# Parameterized Algorithms for Tutte Polynomial Specializations in Graph Orientations

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**Abstract:** The Tutte polynomial is a graph invariant polynomial $T(X, Y)$ which encodes a broad-spectrum of combinatorial information about a graph. In particular, the number of acyclic orientations and the number of totally cyclic orientations of a graph is given by the evaluations $T(2, 0)$ and $T(0, 2)$, respectively. In this paper, we provide algorithms based on binary decision diagrams to enumerate these orientations (and thus count $T(2, 0)$ and $T(0, 2)$), whose runtime is given by $O^*(2^{(\mathrm{pw}^2/4)+o(\mathrm{pw}^2)})$, where pw denotes the pathwidth of the graph. When viewed as matroids, it is well-known that acyclic and totally cyclic orientations are dual (while when viewed as graphs they can only be said to be viewed as dual in the planar case). The similarity of our algorithms for both orientations suggests a connection between them when viewed in light of their matroid counterparts and their evaluation by Tutte polynomials.

**Keywords:** Acyclic Orientation, Totally Cyclic Orientation, Parameterized Algorithm, FPT Algorithm, Pathwidth, Binary Decision Diagram

## 1. Introduction

An orientation of the edges of a graph $G$ is said to be acyclic if none of its edges is contained in a directed cycle, and is said to be totally cyclic if all of its edges are contained in at least one directed cycle. When $G$ is planar, these two orientations are dual in the sense that acyclic orientation can be transformed in a totally cyclic orientation by by orienting each dual edge by a 90 clockwise turn from the corresponding primal edge, and vice-versa.

These two types of orientation are known to be related to several combinatorial objects. For example, if we denote the chromatic polynomial of $G$ by $\chi_G$ (i.e. $\chi_G(k)$ counts the number of $k$-colorings of $G$), then there exists a non-trivial combinatorial interpretation of $(-1)^n\chi_G(-\lambda)$, where $\lambda > 0$, and, in particular, $(-1)^n\chi_G(-1)$ equals the number of acyclic orientations of $G$ [19]. On the other hand, if we denote the flow polynomial of $G$ by $\chi^*_G$ (i.e. $\chi^*_G(k)$ counts the number of nowhere-zero $k$-flows of $G$), then it is known that $\chi^*_G(-1)$ equals the number of totally cyclic orientations of $G$ [10]. For a given vertex $v$ of $G$, the linear coefficient of $\chi_G$ equals $(-1)^{n+1}$ times the number of acyclic orientations of $G$ such that its only sink is the vertex $v$ [4]. Any graph has at least as many acyclic orientations as Hamiltonian paths, which implies that being able to compute the number of acyclic orientations can be useful for establishing lower bounds on the computational complexity of determining whether vertex-coloring of a graph is valid and other related decision problems [12]. Furthermore, these results have been extended to hyperplane arrangements [4], point configurations and polytopes [8], among other interpretations related to matroids.

Acyclic and totally cyclic orientations are also related to each other and to other combinatorial entities through the Tutte polynomial, $T_G(X, Y)$, a bivariate generalization of the chromatic and flow polynomials. In particular, they can be computed by evaluating $T_G(2, 0)$ [19] and $T_G(0, 2)$ [10], respectively. For $(x, y) \in \mathbb{C}^2$, it is known that evaluating $T_G(x, y)$ exactly is #P-hard unless $(x, y)$ is an element of $\{(1, 1), (-1, -1), (0, -1), (-1, 0), (i, -i), (-i, i), (j, j^2), (j^2, j)\}$, where $j = \exp(2\pi i/3)$. Several fixed-parameter tractable (FPT) algorithms have been developed with the aim of evaluating $T_G(x, y)$ for a given $(x, y)$, where the fastest ones have time complexity $O^*(2^{\mathrm{tw}^2+o(\mathrm{tw}^2)})$ [14] with respect to the treewidth tw of the graph, and $O^*(2^{\mathrm{tw}\log\mathrm{tw}})$ [18] with respect to the pathwidth pw of the graph, respectively. Here, the notation $O^*$ ignores polynomial terms in $n$.

