

# Mathematical Structure of Finsler Encryption

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**概要** : The public-key encryption scheme called **Finsler encryption** introduced by Nagano and Anada at SecITC2020 is investigated about its mathematical structure. In the previous work they stated the scheme based on the linear parallel displacement of vectors on a curve in a Finsler space and exhibited three algorithms of key generation, encryption and decryption in detail. The base idea of Finsler encryption was stated at CSS2019 and at SCIS2020 by the same authors. In this paper, the public key  $PK$  is represented as a mapping from  $\mathcal{R}^2$  to  $\mathcal{R}^9$  and the secret key  $SK$  is represented as the inverse mapping of  $PK$ .

**キーワード** : Finsler encryption, public-key encryption, Finsler geometry, differential geometry, linear parallel displacement

## 1. Introduction

Finsler encryption was defined by Nagano and Anada during from 2019 to 2020(cf.[8],[9],[10]). In the basic, this encryption has the differential geometry, especially, Finsler geometry(cf.[1],[2],[3],[4],[5],[6]). Thus, all items of Finsler encryption are defined on a real and smooth manifold. Further, in differential geometry(Riemaniann geometry), almost all object have symmetric property. On the other hand, one of the most important notions in the public-key encryption is *one direction property*. And encryption has been ever studied on *discrete mathematics* before(cf.[15],[16],[17],[18],[19],[20]).

Finsler geometry is known as geometry that has asymmetric property different from Riemannian geometry(cf.[7],[11],[12],[13],[14]). Especially, the *linear parallel displacement*(cf.[7]) plays a very important role in Finsler encryption. It is a very interesting notion having asymmetric property which the image of a vector obtained by the linear parallel displacement on a curve  $c$  is different from the image of the same vector obtained by the linear parallel displacement on the inverse curve  $c^{-1}$ . This asymmetric property is used in one direction property in the public-key encryption. Further, in this encryption, al-

most all items are expressed in continuous and real form because that geodesic and the linear parallel displacement are solutions of simultaneous differential equations. In encryption system, it is not good for using them as it is. Thus we must change them to integer and rational forms by a *quantization* of a parameter easily. Fortunately, there are many methods of quantization.

## 2. Finsler Encryption

We call new public-key encryption scheme introduced by T. Nagano and H. Anada at SecITC 2020[10] *Finsler Encryption*. In this paper, we discuss about Finsler encryption of §4 in [10], especially.

### 2.1 Finsler Encryption

Let  $(x, y)$  be the coordinate of the base manifold  $M = \mathcal{R}^2$  and  $(\dot{x}, \dot{y})$  the coordinate of  $T_{(x,y)}M$ , namely,  $x = x^1, y = x^2, \dot{x} = y^1, \dot{y} = y^2$ . The Finsler metric  $F(x, y, \dot{x}, \dot{y})$  is as follows

$$F(x, y, \dot{x}, \dot{y}) = \sqrt{a^2\dot{x}^2 + b^2\dot{y}^2} - h_1x\dot{x} - h_2y\dot{y}, \quad (1)$$

where all  $a, b, h_1, h_2$  are positive numbers.

According to Recipe in §2 of [10], various objects are obtained. However, in the following representation of them are on the geodesic  $c$  only.

**Geodesic:** straight line

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$$c_m(t) = (c^1(t), c^2(t)) = \left( \frac{1}{a\sqrt{1+m^2}}t, \frac{m}{b\sqrt{1+m^2}}t \right) \quad (2)$$

$$p = c_m(t_0), \quad q = c_m(t_1), \quad r = c_m(t),$$

where the equation of the above straight line is  $y = \frac{a}{b}mx$  on  $M$ .

**Linear parallel displacement:**  $\Pi_{c_m}(t)$

$$\Pi_{c_m}(t) = \begin{pmatrix} B_1^1 & B_2^1 \\ B_1^2 & B_2^2 \end{pmatrix}, \quad (3)$$

where

$$B_1^1 = -\frac{1}{(a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0 - (b^2h_1 + a^2h_2m^2)t)^{3/2}} \times \\ \left( a^2(h_2m^2(t+t_0)\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right. \\ \left. - b^2(\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} - (b^2h_1 + a^2h_2m^2)t) \right. \\ \left. + m^2\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right) \\ \left. + b^2h_1t_0\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} - (b^2h_1 + a^2h_2m^2)t \right),$$

$$B_2^1 = \frac{1}{(a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0 - (b^2h_1 + a^2h_2m^2)t)^{3/2}} \times \\ \left( abm(b^2(-\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right. \\ \left. + \sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} - (b^2h_1 + a^2h_2m^2)t) \right. \\ \left. + h_2(t\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right. \\ \left. + t_0\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right. \\ \left. - t_0\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} - (b^2h_1 + a^2h_2m^2)t \right),$$

$$B_1^2 = \frac{1}{(a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0 - (b^2h_1 + a^2h_2m^2)t)^{3/2}} \times \\ \left( abm(a^2(-\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right. \\ \left. + \sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} - (b^2h_1 + a^2h_2m^2)t) \right. \\ \left. + h_1(t\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right. \\ \left. + t_0\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right. \\ \left. - t_0\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} - (b^2h_1 + a^2h_2m^2)t \right),$$

$$B_2^2 = -\frac{1}{(a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0 - (b^2h_1 + a^2h_2m^2)t)^{3/2}} \times \\ \left( -a^2b^2(\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right. \\ \left. + m^2\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} - (b^2h_1 + a^2h_2m^2)t) \right. \\ \left. + b^2h_1(t+t_0)\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} \right. \\ \left. + a^2h_2m^2t_0\sqrt{a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0} - (b^2h_1 + a^2h_2m^2)t \right).$$

