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deBruijnグラフのVLSI分解について

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あらまし
グラフ G の全域部分グラフは，その連結成分がすべて同型であるとき，G の *VLSI* 分解であるといい，各連結成分をビルディングブロックという．小文では，*Viterbi* 復号器の結線構造として用いられている *deBruijn* グラフのビルディングブロックに対する十分条件を与える．

キーワード　deBruijnグラフ，VLSI分解，ビルディングブロック，Viterbi復号器

On VLSI Decompositions for deBruijn Graphs

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Abstract
A VLSI decomposition of a graph G is a collection of isomorphic vertex-disjoint subgraphs (called building blocks) of G which together span G. This paper gives a sufficient condition for a graph to be a building block for deBruijn graphs, which are used to build Viterbi decoders.

key words　deBruijn Graph, VLSI Decomposition, Building Block, Viterbi Decoder

1 Introduction

A VLSI decomposition of a graph G is a collection of isomorphic vertex-disjoint subgraphs of G which together span G. A graph H isomorphic to the subgraphs comprising the decomposition is called a building block for G. The efficiency of H is the fraction of the edges of G which are present in the copies of H. If H is a building block for any graph in a family of graphs $\{G_n\}$, H is called a universal building block for $\{G_n\}$. Finding an efficient building block for G is corresponding to the design of an efficient single VLSI chip with the property that many identical copies of this chip could be wired together to form a circuit represented by G.

A couple of pioneering works on universal building blocks for deBruijn graphs can be found in the literature[1-5]. (The definition of deBruijn graphs will be given in Section 3.) These works were motivated by the need to construct large Viterbi decoders. Schwabe [5] showed that a special kind of subgraph of the nth order deBruijn graph B_n is a universal building block for $\{B_m | m \geq n\}$ with efficiency $1 - O(1/n)$. He also showed that this is asymptotically optimal by proving that the efficiency of any universal building block for $\{B_m | m \geq n\}$ is at most $1 - \Omega(1/n)$. It is conjectured by Dolinar, Ko, and McEliece [2,3] that the efficiency of an optimal universal building block for $\{B_m | m \geq n\}$ is asymptotically equal to $1 - 2/(n+1)$. While optimal universal building blocks for $\{B_m | m \geq n\}$ are known for $n \leq 4$, it remains open to find optimal universal building blocks for larger values of n.

To solve this problem, some necessary conditions and relatively more restrictive sufficient conditions for a graph to be a universal building block for deBruijn graphs have been developed, and based on these sufficient conditions some relatively efficient universal building blocks for deBruijn graphs have been constructed [1,2,3]. However, as far as the authors know, no necessary and sufficient condition is known.

This paper gives a new relatively less restrictive sufficient condition for a graph to be a universal building block for deBruijn graphs, and based on the condition we give some efficient universal building blocks for deBruijn graphs.

2 Preliminaries

2.1 Binary Vectors

We define three mappings L_n, R_n and C_n from $\{0,1\}^n$ to $\{0,1\}^{n-1}$ as follows. If $x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ then

$$\begin{aligned} L_n(x) &= (x_1, x_2, \ldots, x_{n-1}), \\ R_n(x) &= (x_2, x_3, \ldots, x_n), \text{ and} \\ C_n(x) &= (x_1 \oplus x_2, x_2 \oplus x_3, \ldots, x_{n-1} \oplus x_n), \end{aligned}$$

where $\oplus$ is the addition modulo 2. We denote the composition of two mappings f and g by $f \circ g$. We also denote $C_{n-r+1} \circ C_{n-r+2} \circ \ldots \circ C_n$ by C_n^r. For $x, y \in \{0,1\}^n$, $B(x,y)$ is the length of the largest block of consecutive components on which x and y agree. The following two lemmas can be found in the literature.