The Merino-Welsh conjecture [13], originally stated in 1999, corroborates the importance of the relationship between acyclic and totally cyclic orientations. This conjecture states that, for any bridgeless, loopless graph, both the number of acyclic and the number of totally cyclic orientations is bigger than the number of spanning trees. In terms of the Tutte polynomial, it can also be stated as $\max\{T_G(2, 0), T_G(0, 2)\} \geq T_G(1, 1)$. Since the Tutte polynomial is also defined for matroids, the conjecture can naturally be extended to matroids. It can also be further extended by considering the convexity of the Tutte polynomial: the convex version of the Merino-Welsh conjecture states that for any matroid $\mathcal{M}$, $T_M(x, 2 - x)$ is convex for $x \in [0, 2]$. Some partial results of the conjecture have been recently obtained: it holds for (i) $n$-vertex graphs with either at most $16n/15$ edges or at least $4n$ edges [20], (ii) 3-connected graphs with some lower bound on degree [11], and (iii) sereies-parallel graphs [15]. The matroid version has been affirmatively settled for a class of coloopless paving

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matroids [3]. Note that, while it is believed (yet not proved) that, asymptotically speaking, almost all matroids are paving, almost all graphs (or, graphic matroids in more precise terms) do not fall into the class of paving matroids.

The remaining part of this paper is structured as follows: in Sec. 2, we provide the necessary background related to graph and matroid orientations, and give a brief introduction to the Tutte polynomials and binary decision diagrams (BDD). In Sec. 3, we describe our BDD-based algorithms to enumerate acyclic and totally cyclic orientations. In Sec. 4, we prove that our algorithms enumerate the enumerations correctly and give a bound of $O^*(2^{\mathrm{pw}^2/4+o(\mathrm{pw}^2)})$ on the number of nodes of the BDD.

## 2. Preliminaries

If $n$ is a positive integer, then $[n]$ denotes the set $\{1, 2, \ldots, n\}$. If $S$ is a given set, $2^S$ denotes its power set, i.e. the set of all subsets of $S$.

### 2.1 Graph Orientations

Given a graph $G$, we denote its set of vertices and its set of edges by $V(G)$ and $E(G)$, respectively. Furthermore, we write $n(G) := |V(G)|$ and $m(G) := |E(G)|$. When the graph $G$ is clear from the context, we may simply write $V$, $E$, $n$ and $m$.

For each edge $e = \{u, v\} \in E(G)$, where $u, v \in V(G)$, we can *orient e*, obtaining either the oriented edge $(u, v)$ or the oriented edge $(v, u)$. Any directed graph $D$ obtained by orienting all edges $e \in E(G)$ is said to be an *orientation* of $G$. Since each edge of $G$ can be oriented in two ways, there exist $2^m$ orientations of $G$.

Let $A \in 2^{E(G)}$. A *partial orientation P* of $G$ is an edge-disjoint union of an orientation of the subgraph induced by $A$ and the subgraph induced by $E \setminus A$. Observe that there exist $2^{m+1} - 1$ partial orientations of $G$. We will, in general, denote the set of oriented edges of $P$ by $A(P)$. An *acyclic orientation* is an orientation in which no edge belongs to a directed cycle. A *totally cyclic orientation* is an orientation in which each edge belongs to at least one directed cycle.

### 2.2 Matroid Orientations

The concepts of orientation, acyclic orientation and totally cyclic orientation can be extended to oriented matroids. We provide a simplified introduction to these concepts below. For a more detailed account, we refer the reader to [9].

A *matroid* $\mathcal{M} = (E, \rho)$ is a finite set $E$ provided with a *rank function* $\rho : 2^E \longrightarrow \mathbb{N}$ satisfying:
(1) $0 \le \rho(S) \le |S|$ for each $S \in 2^E$.
(2) $\rho(S_1) \le \rho(S_2)$ for each $S_1, S_2 \in 2^E$ where $S_1 \subseteq S_2$.
(3) $\rho(S_1)+\rho(S_2) \ge \rho(S_1 \cup S_2)+\rho(S_1 \cap S_2)$ for each $S_1, S_2 \in 2^E$.
We say that a subset $S$ of $E$ is *independent* if $\rho(S) = |S|$. A *circuit* is a minimal non-null, non-independent set. The *dual matroid* of $\mathcal{M}$, denoted by $\mathcal{M}^* = (E, \rho^*)$, is the matroid defined on $E$ where

$$\rho^*(S) = |S| + \rho(E \setminus S) - \rho(E), \text{ for each } S \in 2^E.$$

It can be easily verified that the dual matroid satisfies three conditions given above. We then say that a subset $S$ of $E$ is a *cocircuit* if it is a circuit in the dual. Here we note that circuits in matroids correspond to cycles in graphs.