**Quantization:** We change each equation to forms having rational expressions. For new parameters  $k$  and  $t$ , they are as follows changing.

$$k^2 := a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0 \quad (4)$$

$$t^2 := a^2b^2(1+m^2) - (b^2h_1 + a^2h_2m^2)t_0 - (b^2h_1 + a^2h_2m^2)t. \quad (5)$$

Then as for new parameters  $k, t$ , we have components  $B_1^1, B_2^1, B_1^2, B_2^2$  of  $\Pi$

$$B_1^1 = \frac{1}{(h_1b^2 + a^2h_2m^2)t^3} \left( (h_1b^2 + a^2h_2m^2)(a^2 - h_1t_0)tb^2 + a^2h_2km^2t^2 \right. \\ \left. + a^2km^2((h_1b^2 + a^2h_2m^2)(b^2 - h_2t_0) - h_2k^2) \right),$$

$$B_2^1 = \frac{abm(k-t)(h_2k^2 + h_2tk - (h_1b^2 + a^2h_2m^2)(b^2 - h_2t_0))}{(h_1b^2 + a^2h_2m^2)t^3},$$

$$B_1^2 = \frac{abm(k-t)(-h_2m^2a^4 + h_1(h_2m^2t_0 - b^2)a^2 + h_1(h_1t_0b^2 + k(k+t)))}{(h_1b^2 + a^2h_2m^2)t^3},$$

$$B_2^2 = \frac{1}{(h_1b^2 + a^2h_2m^2)t^3} \left( h_2m^2(kb^2 + m^2(b^2 - h_2t_0)t)a^4 + b^2h_1(b^2(tm^2 + k) \right. \\ \left. - h_2m^2t_0(k+t))a^2 - b^2h_1k(h_1t_0b^2 + k^2 - t^2) \right).$$

Before obtaining the public key and secret key, we must give a regular matrix  $C(\tau)$  as the transformation of  $T_p(M)$  and splitting form of the energy of  $E(v_1)$ . In this paper, we have the same splitting of  $E(v_1)$  as [10].

**Key Generation:** According to [10], we state the outline of its key generation as below.

(1)  $c$ : a geodesic,  $p$ : start point,  $q$ : end point

(2)  $v = (v^1, v^2)$ : a plaintext (a vector in  $\mathcal{R}^2$ ),  $dv = (dv^1, dv^2)$ : a positive difference vector ( $dv^1 > 0, dv^2 > 0$ ),  $v_0 = v + dv$

(3)  $v_1 = C(\tau)v_0$

(4)  $v_2 = \Pi_c(t_2)v_1$

(5)  $E(v_2) = E(v_1) = E_0 + E_1 + E_2$ : a splitting of the energy of  $v_1$

(6)  $E(v_1) = \frac{E_0}{f_0}f_0 + \frac{E_1}{f_1v_0^1}f_1v_0^1 + \frac{E_2}{f_2v_0^2}f_2v_0^2$

(7)  $V_3 = \Pi_c(\tau) \begin{pmatrix} E_1 \\ f_1v_0^1 \end{pmatrix}, \begin{pmatrix} E_2 \\ f_2v_0^2 \end{pmatrix} = \begin{pmatrix} V_3^1 \\ V_3^2 \end{pmatrix}$

(8)  $(\frac{E_0}{f_0}, V_3^1, V_3^2)$ : a ciphertext

After all, we have a public key  $PK = (\frac{E_0}{f_0}, V_3^1, V_3^2)$ .

First of all, let  $(a, b, h_1, h_2, m, t_0, t_1) = (1, 1, 1, 1, 1, \frac{1}{2}, 1)$  be all parameters.

Case (I) Next, let  $C(\tau)$  be

$$C(\tau) = \begin{pmatrix} \tau & -1 \\ 1 & \tau \end{pmatrix}.$$

Then we have the secret key  $SK = \{(f_0, f_1, f_2), \Pi_c(\tau), E(v_1)\}$  is as follows.

$$(f_0, f_1, f_2) = (mh_1, at_0h_2, bt_1h_2^2) = (1, \frac{1}{2}, 1),$$

$$\Pi_c(\tau) = \begin{pmatrix} \frac{\tau+1}{2\tau^2} & -\frac{\tau-1}{2\tau^2} \\ -\frac{\tau-1}{2\tau^2} & \frac{\tau+1}{2\tau^2} \end{pmatrix},$$

$$E(v_1) = G(v_1, v_1) = \frac{1}{8}(3\tau^2 - 2\tau + 3)(v_0^1)^2 + \frac{1}{4}(1 - \tau^2)v_0^1v_0^2 \\ + \frac{1}{8}(3\tau^2 + 2\tau + 3)(v_0^2)^2$$

and  $(\frac{E_0}{f_0}, V_3^1, V_3^2)$  as follows.

