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Some Formulas for Max Nim

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Abstract: In this article, the authors present some theorems of Max Nim that is a combinatorial game. Suppose that there is a pile of n stones, and two players take turns to remove stones from the pile. In each turn, if the number of stones is m the player is allowed to remove at least one and at most $\lceil \frac{m}{d} \rceil$ stones for a fixed positive number d. (Note that d is not necessarily a natural number.) The player who removes the last stone or stones is the winner. The authors present some theorems for the Grundy numbers of this game. Then, the authors modify the standard rules of the game so as to allow for a one-time pass, i.e., a pass move, which may be used at most once in a game and not from a terminal position. Once the pass has been used by either player, it is no longer available. The authors also present a formula for the Grundy numbers of the game with a pass, and they show that there is a simple relation between the Grundy numbers of the variants with and without a pass.

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1. Maximum Nim

Let $Z_{\geq 0}$ and N be the sets of nonnegative numbers and natural numbers, respectively. The classic game of Nim is played with piles of stones. On each of their turns, a player can remove any number of stones from any one pile. The winner is the player to take the last stone. There are many variants of the classic game of Nim, one being Maximum Nim as defined in Definition 1.

Definition 1. Suppose that there is a pile of n stones, and two players take turns to remove stones from the pile. In each turn, if the number of stones is m the player is allowed to remove at least one and at most $f(m)$ stones. The player who removes the last stone or stones is the winner. Here, $f(m)$ is a sequence of nonnegative integers for $m \in Z_{\geq 0}$ such that f is weakly increasing and $0 \leq f(m) - f(m-1) \leq 1$ for any natural number m, and we refer to f as the rule sequence.

The authors present some new facts for the case that $f(m) = \lfloor \frac{m}{d} \rfloor$ for a fixed number $d \in Z_{\geq 0}$.

Definition 2. Let x be the number of stones in the pile. By $move(x)$ we denote the set of all piles of stones that can be created (reached) from the pile of x stones after taking a move. For any $x \in Z_{\geq 0}$, let $move(x) = \{(x-u) : 1 \leq u \leq f(x) \text{ and } u \in N\}$.

Definition 3. We now define the following:
(a) $\mathcal{N}$-positions, from which the next player can force a win, as long as they play correctly at every stage.
(b) $\mathcal{P}$-positions, from which the previous player (the player who will play after the next player) can force a win, as long as they play correctly at every stage.
(c) The *minimum excluded value* (*mex*) of a set S of nonnegative integers is the lowest nonnegative integer that is not in S.
(d) Each position (x) of a game V has an associated Grundy number, and we denote this by $\mathcal{G}(x)$.
The Grundy number is calculated recursively: $\mathcal{G}(x) = mex\{\mathcal{G}(y) : (y) \in move(x)\}$.

Theorem 1. Let $\mathcal{G}$ be the *Grundy* number. Then, (x) is a $\mathcal{P}$-position if and only if $\mathcal{G}(x) = 0$.

For the proof of this theorem, see [1].

Lemma 1. For the Grundy number $\mathcal{G}$, we have the following properties (i) and (ii).
(i) $\mathcal{G}(x) = f(x)$ if $f(x) > f(x-1)$.
(ii) $\mathcal{G}(x) = \mathcal{G}(x - f(x) - 1)$ if $f(x) = f(x-1)$.

This is Lemma 2.1 of [2]. By (ii) of Lemma 1, for any n there exists n' such that $n' < n$ and $\mathcal{G}(n) = \mathcal{G}(n')$.

Our aim is to find a simple function h such that $h(n) > n$ and $\mathcal{G}(n) = \mathcal{G}(h(n))$, so that for any $m \in Z_{\geq 0}$ there exists $t \in Z_{\geq 0}$ such that $\{n : \mathcal{G}(n) = m\} = \{h^p(t) : p \in Z_{\geq 0}\}$, where h^p is the pth functional power of the function h.