Lemma I [2] *If $n \geq 3$ and $r \geq 1$ then*

$$\begin{aligned} C_{n-1}^r \circ L_n(x) &= L_{n-r} \circ C_n^r(x), \text{ and} \\ C_{n-1}^r \circ R_n(x) &= R_{n-r} \circ C_n^r(x) \end{aligned}$$

for any $x \in \{0,1\}^n$.

Lemma II [2] *Let $1 \leq r \leq n-1$ and $x, y \in \{0,1\}^n$. If $C_n^r(x) = C_n^r(y)$ and $B(x,y) \geq r$ then $x = y$.*

2.2 Digraphs

Let G be a digraph (directed graph). We denote the vertex set and arc (directed edge) set of G by $V(G)$ and $A(G)$, respectively. An arc from vertex u to v is denoted by (u,v). Let $X = (x_0, x_1, \ldots, x_k)$ be a sequence of $k+1$ vertices, and $Y = (y_1, y_2, \ldots, y_k)$ be a sequence of k arcs. (X,Y) is called a path of length k if the following two conditions are satisfied:

1. $y_i = (x_i, x_{i-1})$ or $y_i = (x_{i-1}, x_i)$ for any i $(1 \leq i \leq k)$,
2. $x_i \neq x_j$ for any i and j with $0 \leq i < j \leq k$.

A path (X,Y) is called a cycle if $x_0 = x_k$. y_i is called a forward arc if $y_i = (x_{i-1}, x_i)$, and a backward arc otherwise. A dipath is a path with no backward arcs. If a path has f forward arcs and b backward arcs, we define the net length of the path to be $|f - b|$. A cycle is said to be balanced if the net length of the cycle is equal to 0. A cycle is said to be unbalanced if it is not balanced. For a cycle (X,Y), the maximum subnet length of (X,Y) is defined to be the maximum net length of a subpath of (X,Y).

2.3 Building Blocks for Digraphs

A digraph H is called a building block for a digraph G if there exists a spanning subdigraph $\overline{H}$ of G that is a vertex-disjoint union of several copies of H. The efficiency of H as a building block for G, denoted by *eff*$(H : G)$, is defined to be $|A(\overline{H})|/|A(G)|$. It is easy to see the following theorem.

Theorem I [3] *If H is a building block for G then*

$$\mathit{eff}(H:G) = \frac{|V(G)||A(H)|}{|V(H)||A(G)|}.$$

3 deBruijn Graphs and Universal deBruijn Building Blocks

The nth order deBruijn graph B_n is a digraph defined as follows:

$V(B_n) = \{0,1\}^n$,
$A(B_n) = \{(x,y)|x, y \in V(B_n), L_n(x) = R_n(x)\}$.

It should be noted that $|V(B_n)| = 2^n$ and $|A(B_n)| = 2^{n+1}$ by definition. A universal deBruijn building block of order n is a spanning subdigraph of B_n that is a building block for any deBruijn graph B_m with $m \geq n$. It is shown in [3] and easily derived from Theorem I that if H is a universal deBruijn building block of order n then

$$\mathit{eff}(H:B_m) = \frac{|A(H)|}{|A(B_n)|}$$

for all $m \geq n$. This common value which is independent of m is called the efficiency of H as a universal deBruijn building block.

The following conditions for universal deBruijn building blocks can be found in the literature.

Theorem II [3] *If H is a universal deBruijn building block then H does not contain an unbalanced cycle or two vertices u and v such that there are two dipaths of the same length from u to v.*

Theorem III [3] *If H is a spanning subdigraph of B_n with no unbalanced cycles or paths of net length $> n$ then H is a universal deBruijn building block of order n.*

It should be noted that the sufficient condition in Theorem III is more restrictive than the necessary condition in Theorem II. We will give in the next section a relatively less restrictive sufficient condition for universal deBruijn building blocks.

4 Sufficient Condition for Universal deBruijn Building Blocks

We prove in this section the main theorem of the paper. The cycle space of a digraph G is a vector space generated by the cycles of G. A basis of the cycle space is called a fundamental n-basis if the basis is consisting of fundamental cycles with respect to a spanning ditree such that the maximum subnet length of each fundamental cycle is at most n.