$$\begin{aligned} \frac{E_0}{f_0} = & \frac{1}{64t_2^4} \left( (3(\tau-1)^2 t_2^6 - 8(\tau^2 + \tau - 2) t_2^5 - 2((\tau-14)\tau - 5) t_2^4 \right. \\ & + 16(\tau^2 + \tau - 2) t_2^3 + 4(\tau(11\tau - 10) + 17) t_2^2 + 24(\tau - 1)^2 \left. \right) (v_0^1)^2 \\ & + 2(v_0^1)(v_0^2) \left( (t_2(t_2(t_2(3t_2 + 4) - 14) - 8) + 20) t_2^2 + 12(t_2(2t_2^2 \right. \\ & + t_2 - 4) + 2) \tau t_2^2 - ((t_2(t_2(t_2(3t_2 + 4) - 14) - 8) + 20) t_2^2 + 24) \tau^2 + 24 \right) \\ & + (v_0^2)^2 \left( 3(\tau + 1)^2 t_2^6 + 8(2\tau^2 + \tau - 1) t_2^5 + 2(\tau(5\tau - 14) - 1) t_2^4 \right. \\ & \left. - 16(2\tau^2 + \tau - 1) t_2^3 + 4(\tau(17\tau + 10) + 11) t_2^2 + 24(\tau + 1)^2 \right), \end{aligned} \quad (6)$$

$$\begin{aligned} V_3^1 = & \frac{1}{64\tau^2 t_2^4 (v_0^1)(v_0^2)} \left( -2(\tau - 1) \left( (t_2^2 - 2) (t_2(t_2 + 2) - 4) t_2^2 + (t_2^6 - 4t_2^4 + 8) \tau^2 \right. \right. \\ & - 2(t_2 + 2) (t_2^2 - 2) (t_2^3 - t_2^2 + t_2 - 2) \tau + 8 \left. \right) (v_0^1)^3 + (v_0^1)^2 (v_0^2) \left( 3(\tau - 1)^2 \right. \\ & (\tau + 1) t_2^6 - 4(\tau - 1)(3\tau(\tau + 2) + 5) t_2^5 + 2(\tau(3(1 - 5\tau)\tau + 13) - 13) t_2^4 \\ & + 8(\tau - 1)(3\tau(\tau + 2) + 5) t_2^3 + 4(\tau(\tau(13\tau + 11) + 9) + 27) t_2^2 \\ & + 24(\tau - 1)^2 (\tau + 1) - 4(v_0^1)(v_0^2)^2 \left( 3(\tau - 1)(\tau + 1)^2 t_2^6 + (\tau + 1)(3(\tau - 5)\tau \right. \\ & - 4) t_2^5 + (\tau(14 - \tau(14\tau + 19)) + 15) t_2^4 - 2(\tau + 1)(3(\tau - 5)\tau - 4) t_2^3 \\ & + 2(\tau((\tau - 8)\tau - 18) - 3) t_2^2 + 24(\tau - 1)(\tau + 1)^2 \left. \right) + (v_0^2)^3 (\tau + 1) \\ & \left. (7(\tau + 1)^2 t_2^6 + 8(\tau + 1)(3\tau - 1) t_2^5 - 2(\tau(19\tau + 34) + 21) t_2^4 - 16(\tau + 1) \right. \\ & \left. (3\tau - 1) t_2^3 + 4(\tau(25\tau + 6) + 11) t_2^2 + 56(\tau + 1)^2 \right), \end{aligned} \quad (7)$$

$$\begin{aligned} V_3^2 = & \frac{1}{64\tau^2 t_2^4 (v_0^1)(v_0^2)} \left( 2(\tau + 1) \left( (t_2^2 - 2) (t_2(t_2 + 2) - 4) t_2^2 + (t_2^6 - 4t_2^4 + 8) \tau^2 \right. \right. \\ & - 2(t_2 + 2) (t_2^2 - 2) (t_2^3 - t_2^2 + t_2 - 2) \tau + 8 \left. \right) (v_0^1)^3 + (v_0^1)^2 (v_0^2) ((\tau - 1) \\ & (\tau(3\tau - 22) + 3) t_2^6 - 4(3\tau - 5)(\tau(\tau + 2) - 1) t_2^5 + 2(13 - 3\tau(\tau(5\tau \\ & - 19) + 21)) t_2^4 + 8(3\tau - 5)(\tau(\tau + 2) - 1) t_2^3 + 4(\tau(\tau(13\tau - 7) + 37) \\ & - 27) t_2^2 + 8(\tau - 1)(\tau(3\tau - 22) + 3) - 4(v_0^1)(v_0^2)^2 \left( (\tau + 1)(\tau(3\tau - 8) + 3) t_2^6 \right. \\ & + (\tau(\tau(3\tau - 22) + 11) + 4) t_2^5 + (\tau(7(3 - 2\tau)\tau + 24) - 15) t_2^4 - 2(\tau(\tau(3\tau \\ & - 22) + 11) + 4) t_2^3 + 2(\tau((\tau - 18)\tau + 12) + 3) t_2^2 + 8(\tau + 1)(\tau(3\tau - 8) + 3) \left. \right) \\ & + (v_0^2)^3 (\tau - 1) \left( 7(\tau + 1)^2 t_2^6 + 8(\tau + 1)(3\tau - 1) t_2^5 - 2(\tau(19\tau + 34) + 21) t_2^4 \right. \\ & \left. - 16(\tau + 1)(3\tau - 1) t_2^3 + 4(\tau(25\tau + 6) + 11) t_2^2 + 56(\tau + 1)^2 \right). \end{aligned} \quad (8)$$