2. When the Rule Sequence $\mathbf{f(m) = \lceil \frac{m}{d} \rceil}$ for a Nonnegative Number d.

In the remainder of this article, we will assume that the rule sequence $f(m) = \lceil \frac{m}{d} \rceil$ for $d \in Z_{\geq 0}$ with $d > 1$.

Definition 4. For $x \in N$ and a positive number y such that $y > 2$, we let $p = \lfloor \frac{x}{y} \rfloor$.
We define a function $g(x, y)$ by the following properties (i) and (ii).
(i) $g(x, y) = x - py$ if $x - py \leq y - 1$.
(ii) $g(x, y) = x - (p+1)y$ if $x - py > y - 1$.

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Definition 5. We define a function $h_d(n)$ for a nonnegative integer n as
$h_d(n) = \frac{d}{d-1}n - \frac{1}{d-1}g(n, d-1) + 2$.

Theorem 2. Suppose that $d > 2$. Then, the following hold:
(a) $h_d(n) = \lceil \frac{h_d(n)}{d} \rceil + n + 1$, where h_d is the function defined in Definition 5 for $d > 2$.
(b) $\mathcal{G}(h_d(n)) = \mathcal{G}(n)$ for $n \in Z_{\geq 0}$.
(c) For any $m \in Z_{\geq 0}$ there exists $t \in Z_{\geq 0}$ such that $\{n : \mathcal{G}(n) = m\} = \{h_d^p(t) : p \in Z_{\geq 0}\}$.

Proof.

$$\text{Let } p = \lfloor \frac{n}{d-1} \rfloor. \tag{2.1}$$

Case(i) Suppose that

$$n - p(d-1) \leq d-2. \tag{2.2}$$

Then, by Definition 4 and 5 we have

$$g(n, d-1) = n - p(d-1)$$

$$\text{and } h_d(n) = \frac{d}{d-1}n - \frac{1}{d-1}(n - p(d-1)) + 2$$

$$= n + p + 2. \tag{2.3}$$

By (2.1), (2.2)

$$p(d-1) \leq n \leq p(d-1) + d - 2. \tag{2.4}$$

By (2.3) and (2.4)

$$p(d-1) + p + 2 \leq h_d(n) \leq p(d-1) + d - 2 + p + 2,$$

and we have

$$pd + 2 \leq h_d(n) \leq pd + d.$$

Then,

$$p + \frac{2}{d} \leq \frac{h_d(n)}{d} \leq p + 1, \tag{2.5}$$

$$\text{and hence} \lceil \frac{h_d(n)}{d} \rceil = p + 1. \tag{2.6}$$

Therefore, by (2.3) and (2.6) we have

$$h_d(n) = \lceil \frac{h_d(n)}{d} \rceil + n + 1. \tag{2.7}$$

Case(ii) Suppose that

$$n - p(d-1) > d - 2. \tag{2.8}$$

Then, by Definition 4 $g(n, d-1) = n - (p+1)(d-1)$,

and hence by Definition 5 (2.9)

$$h_d(n) = \frac{d}{d-1}n - \frac{1}{d-1}(n - (p+1)(d-1)) + 2$$

$$= n + p + 3. \tag{2.10}$$

By (2.8) and (2.1)

$p(d-1) + d - 2 < n < (p+1)(d-1)$, and hence by (2.10)
$p(d-1) + d - 2 + p + 3 < h_d(n) < (p+1)(d-1) + p + 3$.

Then,

$$pd + d + 1 < h_d(n) < pd + d + 2.$$

Therefore

$$p + 1 + \frac{1}{d} < \frac{h_d(n)}{d} < p + 1 + \frac{2}{d}, \tag{2.11}$$

and hence by the fact that $d > 2$,

$$\lceil \frac{h_d(n)}{d} \rceil = p + 2. \tag{2.12}$$

Then we have

$$h_d(n) - \lceil \frac{h_d(n)}{d} \rceil = n + 1. \tag{2.13}$$

Next, we prove (b) of this theorem.
Case (i) Suppose that (2.2) is valid. Then, by (2.5)

