4]. However, an exhaustive search is too slow. First Fit Decreasing algorithm (FFD) which is a well-known approximation algorithm of the Bin-Packing problem is often used [3][7], for it is very fast and provides a good solution. Moreover, the authors have found that the FFD provides an optimal solution if the capacity is 6 or less, and furthermore, improved to provide an optimal solution with capacity up to 8, without increasing substantial computational complexity [8].

However, the Cube-Packing problem looks to be far harder than the Bin-Packing problem. Actually, we can prove in this paper that the decision version of Cube-Packing problem is NP-complete even if the capacity is fixed to 3 or more, while the Bin-Packing problem with fixed capacity is pseudo-polynomially solvable.

2 Problem Formulation

Let U be a set of elements, called the *variables*. A *cube* on U is a subset of U. Let $\mathcal{C}$ be a given set of cubes. The size of a subset γ of $\mathcal{C}$ is $|\{u \mid u \in C, C \in \gamma\}|$. Given a positive integer

k, a *cube packing* of $\mathcal{C}$ is a set Γ of disjoint subsets of $\mathcal{C}$ such that $\bigcup_{\gamma\in\Gamma}\gamma = \mathcal{C}$ and the size of any $\gamma \in \Gamma$ is at most k. We can view $\gamma \in \Gamma$ as specifying a set of cubes to be filled in a *bin* of *capacity* k.

CUBE PACKING (CP)
Instance : A set U of variables, a set $\mathcal{C}$ of cubes, capacity k, and a positive integer K.
Question : Is there any cube packing of $\mathcal{C}$ into K or less bins where the capacity of each bin is k ?

CUBE PACKING WITH CAPACITY k (kCP) is a subproblem of CP whose bin capacity is fixed.

For the proof of NP-completeness of kCP, we introduce two classes of problems.

Given a finite set X with $|X| = kq$ where k and q are positive integers, and a set $\mathcal{M}$ of k-tuples of X, *exact cover* of X is a subset Λ of $\mathcal{M}$ such that every element of X is included in exactly one member of Λ.

EXACT COVER BY k-SETS (XkC)
Instance : A finite set X with $|X| = kq$ and a set $\mathcal{M}$ of k-tuples of X.
Question : Does $\mathcal{M}$ contain an exact cover of X?

XkC is a generalized version of the X3C.

For an undirected graph G, we denote the set of vertices in G as $V(G)$ and the set of edges as $E(G)$. A set of edges including as their endpoints at most k vertices is called a *k-cluster.*

k VERTEX EDGE PARTITION (kEP)
Instance : An undirected graph G and a positive integer K.
Question : Is there any partition of $E(G)$ into disjoint K or less k-clusters?

An edge-induced subgraph of G on $E' \subseteq E(G)$ is denoted by $G[E']$. An edge set E' is said to be connected, if $G[E']$ is connected.

3 Results

This section presents our results on the NP-completeness of kCP. We start with XkC which is a generalized version of X3C. X3C is known to be NP-complete [5] and therefore we have

Fact 1 XkC *for $k \geq 3$ is NP-complete.*

We reduce XkC to kEP.

Lemma 1 kEP *for $k \geq 3$ is NP-complete.*

Proof. It is easy to see that kEP $\in$ NP. We reduce XkC to kEP. Let $I_{XC} = (X, \mathcal{M})$ be an instance of XkC where $|X| = kq$ and $|\mathcal{M}| = p$. Let $X = \{x_1, x_2, \dots, x_{kq}\}$ and $\mathcal{M} = \{M_1, M_2, \dots, M_p\}$. We define $\mu(x) = \{M \mid M \in \mathcal{M}, x \in M\}$. We denote $M_i \in \mu(x)$ by $\mu(x, \ell)$ if $|\{M_j \mid M_j \in \mu(x), j \leq i\}| = \ell$ and denote $x_i \in M$ by $\chi(M, \ell)$ if $|\{x_j \mid x_j \in M, j \leq i\}| = \ell$. If $|\mu(x)| < 1$ for some $x \in X$, I_{XC} has no exact cover and we immediately generate some instance with no partition of $E(G)$ into K or less k-clusters. So, we assume $|\mu(x)| \geq 1$ for all $x \in X$ in the following, and then $p \geq q$.