Theorem 1 *If H is a connected spanning subdigraph of B_n with a fundamental n-basis of the cycle space and without an unbalanced cycle then H is a universal deBruijn building block of order n.*

It should be noted that the condition in Theorem 1 is less restrictive than the condition in Theorem III. The rest of the section is devoted to the proof of Theorem 1.

4.1 Proof of Theorem 1

4.1.1 Spanning Ditrees

First of all, we show the following theorem.

Theorem 2 *Any spanning ditree of B_n is a universal deBruijn building block of order n.*

Proof of Theorem 2 Before proving the theorem, we need some technical lemmas.

Lemma 1 *Let i be a fixed integer ($1 \leq i \leq n+1$). For any $x \in \{0,1\}^n$ and $y \in \{0,1\}^r$, there exists a unique $z = (z_1, z_2, \ldots, z_{n+r}) \in \{0,1\}^{n+r}$ such that $C^r_{n+r}(z) = x$ and $(z_i, z_{i+1}, \ldots, z_{i+r-1}) = y$.*

Proof of Lemma 1 Let $z = (z_1, z_2, \ldots, z_{n+r}) \in \{0,1\}^{n+r}$, and let denote $(z_i, z_{i+1}, \ldots, z_{i+r-1})$ by $z(i : i+r-1)$. For any $z \in \{0,1\}^{n+r}$, we associate a pair

$$\langle C^r_{n+r}(z), z(i : i+r-1)\rangle \in \{0,1\}^n \times \{0,1\}^r.$$

By Lemma II, if $z \neq z'$ then

$$\langle C^r_{n+r}(z), z(i : i+r-1)\rangle \neq \langle C^r_{n+r}(z'), z'(i : i+r-1)\rangle.$$

Since

$$|\{0,1\}^{n+r}| = |\{0,1\}^n \times \{0,1\}^r| = 2^{n+r},$$

we conclude that for any $\langle x, y\rangle \in \{0,1\}^n \times \{0,1\}^r$, there exists a unique $z \in \{0,1\}^{n+r}$ such that $C^r_{n+r}(z) = x$ and $z(i : i+r-1) = y$. □

Lemma 2 *For any $x \in \{0,1\}^n$,*

$$|\{z|z \in \{0,1\}^{n+r}, C^r_{n+r}(z) = x\}| = 2^r.$$

Proof of Lemma 2 For any $x \in \{0,1\}^n$, let

$$S_x = \{z|z \in \{0,1\}^{n+r}, C^r_{n+r}(z) = x\}.$$

For any $x \in \{0,1\}^n$ and $y \in \{0,1\}^r$, define

$$T_{x,y} = \{z|z \in \{0,1\}^{n+r}, C^r_{n+r}(z) = x, (z_1, \ldots, z_r) = y\}.$$

input H: subdigraph of B_n,
r: natural number

Step 1: Set $S = \emptyset$ and $V_i = \emptyset (1 \le i \le 2^r)$.

Step 2: Choose $x \in V(H)$. Let z_i be a vertex in $V(B_{n+r})$ such that $C^r_{n+r}(z_i) = x (1 \le i \le 2^r)$.

Step 3: Set $S = S \cup \{x\}$ and $V_i = V_i \cup \{z_i\}$ $(1 \le i \le 2^r)$.

Step 4: If there exists no vertex in $V(H) - S$ connected with a vertex in S by an arc then go to Step 9.

Step 5: Choose $x \in V(H) - S$ connected with a vertex, say y, in S.

Step 6: If $(y, x) \in A(H)$ then let z_i be a vertex in $V(B_{n+r})$ such that $C^r_{n+r}(z_i) = x$ and $(w_i, z_i) \in A(B_{n+r})$ for $w_i \in V_i$ with $C^r_{n+r}(w_i) = y (1 \le i \le 2^r)$.