Graphs and matroids are related by the the following fact: each graph $G$ has a canonical matroid $M(G)$ associated to it, called the *graphic matroid* of $G$, defined on $E(G)$, with $\rho(S) = n-c$ for each $S \in 2^{E(G)}$, where $c$ denotes the number of connected components of the graph induced by $S$. In general, a matroid is called *graphic* if it is the graphic matroid of some graph. A *co-graphic matroid* is a matroid whose dual is a graphic matroid. It can be shown that a matroid is graphic and co-graphic simultaneously if and only if it is the graphic matroid of a planar graph [21].

From here on, in order to simplify the exposition, we restrict ourselves to linear matroids over the reals, i.e. matroids that can be expressed as $\mathcal{M}_A = (E, \mathcal{J}_A)$, where $E = [n]$, $A = (a_j) \in \mathbb{R}^{m\times n}$ and, for each $S \in 2^E$, $\mathcal{J}_A(S)$ denotes the number of linearly independent vectors in $\{a_j : j \in S\}$.

We can then define the analog of orientations and cycles in graphs for matroids. An *orientation* of the matroid $\mathcal{M}_A = (E, \mathcal{J}_A)$ induced by the vector $\sigma \in \{-1, +1\}^E$ is the matroid $\mathcal{M}_{A(\sigma)}$, where $A(\sigma) := (\sigma(j)\, a_j)_{j\in E}$.

We can also define the analog of acyclic and totally cyclic orientation for matroids. Let $S \in 2^E$. A circuit $(a_j)_{j\in S}$ in $\mathcal{M}_A$ is *positive* if and only if

$$\sum_{j\in S} \lambda_j a_j = 0 \text{ for some } \lambda_j > 0.$$

One can verify that every positive circuit in $\mathcal{M}_A$ is a positive cocircuit in its dual $\mathcal{M}_A{}^*$. We then say that $\mathcal{M}_{A(\sigma)}$ is *acyclic* if it contains no positive circuit, and it is *totally cyclic* if every element $a_i$, where $i \in E$, belongs to some positive circuit.

Acyclic and totally cyclic orientations of matroids are related by the following relation: an orientation $\mathcal{M}_{A(\sigma)}$ is acyclic if and only if it dual $\mathcal{M}_{A(\sigma)}$ is totally cyclic [16]. This also holds when we restrict ourselves to planar graphs instead of linear matroids in general. However, unlike in graphs, the concept of dual is defined for any matroid and not only those which are planar.

### 2.3 Tutte Polynomial

Each matroid $\mathcal{M}$ has an important bivariate polynomial associated to it, called its *Tutte polynomial*, defined as

$$T_{\mathcal{M}}(X, Y) = \sum_{S\in 2^E} (X-1)^{\rho(E)-\rho(S)}(Y-1)^{|S|-\rho(S)},$$

where the definition extendeds to Tutte polynomial of graphs by considering their graphic matroids. Some notable properties of Tutte polynomials are: $T_{\mathcal{M}}(X, Y) = T_{\mathcal{M}^*}(Y, X)$ for all matroids $\mathcal{M}$; $T_{\mathcal{M}}(2, 0)$ counts the number of acyclic orientations of the matriod (or graph) $\mathcal{M}$; and $T_{\mathcal{M}}(0, 2)$ counts the number of totally cyclic orientations of the matroid (or graph) $\mathcal{M}$.

### 2.4 Binary Decision Diagrams

In this technical report, we will use a data-structure known as *binary decision diagram* (BDD) to efficiently enumerate all the acyclic and totally cyclic orientation of graphs. BDDs were created with the intention of compactly representing Boolean functions in the seminal paper [2]. Sekine, Imai, Tani [18] proposed constructing the BDD in a top-down manner, instead of the traditional bottom-up fashion given in [2], and we follow the same approach in this report.