$$p + \frac{1}{d} \leq \frac{h_d(n) - 1}{d} < \frac{h_d(n)}{d} \leq p + 1, \tag{2.14}$$

and hence (2.15)

$$f(h_d(n) - 1) = \lceil \frac{h_d(n) - 1}{d} \rceil$$

$$= \lceil \frac{h_d(n)}{d} \rceil = f(h_d(n)) = p + 1 \tag{2.16}$$

(Note that $f(x) = \lceil \frac{x}{d} \rceil$.)
By (2.16), (ii) of Lemma 1 and (i) of this proof,

$$\mathcal{G}(h_d(n)) = \mathcal{G}(h_d(n) - f(h_d(n)) - 1)$$

$$= \mathcal{G}(h_d(n) - \lceil \frac{h_d(n)}{d} \rceil - 1) = \mathcal{G}(n).$$

Case (ii) Suppose that (2.8) is valid. By (2.11),

$$p + 1 < \frac{h_d(n) - 1}{d} < \frac{h_d(n)}{d} < p + 1 + \frac{2}{d},$$

and hence by the fact that $d > 2$,

$$f(h_d(n) - 1) = \lceil \frac{h_d(n) - 1}{d} \rceil = \lceil \frac{h_d(n)}{d} \rceil$$

$$= f(h_d(n)) = p + 2 \tag{2.17}$$

By (2.17), (ii) of Lemma 1 and (i) of this proof,

$$\mathcal{G}(h_d(n)) = \mathcal{G}(h_d(n) - f(h_d(n)) - 1)$$

$$= \mathcal{G}(h_d(n) - \lceil \frac{h_d(n)}{d} \rceil - 1) = \mathcal{G}(n).$$

(c) is direct from (b). □

Definition 6. For a positive number d such that $1 < d \leq 2$ and a natural number x, let $L(x, d) = \{(m, p) : x = (d-1)p + m \text{ and } -2 < m \leq d-2\}$.

Definition 7. We define a function $h_d(n)$ for a nonnegative integer n as
$h_d(n) = \min(\{dp + m + 2 : (m, p) \in L(n, d)\})$.

Theorem 3. Suppose that $1 < d \leq 2$. Then, the following hold:
(i) $h_d(n) = \lceil \frac{h_d(n)}{d} \rceil + n + 1$, where h_d is the function defined in Definition 7 for $2 \geq d > 1$.
(ii) $\mathcal{G}(h_d(n)) = \mathcal{G}(n)$ for $n \in Z_{\geq 0}$.
(iii) For any $m \in Z_{\geq 0}$ there exists $t \in Z_{\geq 0}$ such that $\{n : \mathcal{G}(n) = m\} = \{h_d^p(t) : p \in Z_{\geq 0}\}$

Proof. $h_d(n) = \min(\{dp + m + 2 : (m, p) \in L(n, d)\})$, where

Table 1 The case that $h(m) = \lceil \frac{m}{d} \rceil$ with $d = \frac{7}{4}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathcal{G}_d$	0	1	2	0	3	1	4	2	5	6	0	7	3	8	1	9	10	4	11	2	12
$\mathcal{G}_{d,p}$	0	2	1	3	0	2	5	1	4	7	3	6	0	9	2	8	11	5	10	1	13

$L(n,d) = \{(m,p) : n = (d-1)p+m \text{ and } -2 < m \le d-2\}$. Suppose that $h_d(n) = dp + m + 2$ and $(m,p) \in L(n,d)$. If $-2 < m+(d-1) \le d-2$, then $(m+(d-1), p-1) \in L(n,d)$ and $d(p-1) + m + (d-1) + 2 = dp + m + 1 < h_d(n)$. This contradicts the definition of the function $h_d(n)$. Therefore, we have $m + (d-1) > d-2$. Then,

$$m > -1. \tag{2.18}$$

Since $n = (d-1)p + m$ and $-2 < m \le d-2$, by (2.18)

$$\lceil \frac{dp}{d} \rceil < \lceil \frac{dp+m+1}{d} \rceil = f(h_d(n)-1)$$
$$\le \lceil \frac{dp+m+2}{d} \rceil = f(h_d(n)) \le \lceil \frac{dp+d}{d} \rceil = p+1. \tag{2.19}$$