Step 7: If $(x, y) \in A(H)$ then let z_i be a vertex in $V(B_{n+r})$ such that $C^r_{n+r}(z_i) = x$ and $(z_i, w_i) \in A(B_{n+r})$ for $w_i \in V_i$ with $C^r_{n+r}(w_i) = y (1 \le i \le 2^r)$.

Step 8: Go to step 3.

Step 9: If $S = V(H)$ then output $(V_1, V_2, \ldots, V_{2^r})$ and halt.

Step 10: Go to step 2.

Figure 1. Algorithm PARTITION.

By Lemma 1,

$$|T_{x,y}| = 1.$$

Since

$$S_x = \bigcup_{y \in \{0,1\}^r} T_{x,y}, \text{ and}$$

$$T_{x,y} \cap T_{x,y'} = \emptyset \text{ if } y \ne y',$$

we conclude that

$$|S_x| = \sum_{y \in \{0,1\}^r} |T_{x,y}| = 2^r.$$

□

Lemma 3 *For any $(y, x) \in A(B_n)$ and for any $w \in V(B_{n+r})$ with $C^r_{n+r}(w) = y$, there exists a unique $z \in V(B_{n+r})$ such that $C^r_{n+r}(z) = x$ and $(w, z) \in A(B_{n+r})$.*

Proof of Lemma 3 By Lemma 1, there exists a unique $z = (z_1, z_2, \ldots, z_{n+r}) \in \{0,1\}^{n+r}$ such that $C^r_{n+r}(z) = x$ and $(z_2, z_3, \ldots, z_{r+1}) = (w_1, w_2, \ldots, w_r)$. By the definition of z,

$$B(L_{n+r}(w), R_{n+r}(z)) \ge r.$$

Since $C^r_{n+r}(z) = x$, $C^r_{n+r}(w) = y$, and $(y, x) \in A(B_n)$, we have

$$L_n \circ C^r_{n+r}(w) = R_n \circ C^r_{n+r}(z).$$

Thus, by Lemma I,

$$C^r_{n+r-1} \circ L_{n+r}(w) = C^r_{n+r-1} \circ R_{n+r}(z).$$

Therefore we conclude by Lemma II that

$$L_{n+r}(w) = R_{n+r}(z),$$

which means that $(w, z) \in A(B_{n+r})$.

It remains to show that such z is unique. Assume that $C^r_{n+r}(z_1) = C^r_{n+r}(z_2) = x$ and $(w, z_1), (w, z_2) \in A(B_{n+r})$ for some $z_1, z_2 \in \{0,1\}^{n+r}$. Since $L_{n+r}(w) = R_{n+r}(z_1) = R_{n+r}(z_2)$,

$$B(z_1, z_2) \ge n + r - 1 \ge r.$$

Thus we conclude that $z_1 = z_2$ by Lemma II. □

Similarly, we can prove the following.

Lemma 4 *For any $(x, y) \in A(B_n)$ and for any $w \in V(B_{n+r})$ with $C^r_{n+r}(w) = y$, there exists a unique $z \in V(B_{n+r})$ such that $C^r_{n+r}(z) = x$ and $(z, w) \in A(B_{n+r})$.* □

Now we are ready to prove Theorem 2. It is sufficient to prove the following.

Theorem 3 *For any spanning ditree H of B_n and a natural number r, there exists a partition $(V_1, V_2, \ldots, V_{2^r})$ of $V(B_{n+r})$ that satisfies the following two conditions:*

1. *$\phi_i : V_i \to V(H)$ is a bijection for any $i (1 \le i \le 2^r)$, where ϕ_i is a restriction of C^r_{n+r} to V_i,*
2. *If $(x, y) \in A(H)$ then $(\phi_i^{-1}(x), \phi_i^{-1}(y)) \in A(B_{n+r})$ for any $i (1 \le i \le 2^r)$.*

Proof of Theorem 3 Consider an algorithm PARTITION shown in Figure 1. We will show that if an input H is a spanning ditree of B_n then the output $\Pi = (V_1, V_2, \ldots, V_{2^r})$ of PARTITION is a desired partition of $V(B_n)$.