By using such an approach, our algorithms, to be described in Sec. 3, construct the BDDs of all acyclic and all totally cyclic orientations of graph in time $O^*(2^{\frac{\mathrm{pw}^2}{4}+o(\mathrm{pw}^2)})$.

Since every path decomposition is also a tree decomposition, the pathwidth is no smaller than the treewidth for any graph. Thus, the algorithm provided by Noble to count the number of acyclic and totally cyclic orientations of a given graph (by evaluating the Tutte polynomial) [14], which takes $O^*(2^{\mathrm{tw}^2+o(\mathrm{tw}^2)})$ time, has better time complexity than the one provided in this report. However, enumerating (i.e. listing) all enumerations is a harder problem than simply counting. Furthermore, there are several advantages in using BDDs to represent the Boolean functions $f_{\mathrm{acyc}}$ and $f_{tcyc}$ over simply counting or even enumerating their solutions. For example, by using BDDs, one is able to sample solutions to these functions such that each of them is equally likely, compute their reliability polynomial and perform operations on them while giving each edge a weight, among other advantages [7].

## 3. Algorithm for Enumeration of Acyclic and Totally Cyclic Orientations

The two Boolean functions we consider for our BDD-based algorithms are the ones described as follows: let $G$ be a graph, fix any ordering of the set of vertices $V$, and take a characteristic vector $v \in \{0,1\}^E$ to indicate the digraph obtained by substituting $\{u,v\}$ by $(u,v)$ if $v(\{u,v\}) = 0$ and by $(v,u)$ otherwise, for each $\{u,v\} \in E$ where $u < v$. The Boolean functions we consider are $f_{\mathrm{acyc}} : \{0,1\}^E \to \{0,1\}$ which takes the value 1 for acyclic orientations and 0 otherwise, and $f_{\mathrm{tcyc}} : \{0,1\}^E \to \{0,1\}$ which takes the value 1 for totally cyclic orientations and 0 otherwise.

**Definition 3.1 (Elimination Front)** Let $P$ be a partial orientation. We say that the *elimination front* of $P$, denoted by $\alpha(P)$, is the set of vertices which are adjacent to at least one edge in $E(P)$ and one directed edge in $A(P)$.

**Definition 3.2 (Reachability Relation)** Let $P$ be a partial orientation of $G$. We say that the *reachability relation* of $P$ is the binary relation $\mathcal{R}_P$ on $\alpha(P)$ given by

$$(u\mathcal{R}_P v \iff u \neq v \text{ and } \exists \text{ directed path from } u \text{ to } v \text{ in } G[A(P)]),$$

for each $(u,v) \in \alpha(P)^2$.

From a graph $G$ and an ordering $e_1, e_2, \ldots, e_m$ of its edges, we construct the BDD for acyclic orientations and totally cyclic orientations as shown in Fig. 1.

We show an example of how the algorithm works in Fig. 2.

## 4. Correctness and Running Time

As we can see in Fig. 2, our algorithms proceed by merging nodes of the BDD. In a BDD, we are allowed to merge two nodes when they are at the same level and, given the same remaining part of the Boolean function, they output the same value.

**Proposition 1 (Correctness for Acyclic Orientations)** At any given level of the BDD for acyclic orientations, we are allowed to merge the partial orientations $P_i$ and $P_j$ if $\mathcal{R}_{P_i} = \mathcal{R}_{P_j}$ and both contain no directed cycles.

**Proof:** Let $\alpha := \alpha(P_i) = \alpha(P_j)$. For an arbitrary orientation of the remaining undirected edges, let $D_i, D_j$ be the digraphs obtained from $P_i, P_j$. By symmetry, it is sufficient to prove that if $D_i$ is cyclic, then $D_j$ is also cyclic. Assume $D_i$ contains a directed cycle $C$. This cycle must necessarily pass through $\alpha$, otherwise it would follow that $D_j$ is acyclic. We can exchange the directed path $C \cap A(P_i)$ by the directed path $C \cap A(P_j)$ in the cycle $C$ to obtain a new cycle $C'$ of $D_j$. □

**Proposition 2 (Correctness for Totally Cyclic Orientations)** At any given level of the BDD for totally cyclic orientations, we are allowed to merge the partial orientations $P_i$ and $P_j$ if $\mathcal{R}_{P_i} = \mathcal{R}_{P_j}$ and, for each one, all directed edges participate in at least one directed path connecting vertices of the elimination front.