Case (II) On the other hand, let  $C(\tau)$  be

$$C(\tau) = \begin{pmatrix} \tau & 1 \\ \tau - 1 & 1 \end{pmatrix}.$$

or

Then we have the same  $(f_0, f_1, f_2)$  and  $\Pi_c$ , however, different  $E(v_1)$  and  $PK$  as follows:

$$E(v_1) = G(v_1, v_1) = \frac{1}{8}(4\tau^2 - 4\tau + 3)(v_0^1)^2 + \frac{1}{2}(2\tau - 1)v_0^1 v_0^2 + \frac{1}{2}(v_0^2)^2 \left( \frac{1}{t} + 1 \right)^2 := a^2 b^2 (1 + m^2) - (b^2 h_1 + a^2 h_2 m^2) t_0 - (b^2 h_1 + a^2 h_2 m^2) t. \quad (10)$$

and  $PK = \left( \frac{E_0}{f_0}, V_3^1, V_3^2 \right)$  is

$$\begin{aligned} \frac{E_0}{f_0} = & \frac{1}{64t_2^4} \left( (v_0^1)^2 \left( t_2^2 (36\tau^2 (t_2^2 + 2) - 24\tau(t_2(t_2(t_2 + 2) - 2) + 4) + t_2(t_2(t_2(3t_2 \right. \right. \\ & + 16) + 10) - 32) + 68) + 24) - 24t_2^2 (v_0^1)(v_0^2) \left( -3\tau(t_2^2 + 2) \right. \\ & \left. + t_2(t_2(t_2 + 2) - 2) + 4) + 36(t_2^2 + 2) t_2^2 (v_0^2)^2 \right), \end{aligned}$$

$$\begin{aligned} V_3^1 = & \frac{1}{64\tau^2 t_2^4 (v_0^1)(v_0^2)} \left( 2(\tau - 1)(v_0^1)^3 \left( t_2^2 (4\tau^2 (t_2^2 - 3) + (2 - t_2^2) (t_2(t_2 + 2) - 4) \right. \right. \\ & + 2\tau(t_2((t_2 - 3)t_2 - 2) + 10)) - 8) + 12(\tau + 1) t_2^2 (t_2^2 - 10) (v_0^2)^3 + 8t_2^2 (v_0^1)(v_0^2)^2 \\ & (-30\tau^2 - 11\tau + 4(\tau + 1) t_2^3 + (3\tau(\tau + 1) - 2) t_2^2 - 8(\tau + 1) t_2 + 25) \\ & + (v_0^1)^2 (v_0^2) \left( -56(\tau + 1) - 7(\tau + 1) t_2^6 + 4(\tau(8\tau + 3) - 7) t_2^5 \right. \\ & \left. + 2(6\tau^3 + 10\tau^2 + \tau + 25) t_2^4 - 8(\tau(8\tau + 3) - 7) t_2^3 + 4(\tau(-30\tau^2 + 2\tau + 41) \right. \\ & \left. - 35) t_2^2 \right), \end{aligned}$$

$$\begin{aligned} V_3^2 = & \frac{1}{64\tau^2 t_2^4 (v_0^1)(v_0^2)} \left( -2(\tau + 1)(v_0^1)^3 \left( t_2^2 (4\tau^2 (t_2^2 - 3) + (2 - t_2^2) (t_2(t_2 + 2) - 4) \right. \right. \\ & + 2\tau(t_2((t_2 - 3)t_2 - 2) + 10)) - 8) - 12(\tau - 1) t_2^2 (t_2^2 - 10) (v_0^2)^3 - 8t_2^2 (v_0^1)(v_0^2)^2 \\ & ((49 - 30\tau)\tau + 4(\tau - 1) t_2^3 + (3(\tau - 1)\tau + 2) t_2^2 - 8\tau t_2 + 8t_2 - 25) \\ & + (v_0^1)^2 (v_0^2) \left( 56(\tau - 1) + 7(\tau - 1) t_2^6 - 4(\tau(8\tau - 13) + 7) t_2^5 + 2(\tau(-6\tau^2 + 2\tau \right. \\ & \left. - 25) + 25) t_2^4 + 8(\tau(8\tau - 13) + 7) t_2^3 + 4(\tau(30\tau^2 - 62\tau + 71) - 35) t_2^2 \right). \end{aligned}$$