Since ϕ_i, a restriction of C^r_{n+r} to V_i, is a bijection for any $i(1 \le i \le 2^r)$ by Lemmas 2, 3, and 4, Π satisfies the first condition. Moreover, since H is a ditree and has no cycles, Π satisfies the second condition by Lemmas 3 and 4.

It remains to show that Π is a partition of $V(B_{n+r})$. It follows from Lemmas 3 and 4 that $V_i \cap V_j = \emptyset$ for any distinct i and j. Since H is a spanning ditree of B_n and ϕ_i is a bijection from V_i to $V(H)$ $(1 \le i \le 2^r)$, $|V_i| = 2^n$ for any $i(1 \le i \le 2^r)$. Thus we have

$$\sum_{i=1}^{2^r} |V_i| = 2^{n+r} = |V(B_{n+r})|.$$

It follows that

$$\bigcup_{i=1}^{2^r} V_i = V(B_{n+r}),$$

and we conclude that Π is a partition of $V(B_{n+r})$. □

This completes the proof of Theorem 2.

4.1.2 Proof of Theorem 1

Now we will complete the proof of Theorem 1.

Lemma 5 *Let H be a connected spanning subdigraph of B_n without unbalanced cycles. Then there exists a mapping $\rho : V(H) \to \mathbb{Z}$ such that $\rho(y) = \rho(x) + 1$ if $(x, y) \in A(H)$.*

Proof of Lemma 5 We define ρ as follows: Choose any $a \in V(H)$ and set $\rho(a) = 0$; If $\rho(x)$ is defined then we define that $\rho(y) = \rho(x) + 1$ if $(x, y) \in A(H)$, and $\rho(y) = \rho(x) - 1$ if $(y, x) \in A(H)$. Since H has no unbalanced cycles, ρ is well-defined. □

The mapping ρ above is called a rank function for H. To prove Theorem 1, it is sufficient to prove the following.

Theorem 4 *Let H be a connected spanning subdigraph of B_n with a fundamental n-basis of the cycle space and without unbalanced cycles. Then for any natural number r, there exists a partition $(V_1, V_2, \ldots, V_{2^r})$ of $V(B_{n+r})$ that satisfies the following two conditions:*

1. *$\phi_i : V_i \to V(H)$ is a bijection for any $i(1 \le i \le 2^r)$, where ϕ_i is a restriction of C^r_{n+r} to V_i.*
2. *If $(x, y) \in A(H)$ then $(\phi_i^{-1}(x), \phi_i^{-1}(y)) \in A(B_{n+r})$ for any $i(1 \le i \le 2^r)$.*

Proof of Theorem 4 The theorem is proved by induction on the dimension of the cycle space, that is, the number of fundamental cycles in a fundamental n-basis.

If a fundamental n-basis of H has no fundamental cycle, that is, if H is a spanning ditree of B_n, the theorem is true for H by Theorem 3.

Assume that the theorem is true for any connected spanning subdigraph of B_n with a fundamental n-basis consisting of $k-1$ fundamental cycles and without unbalanced cycles. Let H be a connected spanning subdigraph of B_n with a fundamental n-basis consisting of k fundamental cycles and without unbalanced cycles. Let H' be the digraph obtained from H by deleting an arc (u, v) which is contained in just one fundamental cycle C in the fundamental n-basis of H. H' is a connected spanning subdigraph of B_n with a fundamental n-basis consisting of $k-1$ fundamental cycles and without unbalanced cycles. Thus, by the induction hypothesis, there exists a partition $(V_1, V_2, \ldots, V_{2^r})$ of $V(B_{n+r})$ which satisfies the conditions in the theorem for H'. We will show that the partition for H' is also a desired partition for H. To prove this, it is sufficient to show that

$$(\phi_i^{-1}(u), \phi_i^{-1}(v)) \in A(B_{n+r})$$

for any $i(1 \le i \le 2^r)$.