**Proof:** Let $P$ denote a partial orientation in which all directed edges participate in at least one directed path connecting vertices of $\alpha(P)$ in the BDD, and let $D(P)$ denote the digraph resulting from a fixed, arbitrary orientation of $E(P)$. To prove the theorem, it is sufficient to show a map $s$ which takes each $P$ to a simplified version $s(P)$ in such a way that (i) $D(s(P))$ is totally cyclic iff $D(P)$ is totally cyclic, and (ii) $D(s(P))$ is totally characterized by $\mathcal{R}_P$. We construct the map as follows: we initially perform vertex identification of each of the cycles of $G[A(P)]$, obtaining $s_1(P)$. Since the elements of any pair of vertices of a cycle are mutually reachable, we have that $D(P)$ is totally cyclic iff $D(s_1(P))$ is totally cyclic. We then define $s(P)$ from $s_1(P)$ by setting $A(s(P)) := \{(u,v) \in \alpha(s_1(P))^2 : u\mathcal{R}_{s_1(P)}v\}$ and deleting the (newly) isolated vertices not contained in $\alpha(s_1(P))$. It follows from the definition that $D(s(P))$ is completely characterized by $\mathcal{R}_P$. Let $\alpha := \alpha(s_1(P)) = \alpha(s(P))$. To prove that a necessary condition for $D(s_1(P))$ to be totally cyclic is that $D(s(P))$ is totally cyclic (sufficiency is immediate), we proceed by contradiction. Assume $D(s_1(P))$ is totally cyclic, but there exists $v_0, v_l \in \alpha$ $(v_0 \neq v_l)$ such that $v_0\mathcal{R}_{s(P)}v_l$ and there exists no path from $v_l$ to $v_0$ on $D(s(P))$. Note that $v_0\mathcal{R}_{s(P)}v_l$ implies that $v_0\mathcal{R}_{s_1(P)}v_l$, and thus there exists a directed path $(v_0,v_1),(v_1,v_2),\ldots,(v_{l-1},v_l)$ contained in $A(s_1(P))$. Since all edges in $A(s_1(P))$ are contained in some cycle after the orientation, there must exist paths $b_i, p_i, a_i$, where $a_i, b_i$ are vertices and $p_i$ is an arbitrary length path, contained in $E(s_1(P))$ and taking each $v_i$ to $v_{i-1}$ $(i = 1, 2, \cdots, l)$ after the orientation. It follows that the path $(v_l, b_l), p_l, (a_l, v_{l-1}), (v_{l-1}, b_{l-1}), \ldots, (a_2, v_1), (v_1, b_1), p_1, (a_1, v_0)$ in $D(s(P))$ (up to vertex identification of cycles) connects $v_l$ to $v_0$, which is a contradiction, completing the proof. □

The *size* of a BDD is defined to be its number of nodes. Many operations that we may want to perform on BDDs grow with its size, so it is in our best interest to keep it as small as possible. It is a well-known that the size of the BDDs depends on the ordering of the Boolean variables [2]. In our case, the Boolean variables are represented by the edges of $G$.

We propose an approach using path decompositions of graphs to find an ordering of edge. This approach allows us to bound the size of the resulting BDD, as will be seen below.

First we define (proper) interval graphs. Given $n$ intervals on the real line, an interval graph is their intersection graph, i.e., each interval is represented by a vertex and two vertices are connected by an edge iff the two corresponding intervals intersect. A proper

**Algorithm 1:** Enumerate all acyclic orientations

**input :** Graph $G$
Ordering of the edges $e_1, e_2, \ldots, e_m$
**output:** BDD enumerating all acyclic orinetations

Add the node $G$ to the 0-th level;
**for** $k \in [m]$ **do**
    Add all the partial orientations obtained by orienting the edge $e_k$ of each node of the $(k-1)$-th level, as well as information about each partial orientation's parent, to a temporary set $T_k$;
    For each node $P$ in $T_k$, if $P$ contains any directed cycle, add $P$ to the $k$-th level (from its parent), map it to **0** and set $T_k := T_k \setminus \{P\}$;
    For each subset $S$ of $T_k$ whose elements possess the same reachability relation, choose one element of $S$ and add it to the $k$-th level (from its parent);
In the $m$-th level, map the remaining nodes to either **0** or **1**

