Reformist Envy-Free Item Allocations: Algorithms and Complexity

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Abstract: We introduce the concept of *reformist envy-free item allocations* when each agent is assigned a single item. Given an envy-free item allocation, we consider an operation to exchange the item of an agent with an unassigned item preferred by the agent that results in another envy-free item allocation. We repeat this operation as long as we can. Then, the resulting allocation is called a reformist envy-free item allocation. We prove that a reformist envy-free item allocation uniquely exists modulo the choice of an initial envy-free item allocation, and can be found in polynomial time. Therefore, we study a shortest sequence to obtain the reformist envy-free item allocation from an initial envy-free item allocation. We prove that the computation of a shortest sequence is computationally hard. On the other hand, we give polynomial-time algorithms for some special cases.

Keywords: item allocation, envy-freeness, combinatorial reconfiguration

1. Introduction

In this paper, we consider the problem of allocating indivisible items to agents that have preferences over acceptable items. Especially, we consider the situation where each agent is assigned a single item. The resulting item allocation is evaluated based on the preferences. This problem is often called the house allocation problem (see, e.g., [1]). Several desirable properties for item allocations have been proposed. For example, Pareto optimality (see, e.g., [2]) is one of the most fundamental properties of item allocations. This property guarantees that we cannot make the situation of any agent better without making that of another agent worse. In this paper, we focus on the property called envy-freeness (see, e.g., [3]). This property guarantees that any agent does not have envy for the other agents on the current item allocation. Envy-freeness of an item allocation has been studied from the algorithmic viewpoint. For example, Gan, Suksompong, and Voudouris [3] considered the problem of checking the existence of an envy-free item allocation in the situation where any agent accepts all the items and the preferences may contain ties. They proved that we can determine whether there exists an envy-free item allocation in polynomial time. Furthermore, Beynier et al. [4] considered envy-freeness on an envy relationship network.

It is not difficult to see that in the house allocation problem, there may exist multiple envy-free item allocations. Thus, even if we are given some envy-free item allocation, the current envy-free item allocation may not be satisfying. Thus, in this paper, we consider the problem of improving the current envy-free item allocation via exchanging items under the condition that we keep the current item allocation envy-free. Here we have to carefully choose the definition of an exchanging operation. This is because it is not realistic for a large number of agents to simultaneously exchange the items. Thus, we should consider more local exchange operations. More concretely, we consider an operation to exchange the item of an agent with an unassigned item preferred by the agent that results in another envy-free item allocation. We repeat this operation as long as we can. Then, we call the resulting allocation a *reformist envy-free item allocation*. We first prove that a reformist envy-free item allocation uniquely exists modulo the choice of an initial envy-free item allocation, and can be found in polynomial time. Therefore, we study a shortest sequence to obtain the reformist envy-free item allocation from an initial envy-free item allocation. We call a sequence to obtain

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the reformist envy-free item allocation a *reformist sequence*, and we call the problem of finding a shortest reformist sequence the *shortest reformist sequence problem.* We prove that the shortest reformist sequence problem is computationally hard. On the other hand, we give polynomial-time algorithms for some special cases.

Our contribution is summarized as follows. We first prove that a reformist envy-free item allocation uniquely exists modulo the choice of an initial envy-free item allocation, and can be found in polynomial time. Then we consider the shortest reformist sequence problem. We define the decision version of the shortest reformist sequence problem as the problem in which we are given a positive integer ℓ and we determine whether there is a reformist sequence of length at most ℓ. In what follows, we denote by n (resp., m) the number of agents (resp., items). Furthermore, for each agent i, we denote by m_i the number of items acceptable to i.

- There is an instance such that the ratio of the output of a simple greedy algorithm to an optimal solution can get worse to any extent.
- The decision version of the shortest reformist sequence problem is NP-complete even for the case when $m_i \leq 4$ for every agent $i \in N$ and each item is acceptable to at most three agents.
- If $m_i \leq 3$ for every agent i, then the shortest reformist sequence problem can be solved in polynomial time.
- If every item is acceptable to at most two agents, then the shortest reformist sequence problem can be solved in polynomial time.
- The shortest reformist sequence problem is inapproximable in polynomial time within a factor of $c \ln n$ for some constant c unless P = NP.
- It is W[1]-hard to determine whether there exists a reformist sequence of length at most $n + k$ when k is a parameter.
- The shortest reformist sequence problem parameterized by $m - 2n$ is fixed-parameter tractable.

Problems of improving a given item allocation via some operations have been considered in the study of item allocations. For example, Gourvés, Lesca, and Wilczynski [5] considered the problem of determining whether a target item allocation can be reached via rational swaps on a social network. Furthermore, they considered that the problem of determining whether some specified agent can get a target item via rational swaps (see also [6], [7]).