Let ρ be a rank function for H'. ρ is also a rank function for H, since H has no unbalanced cycles. Let u' be a vertex of C such that

$$\rho(u') = \min_{x \in V(C)} \rho(x).$$

Since the maximum subnet length of C is at most n, we have

$$(*) \begin{cases} 0 \le \rho(u) - \rho(u') \le n, \text{ and} \\ 0 \le \rho(v) - \rho(u') \le n. \end{cases}$$

Let

$$\begin{aligned} z_i &= \phi_i^{-1}(u) = (z_{i,1}, \ldots, z_{i,n+r}), \\ w_i &= \phi_i^{-1}(v) = (w_{i,1}, \ldots, w_{i,n+r}), \text{ and} \\ z'_i &= \phi_i^{-1}(u') = (z'_{i,1}, \ldots, z'_{i,n+r}), \end{aligned}$$

for any $i(1 \le i \le 2^r)$. It follows from $(*)$ that

$$(\dagger) \begin{cases} (z_{i,\rho(u)-\rho(u')+1}, \ldots, z_{i,\rho(u)-\rho(u')+r}) \\ = (w_{i,\rho(v)-\rho(u')+1}, \ldots, w_{i,\rho(v)-\rho(u')+r}) \\ = (z'_{i,1}, \ldots, z'_{i,r}). \end{cases}$$

Since $(u, v) = (C^r_{n+r}(z_i), C^r_{n+r}(w_i)) \in A(B_n)$, we have $L_n \circ C^r_{n+r}(z_i) = R_n \circ C^r_{n+r}(w_i)$. It follows from Lemma I that

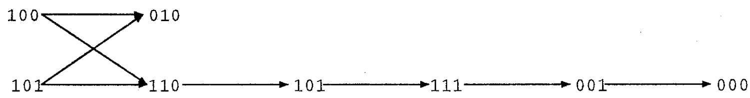

Figure 2. The most efficient universal deBruijn building block of order 3.
The efficiency is 0.5.

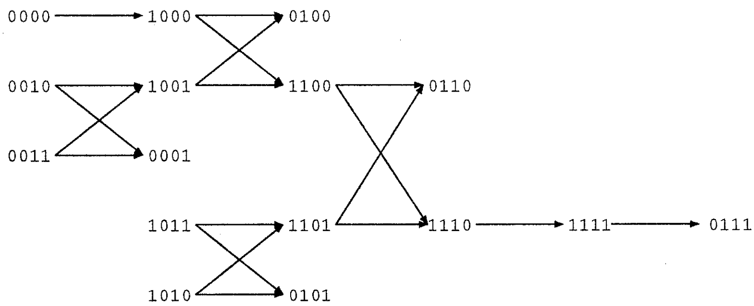

Figure 3. The most efficient universal deBruijn building block of order 4.
The efficiency is 0.594.

$$C^r_{n+r-1} \circ L_{n+r}(z_i) = C^r_{n+r-1} \circ R_{n+r}(w_i)$$

for any $i(1 \leq i \leq 2^r)$. Since

$$\rho(C^r_{n+r}(w_i)) = \rho(C^r_{n+r}(z_i)) + 1,$$

we have from (†) that

$$B(L_{n+r}(z_i), R_{n+r}(w_i)) \geq r.$$

Thus from Lemma II, we conclude that

$$L_{n+r}(z_i) = R_{n+r}(w_i),$$

which means that

$$(z_i, w_i) = (\phi_i^{-1}(u), \phi_i^{-1}(v)) \in A(B_{n+r}),$$

for any $i(1 \leq i \leq 2^r)$. □

This completes the proof of Theorem 1.

Figures 2 and 3 show optimal universal deBruijn building blocks of small orders based on our sufficient condition. It is still open to find optimal universal deBruijn building blocks of larger orders.

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