**Algorithm 2:** Enumerate all totally cyclic orientations

**input :** Graph $G$
Ordering of the edges $e_1, e_2, \ldots, e_m$
**output:** BDD enumerating all totally cyclic orientations

Add the node $G$ to the 0-th level;
**for** $k \in [m]$ **do**
    Add all the partial orientations obtained by orienting the edge $e_k$ of each node of the $(k-1)$-th level, as well as information about each partial orientation's parent, to a temporary set $T_k$;
    For each node $P$ in $T_k$, if $P$ contains an directed edge which is not contained in any directed path connecting vertices of $\alpha(P)$, add $P$ to the $k$-th level (from its parent), map it to **0** and set $T_k := T_k \setminus \{P\}$;
    For each subset $S$ of $T_k$ whose elements possess the same reachability relation, choose one element of $S$ and add it to the $k$-th level (from its parent);
In the $m$-th level, map the remaining nodes to either **0** or **1**

**Fig. 1** BDD-based algorithms to enumerate all acyclic and totally cyclic orientations of a given graph.

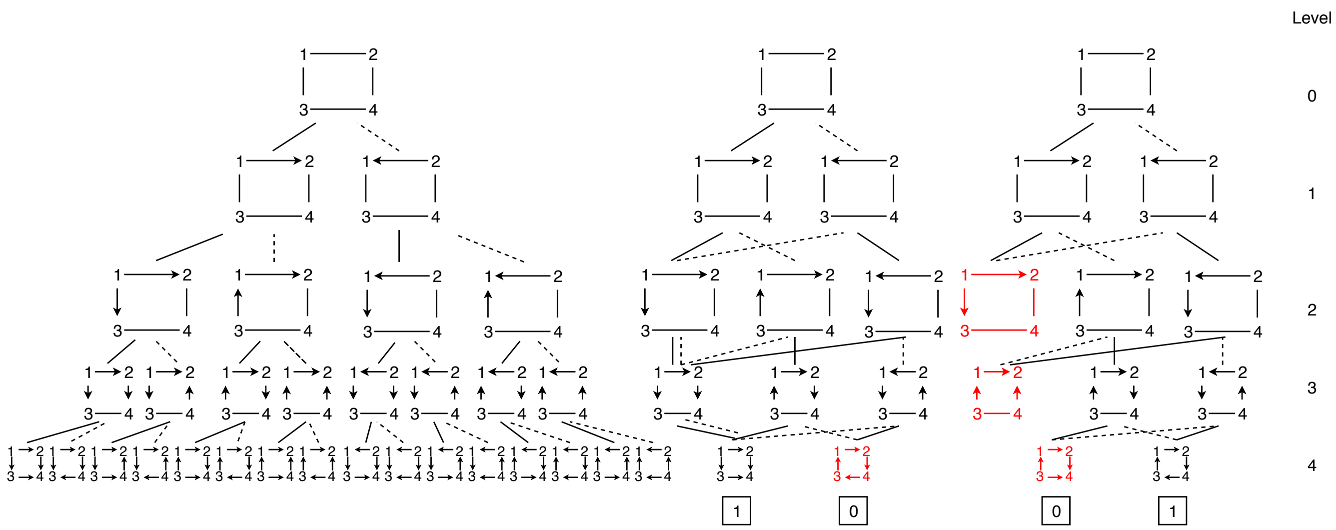


**Fig. 2** (Left) Binary decision tree generated by the cycle graph with four vertices $C_4$. The edges are ordered as follows: $\{1,2\} < \{1,3\} < \{2,4\} < \{3,4\}$. Solid and dotted lines indicate the edge is oriented from smallest to biggest and from biggest so smallest vertex, respectively. (Center) Reduced binary decision diagram used to enumerate acyclic orientations constructed using our algorithm. Red partial orientations denote a direct cycle has been found and there is no need to keep developing the branch emanating from them. (Right) Reduced binary decision diagram used to enumerate totally cyclic orientations constructed from our algorithm. Red partial orientations denote an edge which is not contained in any directed path connecting two elements of $\alpha_k$ has been found and there is no need to keep developing the branch emanating from them.

interval graph is an interval graph where no interval properly contains another interval [5].