We can take any regular matrix  $C(\tau)$ . Whenever applying different  $C(\tau)$ , we can obtain different  $E(v_1)$  and  $(\frac{E_0}{f_0}, V_3^1, V_3^2)$  even if the same parameters  $(a, b, h_1, h_2, m, t_0, t_1)$ . Then we have

**Proposition 2.1** In the Finsler encryption, even if the parameters  $(a, b, h_1, h_2, m, t_0, t_1)$ , the quantization and the splitting form of the energy  $E(v_1)$  are same, there exist different pairs  $(SK, PK)$  of the secret key  $SK$  and the public key  $PK$  depending on the transformation  $C(\tau)$ .

**Remark 2.1** (1) The value  $t_0$  must be chosen a rational number for  $k$  to be a rational number.

(2) The methods of quantization are many. For example

$$\begin{aligned} (t + 1)^2 := & a^2 b^2 (1 + m^2) - (b^2 h_1 + a^2 h_2 m^2) t_0 \\ & - (b^2 h_1 + a^2 h_2 m^2) t \end{aligned} \quad (9)$$

(3) If we change the method of quantization, then we have the arranging matrix  $\Pi_c(\tau)$  in addition.

## 2.2 PK

In this section, we state the mathematical structure of the public key  $PK$  of Finsler encryption.

First, in Case (I) of the previous section, we take  $(\tau, \beta_0, \beta_1, \beta_2) = (1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4})$ . Then we have the public key  $PK_{\tau, \beta_0, \beta_1, \beta_2}$  as follows

$$\begin{aligned}
 & \cdot (\tau = 1, t_2 = \beta_0 = \frac{1}{2}) \\
 pk_0 &= \{pk_{00}, pk_{01}, pk_{02}\} \\
 &= \left\{ \frac{324(v_0^1)^2 + 48(v_0^1)(v_0^2) + 2071(v_0^2)^2}{64}, \right. \\
 &\quad \left. - \frac{468(v_0^1)^2 + 232(v_0^1)(v_0^2) + 4003(v_0^2)^2}{32(v_0^1)}, \right. \\
 &\quad \left. \frac{44(v_0^1)^2 + 46(v_0^1)(v_0^2) + 499(v_0^2)^2}{16(v_0^2)} \right\}, \\
 & \cdot (\tau = 1, t_2 = \beta_1 = \frac{2}{3}) \\
 pk_1 &= \{pk_{10}, pk_{11}, pk_{12}\} \\
 &= \left\{ \frac{99(v_0^1)^2 + 10(v_0^1)(v_0^2) + 388(v_0^2)^2}{32}, \right. \\
 &\quad \left. - \frac{1161(v_0^1)^2 + 498(v_0^1)(v_0^2) + 5858(v_0^2)^2}{144(v_0^1)}, \right. \\
 &\quad \left. \frac{207(v_0^1)^2 + 204(v_0^1)(v_0^2) + 1327(v_0^2)^2}{144(v_0^2)} \right\}, \\
 & \cdot (\tau = 1, t_2 = \beta_2 = \frac{3}{4}) \\
 pk_2 &= \{pk_{20}, pk_{21}, pk_{22}\} \\
 &= \left\{ \frac{17712(v_0^1)^2 + 1872(v_0^1)(v_0^2) + 57209(v_0^2)^2}{6912}, \right. \\
 &\quad \left. - \frac{65232(v_0^1)^2 + 27360(v_0^1)(v_0^2) + 263423(v_0^2)^2}{10368(v_0^1)}, \right. \\
 &\quad \left. \frac{5616(v_0^1)^2 + 5436(v_0^1)(v_0^2) + 28133(v_0^2)^2}{5184(v_0^2)} \right\}.
 \end{aligned} \tag{11}$$

The public key  $PK_{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}} = (pk_0, pk_1, pk_2)$  is true.

We have the plaintext space  $\mathcal{M}$  is a set of lattice points of the first quadrant in  $\mathcal{R}^2$ . From  $PK_{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}}$ , we notice that ciphers are rational points in  $\mathcal{R}^9$  and the set of the ciphers is a certain subset of  $\mathcal{R}^9$ . Thus we have the mapping  $PK_{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}}$  from  $\mathcal{M}$  to  $\mathcal{R}^9$ . In general, for each  $(\tau, \beta_0, \beta_1, \beta_2)$ , we can regard the public key  $PK_{\tau, \beta_0, \beta_1, \beta_2}$  as the mapping from  $\mathcal{M}$  to  $\mathcal{R}^9$

$$\begin{aligned}
 PK_{\tau, \beta_0, \beta_1, \beta_2} : \mathcal{M} \subset \mathcal{R}^2 &\longrightarrow \mathcal{R}^9 \\
 | v_0 = (v_0^1, v_0^2) &\mapsto ct = PK_{\tau, \beta_0, \beta_1, \beta_2}(v_0).
 \end{aligned} \tag{12}$$

**Remark2.2**  $PK_{\tau, \beta_0, \beta_1, \beta_2}$  is the mapping from  $\mathcal{M}$  to  $\mathcal{R}^9$  as the public key, however, as a mapping, on from  $\mathcal{R}^2$  to  $\mathcal{R}^9$ , and excepting axes  $v_0^1 = 0$  and  $v_0^2 = 0$  it is differentiable with respect to  $v_0^1$  and  $v_0^2$ .