We construct a graph G and set the number K such that I_{XC} has an exact cover if and only if G has a partition into K or less k-clusters. For each element $x \in X$, let

$$V_1(x) = \{v_1(x,i) \mid 1 \leq i \leq |\mu(x)|\},$$

$$E_1(x) = \{\{v_1(x,i), v_1(x,i+1)\} \mid 1 \leq i \leq |\mu(x)|-1\}$$

be a set of vertices and a set of edges. For each k-tuple $M \in \mathcal{M}$, let

$$V_2(M) = \{v_2(M,i) \mid 1 \leq i \leq k\}$$

$$E_2(M) = \{\{v_2(M,i), v_2(M, (i \bmod k)+1)\} \mid 1 \leq i \leq k\}$$

be a set of vertices and a set of edges. For each pair of an element $x \in X$ and a k-tuple $M \in \mathcal{M}$, let i and j be integers such that $\mu(x,i) = M$ and $\chi(M,j) = x$, and

$$V_3(x,M) = \{v_3(x,M,\ell) \mid 1 \leq \ell \leq k-3\},$$

$$\begin{aligned} E_3(x,M) = &\{\{v_1(x,i), v_3(x,M,1)\}\} \\ &\cup \{\{v_3(x,M,\ell), v_3(x,M,\ell+1)\} \mid 1 \leq \ell \leq k-4\} \\ &\cup \{\{v_3(x,M,k-3), v_2(M,j)\}\} \end{aligned}$$

be a set of vertices and a set of edges. In the case that $k = 3$, $V_3(x,M)$ is empty and $E_3(x,M)$ has only one edge $\{v_1(x,i), v_2(M,j)\}$.

We define the graph G by

$$V(G) = \bigcup_{x\in X} V_1(x) \cup \bigcup_{M\in\mathcal{M}} V_2(M) \cup \bigcup_{M\in\mathcal{M}, x\in M} V_3(x,M),$$

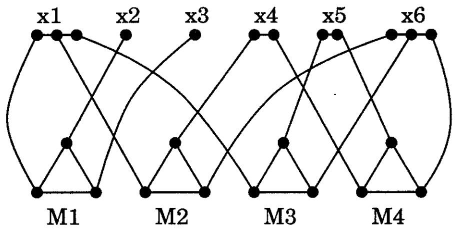

Figure 2: The graph G constructed where $k = 3$, $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\mathcal{M} = \{\{x_1, x_2, x_3\}, \{x_1, x_4, x_6\}, \{x_1, x_5, x_6\}, \{x_4, x_5, x_6\}\}$

$$E(G) = \bigcup_{x \in X} E_1(x) \cup \bigcup_{M \in \mathcal{M}} E_2(M) \cup \bigcup_{M \in \mathcal{M}, x \in M} E_3(x, M).$$

Figure 2 shows an example of the graph constructed. Table 1 shows the numbers of edges in various edge sets. Note that $\sum_{x \in X} |\mu(x)| = kp$. We set the number $K = (k+1)p - q$. Now, we

edge set	# edges
$E_1(x)$	$\lvert\mu(x)\rvert - 1$
$E_2(M)$	k
$E_3(x, M)$	$k-2$
$\bigcup_{x \in X} E_1(x)$	$kp-kq$
$\bigcup_{M \in \mathcal{M}} E_2(M)$	kp
$\bigcup_{M \in \mathcal{M}, x \in M} E_3(x, M)$	$(k-2)kp$
$E(G)$	$k^2p - kq$

Table 1: The number of edges

get an instance of kEP, $I_{EP} = (G, K)$. It is easy to see that this transformation is possible in polynomial time.