Furthermore, the shortest reformist sequence problem is closely related to the study of *combinatorial reconfiguration.* In combinatorial reconfiguration, we consider problems where we are given an initial configuration and a target configuration of some combinatorial objects, and the goal is to check the reachability between these two configurations via some specified operations. The study of algorithmic aspects of combinatorial reconfiguration was initiated in [8]. See, e.g., [9] for a survey of combinatorial reconfiguration.

2. Preliminaries

Throughout this paper, a finite set of n *agents* is denoted by N, and a finite set of m *items* is denoted by M. Each agent $i \in N$ is associated with a subset $M_i \subseteq M$ and a strict total order $\succ_i$ on M_i: M_i represents the set of acceptable items for i, and $\succ_i$ represents the preference of i over M_i. For each agent $i \in N$, we define $m_i := |M_i|$. For each agent $i \in N$ and each pair $x, y \in M$ of items, we write $x \succeq_i y$ if $x \succ_i y$ or $x = y$. Note that $\succ_i$ satisfies transitivity, i.e., if $x \succ_i y$ and $y \succ_i z$, then $x \succ_i z$.

An injective mapping $\mu \colon N \to M$ is called a *matching* if $\mu(i) \in M_i$ for every agent $i \in N$. For each matching μ, an item $x \in M$ is *assigned* if there exists an agent $i \in N$ such that $\mu(i) = x$; otherwise x is *unassigned.* A matching μ is *envy-free* if there exists no pair $i, j \in N$ of distinct agents such that $\mu(j) \in M_i$ and $\mu(j) \succ_i \mu(i)$. For each matching μ, we denote the set of unassigned items for μ by $\overline{M}_\mu$.

For two envy-free matchings μ, σ, we write $\mu \rightsquigarrow \sigma$ if there exists an agent $i \in N$ with the following two conditions:

(E1) $\sigma(i) \succ_i \mu(i)$;

(E2) $\mu(j) = \sigma(j)$ for every agent $j \in N \setminus \{i\}$.

Intuitively, if items are assigned to the agents according to μ and $\mu \rightsquigarrow \sigma$, then $\sigma(i) \in \overline{M}_\mu$ and i has an incentive to exchange her item $\mu(i)$ with $\sigma(i)$ and the resulting matching is still envy-free. This way, the operation "$\rightsquigarrow$" unilaterally improves the current envy-free matching μ to a new envy-free matching σ.

Let μ, σ be envy-free matchings. If there exist envy-free matchings $\mu_0, \mu_1, \dots, \mu_\ell$ such that

- $\mu_0 = \mu$, $\mu_\ell = \sigma$,
- $\mu_t \rightsquigarrow \mu_{t+1}$ for each $t \in \{0, 1, \dots, \ell - 1\}$, and
- there exists no envy-free matching μ' such that $\mu_\ell \rightsquigarrow \mu'$,

then σ is called a *reformist envy-free matching* with respect to μ. Intuitively, a reformist envy-free matching with respect to μ is an envy-free matching that is obtained from μ as an outcome of the iterative improvement.

3. Existence and Uniqueness

Here we prove the existence and uniqueness of a reformist envy-free matching with respect to a given envy-free matching.

Theorem 1. *Let μ be an envy-free matching. A reformist envy-free matching with respect to μ uniquely exists.*

Proof. The existence is immediate from the definition. We prove the uniqueness. Suppose to the contrary that there exist reformist envy-free matchings σ, τ with respect to μ such that $\sigma \neq \tau$. Without loss of generality, we can assume that there exists an agent $i \in N$ such that $\sigma(i) \succ_i \tau(i)$. Suppose that

$$\mu = \sigma_0 \rightsquigarrow \sigma_1 \rightsquigarrow \sigma_2 \rightsquigarrow \cdots \rightsquigarrow \sigma_\ell = \sigma.$$

Since τ is a reformist envy-free matching with respect to μ, we have $\tau(j) \succeq_j \sigma_0(j)$ for every agent $j \in N$. Let t be the minimum integer in $\{1, 2, \dots, \ell\}$ such that $\sigma_t(i) \succ_i \tau(i)$ for some agent $i \in N$. Then it holds that $\tau(j) \succeq_j \sigma_t(j)$ for every agent $j \in N \setminus \{i\}$.