### 2.3 SK

First, we put a ciphertext  $ct = PK_{\tau, \beta_0, \beta_1, \beta_2}(v_0) = (ct_0, ct_1, ct_2)$ , where

$$\begin{aligned}
 ct_0 &= pk_0(v_0) = (ct_{00}, ct_{01}, ct_{02}), \\
 ct_1 &= pk_1(v_0) = (ct_{10}, ct_{11}, ct_{12}), \\
 ct_2 &= pk_2(v_0) = (ct_{20}, ct_{21}, ct_{22}).
 \end{aligned} \tag{13}$$

According to the algorithm of the decryption,  $(v_0^1, v_0^2)$  is the solution of the following simultaneous linear system of  $(X, Y)$

$$\begin{cases}
 \frac{1}{4}((-ct_{01} + ct_{02} + ct_{11} - ct_{12})\tau + (-ct_{01} - ct_{02} + ct_{11} + ct_{12})\tau^2)X \\
 + \frac{1}{2}((ct_{01} - ct_{02} - ct_{11} + ct_{12})\tau + (-ct_{01} - ct_{02} + ct_{11} + ct_{12})\tau^2)Y \\
 = ct_{00} - ct_{10} \\
 \frac{1}{4}((-ct_{01} + ct_{02} + ct_{21} - ct_{22})\tau + (-ct_{01} - ct_{02} + ct_{21} + ct_{22})\tau^2)X \\
 + \frac{1}{2}((ct_{01} - ct_{02} - ct_{21} + ct_{22})\tau + (-ct_{01} - ct_{02} + ct_{21} + ct_{22})\tau^2)Y \\
 = ct_{00} - ct_{20}.
 \end{cases} \tag{14}$$

If the following the determinant  $Det$  of the above system

$$Det = \frac{1}{2}(ct_{01}ct_{12} - ct_{01}ct_{22} + ct_{02}ct_{21} - ct_{02}ct_{11} + ct_{11}ct_{22} - ct_{12}ct_{21})\tau^3 \tag{15}$$

is non-zero, then the answer  $(X, Y)$ , namely,  $(v_0^1, v_0^2)$  is rewrote by

$$\begin{cases}
 v_0^1 = \frac{A - B\tau}{(ct_{01}ct_{12} - ct_{01}ct_{22} + ct_{02}ct_{21} - ct_{02}ct_{11} + ct_{11}ct_{22} - ct_{12}ct_{21})\tau^2} \\
 v_0^2 = \frac{A + B\tau}{2(ct_{01}ct_{12} - ct_{01}ct_{22} + ct_{02}ct_{21} - ct_{02}ct_{11} + ct_{11}ct_{22} - ct_{12}ct_{21})\tau^2},
 \end{cases} \tag{16}$$

where

$$\begin{aligned}
 A &= ct_{01}ct_{10} - ct_{02}ct_{10} - ct_{00}ct_{11} + ct_{00}ct_{12} - ct_{01}ct_{20} + ct_{02}ct_{20} \\
 &\quad + ct_{11}ct_{20} - ct_{12}ct_{20} + ct_{00}ct_{21} - ct_{10}ct_{21} - ct_{00}ct_{22} + ct_{10}ct_{22},
 \end{aligned}$$

$$\begin{aligned}
 B &= ct_{01}ct_{10} + ct_{02}ct_{10} - ct_{00}ct_{11} - ct_{00}ct_{12} - ct_{01}ct_{20} - ct_{02}ct_{20} \\
 &\quad + ct_{11}ct_{20} + ct_{12}ct_{20} + ct_{00}ct_{21} - ct_{10}ct_{21} + ct_{00}ct_{22} - ct_{10}ct_{22}.
 \end{aligned}$$

**Proposition2.2** If  $(ct_{00}, ct_{01}, ct_{02}, \dots, ct_{20}, ct_{21}, ct_{22})$  is a ciphertext  $ct$  of a plaintext  $v_0 = (v_0^1, v_0^2)$  and  $Det$  of (15) is non-zero, then the simultaneous linear system (14) has unique solution  $(X, Y)$  and it is the plaintext  $v_0 = (v_0^1, v_0^2)$ , namely,

$$(X, Y) = (v_0^1, v_0^2)$$

is satisfied. (**Decryption of Finsler encryption**).