We claim that I_{EP} has a partition of $E(G)$ into K or less k-clusters if and only if I_{XC} has an exact cover. Suppose that $\Lambda \subseteq \mathcal{M}$ is an exact cover for I_{XC}. For each $x \in X$, let M be a k-tuple in $\Lambda \cap \mu(x)$ and let i and j be integers such that $\mu(x, i) = M$ and $\chi(M, j) = x$. We define a set of edges $E'(x, \ell)$ for each $1 \le \ell \le |\mu(x)|$ by

$$E_3(x, \mu(x, \ell)) \cup \{\{v_1(x, \ell), v_1(x, \ell+1)\}\} \quad \text{if } 1 \le \ell < i,$$
$$E_3(x, \mu(x, \ell)) \cup \{\{v_2(M, j), v_2(M, (j \bmod k)+1)\}\} \quad \text{if } \ell = i,$$
$$E_3(x, \mu(x, \ell)) \cup \{\{v_1(x, \ell-1), v_1(x, \ell)\}\} \quad \text{if } i < \ell \le |\mu(x)|.$$

Figure 3 shows an example of $E'(x, \ell)$s for $x \in X$ where the third k-tuple of x ($i = 3$) is in Λ. Note that $\bigcup_{x \in X, 1 \le \ell \le |\mu(x)|} E'(x, \ell) = E(G) \setminus \bigcup_{M \notin \Lambda} E_2(M)$. Notice also that each edge set $E'(x, \ell)$ has $k-1$ edges and k vertices and $E_2(M)$ for $M \notin \Lambda$ has k edges and k vertices. Thus,

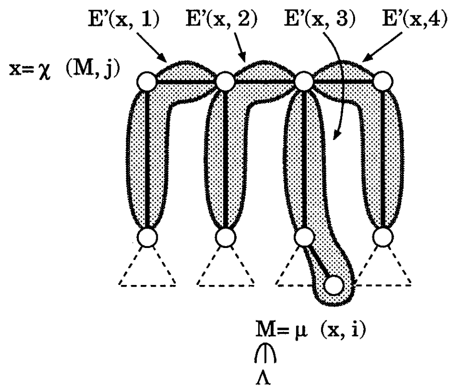

Figure 3: $E'(x, \ell)$s for $x \in X$

$$\Pi = \{E'(x, \ell) \mid x \in X, 1 \le \ell \le |\mu(x)|\} \cup \{E_2(M) \mid M \notin \Lambda\}$$

is a partition of $E(G)$ into k-clusters. The number of clusters in Π is

$$|\Pi| = \sum_{x \in X} |\mu(x)| + (p-q) = (k+1)p - q.$$

Thus, G has a partition into K k-clusters.

Conversely, suppose that Π is a partition of $E(G)$ into k-clusters and $|\Pi| \le K$. Since the length of each cycle in G is k or more, every k-cluster has at most k edges. Since every cycle with length k consists of edges in $E_2(M)$ for some $M \in \mathcal{M}$, a k-cluster π has k edges if and only if $\pi = E_2(M)$ for some $M \in \mathcal{M}$. A k-cluster π has $k-1$ edges if and only if $\pi \neq E_2(M)$ for any $M \in \mathcal{M}$ and π is connected.

Let $\mathcal{M}'$ be the set of $M \in \mathcal{M}$ such that $E_2(M) \in \Pi$, and $\overline{\mathcal{M}'} = \mathcal{M} \backslash \mathcal{M}'$. Let X' be a set of $x \in X$ such that $\mu(x) \subseteq \mathcal{M}'$, and $\overline{X'} = X \backslash X'$. In the following, we show that the number of k-tuples in $\mathcal{M}'$ denoted by n_1 is $p-q$ and the number of elements in X' denoted by n_2 is 0. First, we show the upper bound of n_1. Since $\mu(x) \not\subseteq \mathcal{M}'$ if $x \in M \in \overline{\mathcal{M}'}$,

$$\bigcup_{M \in \overline{\mathcal{M}'}} M = \overline{X'}$$

and the number of elements included in some $M \in \overline{\mathcal{M}'}$ satisfies

$$\left| \bigcup_{M \in \overline{\mathcal{M}'}} M \right| \leq k|\overline{\mathcal{M}'}| = k(p-n_1).$$