The pathwidth of a graph can be characterized by interval graphs [1]. Theorem 29 in the last reference states that, for a graph $G$, the pathwidth of $G$ is at most $k-1$ if and only if interval thickness is at most $k$, where the interval thickness of $G$ is the smallest maximum clique size of an interval graph containing $G$ as its subgraph.

**Proposition 3 (Bound on Size of the Elimination Front [17])** For any graph $G$ of $n$ vertices, there exists an ordering of the edges in which the size of the elimination front is bounded by $\mathrm{pw}+1$.

A binary relation is said to be a *strict partial order* if it is irreflexive, transitive and asymmetric. It is a well known and easy to verify fact that, for any finite set $S$, the set of all partial orders of $S$ is in bijection with the set of all strict partial orders of $S$ (we need only take away the elements of the form $(a,a)$ $(a \in S)$ from the binary relations of the former set).

**Theorem 1** The width of the BDDs described in this section is bounded by

$$2^{\frac{\mathrm{pw}^2}{4}+o(\mathrm{pw}^2)}$$

for a pathwidth ordering of the edges, where pw denotes the pathwidth of the graph.

**Proof:** Let us first consider the BDD for acyclic orientations. For a given partial orientation $P$, where $A(P)$ is acyclic, it is simple to check that the reachability relation $\mathcal{R}_P$ is a strict partial order on the elimination front $\alpha(P)$. As explained above, there exists a one-to-one correspondence between the set of strict partial orders on $\alpha(P)$ and the set of partial orders on $\alpha(P)$. The following bound to the number of partial orders $P_n$ of a finite set of $n$ elements is given in [6]:

$$\log_2 P_n = \frac{n^2}{4} + \frac{3n}{2} + O(\log n),$$

from which the desired result follows.

In the case of totally cyclic orientations, there may exist subsets of vertices in the elimination front whose elements are mutually reachable. If we identify vertices of the elimination front that are mutually reachable, we can use Kleitman and Rothschild's result by considering the reachability relations on sets of cardinality $1, 2, \dots, \mathrm{pw}$. The number of possible reachability relations becomes bounded by

$$\begin{aligned} & 2^{\frac{1^2}{4}+o(\mathrm{pw}^2)} + 2^{\frac{2^2}{4}+o(\mathrm{pw}^2)} + \cdots + 2^{\frac{\mathrm{pw}^2}{4}+o(\mathrm{pw}^2)} \\ &= 2^{\frac{\mathrm{pw}^2}{4}+o(\mathrm{pw}^2)} + (\mathrm{pw}-1)2^{\frac{(\mathrm{pw}-1)^2}{4}+o(\mathrm{pw}^2)} \\ &= 2^{\frac{\mathrm{pw}^2}{4}+o(\mathrm{pw}^2)} + 2^{\frac{2\,\mathrm{pw}-1}{4}}2^{\frac{(\mathrm{pw}-1)^2}{4}+o(\mathrm{pw}^2)} \\ &= 2^{\frac{\mathrm{pw}^2}{4}+o(\mathrm{pw}^2)}. \end{aligned}$$

□

Pathwidth is a graph invariant which indicates how close to a path graph a certain graph is. For example, any path graph $P_n$ has pathwidth 1, while trees $T$, lattice graphs $L_{h\times k}$ and lattice graphs ${P_k}^3$ have pathwidth $\log(|T|)$, $\min\{h, k\}$ and $3k^2/4 + O(k)$, respectively.

## 5. Conclusion

In this report, we have proposed BDD-based algorithms to enumerate all acyclic and totally cyclic orientations of a graph, fixed-parameter tractable with respect to the pathwidth of the graph. While it is known that acyclic orientations and totally cyclic orientations are closely related, this relationship is not completely understood (e.g. the Merino-Welsh conjecture is still open at the time of writing). In this regard, our algorithms provide evidence that this relationship may manifest itself as similarity in methods used to compute these quantities. Furthermore, our enumerating methods, in virtue of being based in binary decision diagrams, may be used not only for counting the orientations, which is an easier problem than enumerating, but also for the other advantages provided by BDDs, as explained in the previous sections. Possible directions of further research include developing fixed-parameter algorithms which are tractable with respect to other parameters, especially those which are robust to graph density.

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