In general, from a point  $(ct_{00}, ct_{01}, ct_{02}, \dots, ct_{20}, ct_{21}, ct_{22}) \in \mathcal{R}^9$  and a value  $\tau (\neq 0)$ , we take the following matrix

$$ppk_\tau = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}, \tag{17}$$

where

$$\pi_{11} = \frac{1}{4}((-ct_{01} + ct_{02} + ct_{11} - ct_{12})\tau + (-ct_{01} - ct_{02} + ct_{11} + ct_{12})\tau^2)$$

$$\pi_{12} = \frac{1}{2}((ct_{01} - ct_{02} - ct_{11} + ct_{12})\tau + (-ct_{01} - ct_{02} + ct_{11} + ct_{12})\tau^2)$$

$$\pi_{21} = \frac{1}{4}((-ct_{01} + ct_{02} + ct_{21} - ct_{22})\tau + (-ct_{01} - ct_{02} + ct_{21} + ct_{22})\tau^2)$$

$$\pi_{22} = \frac{1}{2}((ct_{01} - ct_{02} - ct_{21} + ct_{22})\tau + (-ct_{01} - ct_{02} + ct_{21} + ct_{22})\tau^2)$$

and can consider a linear mapping  $ppk_\tau$  as following

$$ppk_\tau : \mathcal{R}^2 \longrightarrow \mathcal{R}^2 | v_0 = (v_0^1, v_0^2) \in \mathcal{R}^2 \mapsto ppk_\tau(v_0) \in \mathcal{R}^2. \tag{18}$$

Under the determinant  $Det(ppk_\tau) \neq 0$ , we can regard

the linear mapping  $ppk_\tau$  as the mapping  $PK_{\tau,\beta_0,\beta_1,\beta_2}$ .

( $\therefore$ ) We consider a projection  $pr$  from  $\mathcal{R}^9$  to  $\mathcal{R}^2$  as follows

$$pr : (ct_{00}, ct_{01}, ct_{02}, ct_{10}, ct_{11}, ct_{12}, ct_{20}, ct_{21}, ct_{22}) \\ \mapsto (ct_{00} - ct_{10}, ct_{00} - ct_{20}).$$

Then, for a ciphertext  $ct$ , we can commute  $ppk_\tau$  into  $pr \circ PK_{\tau,\beta_0,\beta_1,\beta_2}$ , namely,

$$ppk_\tau = pr \circ PK_{\tau,\beta_0,\beta_1,\beta_2} \quad (19)$$

by using the value  $\tau$  decided by the energy  $E(v_1)$ (See Appendix).

The projection  $pr$  is not injection, in general. Even if  $PK_{\tau,\beta_0,\beta_1,\beta_2}$  is injective,  $ppk_\tau$  is not necessarily injective. However, if  $Det(ppk_\tau) \neq 0$ , then  $ppk_\tau$  is regular, namely,

$$ppk_\tau^{-1}(pr(ct)) = ppk_\tau^{-1}(pr(PK_{\tau,\beta_0,\beta_1,\beta_2}(v_0))) \\ = (ppk_\tau^{-1} \circ pr \circ PK_{\tau,\beta_0,\beta_1,\beta_2})(v_0) = v_0 \quad ((\therefore) \text{ Proposition 2.2})$$

is satisfied, where  $pr(ct) = (ct_{00} - ct_{10}, ct_{00} - ct_{20})$ . Thus, we have  $PK_{\tau,\beta_0,\beta_1,\beta_2}^{-1} = ppk_\tau^{-1} \circ pr$ , namely, the secret key  $SK_{\tau,\beta_0,\beta_1,\beta_2}$

$$SK_{\tau,\beta_0,\beta_1,\beta_2} = ppk_\tau^{-1} \circ pr \quad (20)$$

satisfies.  $\square$

Then we have

**Proposition 2.3** If the linear mapping  $ppk_\tau$  of (18) is regular, for a ciphertext  $ct$ , the secret key  $SK_{\tau,\beta_0,\beta_1,\beta_2}$  satisfies the relation (20). Thus  $SK_{\tau,\beta_0,\beta_1,\beta_2}$  is a mapping from  $\mathcal{R}^9$  to  $\mathcal{R}^2$ .

**Remark 2.3** (1) The general form of  $ppk_\tau$  depends on  $\Pi$  and  $(f_0, f_1, f_2)$ .

(2) The equation of  $\tau$  obtained by  $E(v_1)$  is a polynomial equation of a certain degree.

(3) The general form of  $\Pi$  and the degree of  $E(v_1)$  with respect to  $\tau$  depend on the quantization of the parameter  $t$ .

(4) The general form of  $E(v_1)$  depends on the transformation  $C(\tau)$ .

(5) When  $(\tau, \beta_0, \beta_1, \beta_2) = (4, 1, 2, 3)$ ,

$$Det(ppk_4) = \frac{1}{2654208v_0^1v_0^2} (3v_0^1 - 5v_0^2)(497853(v_0^1)^3 - 1335798(v_0^1)^2(v_0^2) \\ + 1552365(v_0^1)(v_0^2)^2 - 415172(v_0^2)^3)$$

Thus the plaintext  $v_0 = (v_0^1, v_0^2)$  that satisfies  $3v_0^1 = 5v_0^2$  has  $Det(ppk_4) = 0$ . There is the ciphertext  $ct = PK_{4,1,2,3}(v_0)$ , however, the plaintext  $v_0$  is not obtained by the linear mapping  $ppk_4$ . Indeed, for any  $(\tau, \beta_0, \beta_1, \beta_2)$ , the mapping  $ppk_\tau$  always has non-regular plaintext  $v_0$ , where such  $v_0$  is not necessary in the first quadrant of  $\mathcal{R}^2$ , in general.