Since

$$\begin{aligned} n_2 &= |X'| \\ &= |X| - |\overline{X'}| \\ &= |X| - \left| \bigcup_{M \in \overline{\mathcal{M}'}} M \right| \\ &\geq kq - k(p-n_1), \end{aligned}$$

we have

$$n_1 \leq p-q+\frac{n_2}{k}. \qquad (1)$$

Next, we show the lower bound of n_1. Let

$$\Pi_i = \{\pi \mid \pi \in \Pi, |\pi| = i\},$$

$$\Pi_{\leq i} = \{\pi \mid \pi \in \Pi, |\pi| \leq i\},$$

for $1 \leq i \leq k$. For each $x \in X'$, let

$$\mathcal{E} = E_1(x) \cup \bigcup_{M \in \mu(x)} E_3(x, M),$$

$$\Pi' = \{\pi \mid \pi \in \Pi_{k-1}, \pi \cap \mathcal{E} \neq \emptyset\},$$

$$\mathcal{E}' = \{e \mid e \in \pi \cap \mathcal{E}, \pi \in \Pi'\}.$$

Note that $\mathcal{E}$ has $(k-1)|\mu(x)|-1$ edges. Since any k-cluster in Π_{k-1} must be connected, and since $E_2(M)$ for every $M \in \mu(x)$ is a k-cluster in Π, for any k-cluster $\pi \in \Pi'$, $\pi \subseteq \mathcal{E}$. Then, the number of k-clusters in Π' is at most $\left\lfloor \frac{|\mathcal{E}|}{k-1} \right\rfloor = |\mu(x)|-1$. Thus, $\mathcal{E}'$ has at most $(k-1)(|\mu(x)|-1)$ edges. Since edges in $\mathcal{E}$ can not be in any k-cluster in Π_k, each edge in $\mathcal{E} \backslash \mathcal{E}'$ is in some k-cluster in $\Pi_{\leq k-2}$. The number of such edges is $|\mathcal{E}| - |\mathcal{E}'| = (k-1)|\mu(x)|-1-(k-1)(|\mu(x)|-1) \geq k-2$. Totally, the number of edges included in some k-cluster in $\Pi_{\leq k-2}$ is at least $\sum_{x \in X'} (k-2) = (k-2)n_2$. Thus, $|\Pi_{\leq k-2}| \geq n_2$. We have

$$\begin{aligned} |E(G)| &= \sum_{i=1}^{k} i|\Pi_i| \\ &\leq k|\Pi_k| + (k-1)|\Pi_{k-1}| + (k-2)|\Pi_{\leq k-2}| \\ &\leq kn_1 + (k-1)(|\Pi_{\leq k-1}| - n_2) + (k-2)n_2 \\ &= kn_1 + (k-1)(|\Pi| - n_1 - n_2) + (k-2)n_2. \end{aligned}$$

Since $|\Pi| \leq K = (k+1)p - q$ and $|E(G)| = k^2p - kq$,

$$n_1 \geq p-q+n_2. \qquad (2)$$

Finally, from inequalities (1),(2), and $n_2 \geq 0$, $k \geq 3$, we have

$$n_1 = p-q, \ n_2 = 0.$$

Thus, $|\overline{\mathcal{M}'}| = q$ and every $x \in X = \overline{X'}$ is included in some $M \in \overline{\mathcal{M}'}$. So, $\overline{\mathcal{M}'}$ is an exact cover of I_{XC}. □

Since the degree of each vertex of the graph G we constructed in the proof of Lemma 1 is at most 3, we have the following corollary.

Corollary 1.1 kEP *for* $k \geq 3$ *is NP-complete even if* G *has no vertex with degree exceeding 3.*

Now we arrive at our original problem, cube packing with fixed bin capacity.

Theorem 2 kCP *for* $k \geq 3$ *is NP-complete.*

Proof. kCP can be restricted to kEP by allowing only instances with cubes of two variables and regarding the variables as vertices and the cubes as edges. □

The proof of Theorem 2 will not change if instances with only cubes of two variables are allowed. Applying Corollary 1.1, we can allow only instances such that each variable occurs in at most three cubes. This leads the following corollary.

Corollary 2.2 kCP *for* $k \geq 3$ *is NP-complete even if each cube consists of two variables and each variable occurs in at most three cubes.*

4 Conclusion

We proved that the Cube-Packing problem is NP-complete even if the capacity k of bins is fixed for $k \geq 3$. Furthermore, the Cube-Packing with fixed bin capacity remains NP-complete even if each cube consists of two variables and each variable occurs in at most three cubes. It is easy to see that the problem is polynomially solvable, if the capacity is 3 and each variable occurs in at most two cubes. However, the case that $k \geq 4$ is an open problem.

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