If there exists an agent $j \in N \setminus \{i\}$ such that $\tau(j) = \sigma_t(i)$, then $\tau(j) \succ_i \tau(i)$, which contradicts that τ is envy-free. Thus, $\tau(j) \neq \sigma_t(i)$ holds for every agent $j \in N \setminus \{i\}$. Hence, under the matching τ, the agent i can exchange $\tau(i)$ with $\sigma_t(i)$. Since τ is a reformist envy-free matching, the resulting matching, denoted by τ', is not envy-free. That is, there exists an agent $j \in N \setminus \{i\}$ such that $\tau'(i) \succ_j \tau(j)$. For such an agent $j \in N \setminus \{i\}$, we have

$$\sigma_t(i) = \tau'(i) \succ_j \tau(j) \succeq_j \sigma_t(j).$$

However, this means that the agent j has envy for i on σ_t, which contradicts the fact that σ_t is envy-free. This completes the proof. $\square$

4. Shortest Reformist Sequence

The definition of a reformist envy-free matching gives a decentralized algorithm. Namely, given an envy-free matching, while there exists an agent who has an item among the unassigned items that she prefers to the currently assigned item, she exchanges the items as long as the exchange does not violate envy-freeness. This process eventually terminates, and the obtained envy-free matching is a reformist envy-free matching, which is unique by Theorem 1. However, the number of steps in this process is not discussed yet.

With a decentralized algorithm, we may end up with an extremely long sequence of envy-free matchings until we obtain a reformist envy-free matching. On the other hand, if there is coordination among the agents, they may quickly obtain a reformist envy-free matching. Coordination is modeled as a centralized algorithm in which a central authority declares who should exchange an item next, and agents obey the declarations of the central authority. Since a reformist envy-free matching is unique (Theorem 1), there is no reason for agents to deviate from the orders of the central authority.

To formalize the discussion, we consider the following type of algorithms. Until a reformist envy-free matching is obtained, an agent is nominated at each step. Let i be the nominated agent. Then, i exchanges the currently assigned item with an unassigned item that is most preferred by i such that after exchange the resulting matching is still envy-free. Namely, if the current envy-free matching is μ, then we define the envy-free matching μ' that satisfies $\mu \rightsquigarrow \mu'$ and $\mu'(i) \succeq_i x$ for all $x \in \overline{M}_\mu$ such that exchanging $\mu(i)$ with x yields an envy-free matching. Note that exchanging $\mu(i)$ with some item $x' \prec_i \mu'(i)$ is a redundant step, i.e., exchanging $\mu(i)$ with x' can be replaced with $\mu \rightsquigarrow \mu'$ without increasing the number of steps.

The choice of nominated agents can change the number of steps. In the decentralized setting the choice will be done arbitrarily while in the centralized setting the choice is supposed to be done cleverly to minimize the number of steps. Thus, we examine the minimum number of steps to obtain a reformist envy-free matching with respect to a given envy-free matching.

In what follows, we call a sequence of exchanges to obtain the reformist envy-free matching a *reformist sequence*, and we call the problem of finding a shortest reformist sequence the *shortest reformist sequence problem*. Furthermore, we define the decision version of the shortest reformist sequence problem as the problem in which we are given a positive integer ℓ and we determine whether there is a reformist sequence of length at most ℓ.

We first show that coordination sometimes makes sense by giving an example in which the maximum number of steps can be arbitrarily larger than the minimum number of steps. On the other hand, we later prove that the minimum number of steps is hard to compute.

Theorem 2. *For any positive integer p, there exists an instance of the shortest reformist sequence problem with 3 agents and $2p+3$ items such that a decentralized algorithm may take $2p-1$ steps to obtain a reformist matching while the optimal reformist sequence has length 4.*

5. Results

First, we state our hardness results on the shortest reformist sequence problem. We prove the NP-completeness of the decision version of the shortest reformist sequence problem by reduction from the vertex cover problem.

Theorem 3. *The decision version of the shortest reformist sequence problem is NP-complete even when $m_i \leq 4$ for every agent $i \in N$ and $|\{i \in N \mid x \in M_i\}| \leq 3$ for every item $x \in M$.*

Then we prove the W[1]-hardness of the decision version of the shortest reformist sequence problem by reduction from the multi-colored clique problem.

Theorem 4. *It is W[1]-hard to decide whether there exists a reformist sequence of length at most $n + k$ when k is a parameter.*

We also prove the inapproximability of the shortest reformist sequence problem by reduction from the set cover problem.

Theorem 5. *The shortest reformist sequence problem is inapproximable in polynomial time within a factor of $c \ln n$ for some constant $c > 0$, unless P = NP.*

Next, we state our positive results for some special cases. The following positive results can be considered as the complement to Theorem 3.

Theorem 6. *If $m_i \leq 3$ for every agent $i \in N$, then the shortest reformist sequence problem can be solved in polynomial time.*

Theorem 7. *If $|\{i \in N \mid x \in M_i\}| \leq 2$ for every item $x \in M$, then the shortest reformist sequence problem can be solved in polynomial time.*

Finally, we prove a fixed-parameter algorithm for the shortest reformist sequence problem.

Theorem 8. *The shortest reformist sequence problem parameterized by $m - 2n$ is fixed-parameter tractable.*

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