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## 付 録

We exhibit the calculation that obtain the value of  $\tau$  under the following data.

$$\begin{aligned}(\tau, \beta_0, \beta_1, \beta_2) &= \left(1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right), \\ v_0 &= (1516, 7084), \\ ct &= (ct_0, ct_1, ct_2),\end{aligned}$$

where

$$\begin{aligned}ct_0 &= (ct_{00}, ct_{01}, ct_{02}) \\ &= \left(\frac{6574327027}{4}, \frac{-12778117799}{3032}, \frac{145661807}{644}\right)\end{aligned}$$

$$\begin{aligned}ct_1 &= (ct_{10}, ct_{11}, ct_{12}) \\ &= \left(\frac{1237871657}{2}, \frac{-18874300661}{13644}, \frac{94102555}{1386}\right)\end{aligned}$$

$$\begin{aligned}ct_2 &= (ct_{20}, ct_{21}, ct_{22}) \\ &= \left(\frac{183233325809}{432}, \frac{-853944965495}{982368}, \frac{92692874633}{2295216}\right)\end{aligned}$$

### Start!

Now, from (15), by using  $ct$

$$\text{Det}(ppk_\tau) = \frac{5140375407580567\tau^3}{2947104} \neq 0 \text{ (because of } \tau \neq 0)$$

From (16), by using  $ct$ , we have a formal solution  $(v_0^1, v_0^1)$  as follow

$$\begin{cases} v_0^1 = \frac{-2(-3921 + 3163\tau)}{\tau^2} \\ v_0^2 = \frac{3921 + 3163\tau}{\tau^2}. \end{cases}$$

Input  $(v_0^1, v_0^2)$  to the following energy equation(\*).

$$\begin{aligned}&\frac{1}{8}(3\tau^2 - 2\tau + 3)(v_0^1)^2 + \frac{1}{4}(1 - \tau^2)v_0^1v_0^2 + \frac{1}{8}(3\tau^2 + 2\tau + 3)(v_0^2)^2 \\ &= \frac{1}{1952608}(3209270886884104 - 2167688615345v_0^1\tau \\ &+ 4335377230690v_0^2\tau - 1946865315933v_0^1\tau^2 - 3893730631866v_0^2\tau^2).\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{1}{8\tau^4}(-292110579 + 315483660\tau - 527209370\tau^2 + 283265628\tau^3 \\ &+ 220570661\tau^4) = 0.\end{aligned}$$

By solving this equation, we obtain  $\tau = 1$  because that  $\tau$  is a rational number.  $\square$

### (\* About the energy equation.

From  $ct_0$ , we have

$$\begin{aligned}\Pi_c(\tau)^{-1} \begin{pmatrix} ct_{01} \\ ct_{02} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}\tau(1 + \tau) & \frac{1}{2}\tau(-1 + \tau) \\ \frac{1}{2}\tau(-1 + \tau) & \frac{1}{2}\tau(1 + \tau) \end{pmatrix} \begin{pmatrix} ct_{01} \\ ct_{02} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\tau(1 + \tau)ct_{01} + \frac{1}{2}\tau(-1 + \tau)ct_{02} \\ \frac{1}{2}\tau(-1 + \tau)ct_{01} + \frac{1}{2}\tau(1 + \tau)ct_{02} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{12778117799\tau(1+\tau)}{6064} + \frac{145661807\tau(-1+\tau)}{1288} \\ -\frac{12778117799\tau(-1+\tau)}{6064} + \frac{145661807\tau(1+\tau)}{1288} \end{pmatrix}\end{aligned}$$

Thus, we have the energy equation as follows, by using the first component  $ct_{00}$  and above 2-dimensional vector,

$$\begin{aligned}E(v_1) &= \left(\frac{6574327027}{4}, -\frac{12778117799\tau(1 + \tau)}{6064} \right. \\ &\quad \left. + \frac{145661807\tau(-1 + \tau)}{1288}, \right. \\ &\quad \left. -\frac{12778117799\tau(-1 + \tau)}{6064} + \frac{145661807\tau(1 + \tau)}{1288}\right) \begin{pmatrix} 1 \\ \frac{1}{2}v_0^1 \\ v_0^2 \end{pmatrix}.\end{aligned}$$

This equation is

$$\begin{aligned}&\frac{1}{8}(3\tau^2 - 2\tau + 3)(v_0^1)^2 + \frac{1}{4}(1 - \tau^2)v_0^1v_0^2 + \frac{1}{8}(3\tau^2 + 2\tau + 3)(v_0^2)^2 \\ &= \frac{1}{1952608}(3209270886884104 - 2167688615345v_0^1\tau \\ &+ 4335377230690v_0^2\tau - 1946865315933v_0^1\tau^2 - 3893730631866v_0^2\tau^2).\end{aligned}$$

In addition, from  $ct_1$  and  $ct_2$ , we also have the same equation.