By (2.19), (ii) of Lemma 1 and (i) of this proof,

$$\mathcal{G}(h_d(n)) = \mathcal{G}(h_d(n) - f(h_d(n)) - 1)$$
$$= \mathcal{G}(h_d(n) - \lceil \frac{h_d(n)}{d} \rceil - 1) = \mathcal{G}(n).$$

(c) is direct from (b). □

3. The Case with a Pass Move

In this section we study a game with a pass move that may be used at most once in a game, but not from a terminal position. Let $\mathcal{G}_d$ and $\mathcal{G}_{d,p}$ be the Grundy numbers for games without and with a pass move, respectively. Table 1 lists values for $\mathcal{G}_d$ and $\mathcal{G}_{d,p}$. It is clear that there is a simple relation between these two types of Grundy number. Theorem 4 presents relations between $\mathcal{G}_d$ and $\mathcal{G}_{d,p}$, and the following tables illustrate the relations presented in the theorem.

Theorem 4. (i) If $d \ge 3$, then we have the following properties (a) and (b):
(a) $\mathcal{G}_{d,p} = 1$ or 0 when $\mathcal{G}_d = 0$ or 1, respectively.
(b) $\mathcal{G}_{d,p} = 2m+1$ or $2m$ when $\mathcal{G}_d = 2m$ or $2m+1$, respectively.
(ii) If $3 > d \ge 2$, then we have the following properties (a) and (b):
(a) $\mathcal{G}_{d,p} = 1, 2,$ or 0 when $\mathcal{G}_d = 0, 1,$ or 2, respectively.
(b) $\mathcal{G}_{d,p} = 2m+2$ or $2m+1$ when $\mathcal{G}_d = 2m+1$ or $2m+2$, respectively.
(iii) If $2 > d > 1$, then there exists $a \in N$ such that $\frac{a}{a-1} > d \ge \frac{a+1}{a}$. We then have the cases $(iii.1)$ and $(iii.2)$.
$(iii.1)$ Suppose that a is odd. Then, we have the following properties (a) and (b).
(a) $\mathcal{G}_{d,p} = a, a+1,$ or 0 when $\mathcal{G}_d = 0, a,$ or $a+1$, respectively.
(b) $\mathcal{G}_{d,p} = 2m+2$ or $2m+1$ when $\mathcal{G}_d = 2m+1$ or $2m+2$ for $m \ne a, a+1$, respectively.

Table 2 $d \ge 3$

$\mathcal{G}_d$	0	1	2m	2m+1
$\mathcal{G}_{d,p}$	1	0	2m+1	2m

Table 3 $3 > d \ge 2$

$\mathcal{G}_d$	0	1	2	2m+1	2m+2
$\mathcal{G}_{d,p}$	1	2	0	2m+2	2m+1

Table 4 $2 > d > 1$ and a is odd

$\mathcal{G}_d$	0	a	a+1	2m+1	2m+2
$\mathcal{G}_{d,p}$	a	a+1	0	2m+2	2m+1

Table 5 $m > a+1$

$\mathcal{G}_d$	0	a+1	2m	2m+1
$\mathcal{G}_{d,p}$	a+1	0	2m+1	2m

Table 6 $m < a$

$\mathcal{G}_d$	0	a+1	2m+1	2m+2
$\mathcal{G}_{d,p}$	a+1	0	2m+2	2m+1

$(iii.2)$ Suppose that a is even. Then, we have the following properties (a) and (b).
(a) $\mathcal{G}_{d,p} = a+1$ or 0 when $\mathcal{G}_d = 0$ or $a+1$, respectively.
(b) For $m > a+1$, $\mathcal{G}_{d,p} = 2m+1$ or $2m$ when $\mathcal{G}_d = 2m$ or $2m+1$, respectively.
(c) For $m < a$, $\mathcal{G}_{d,p} = 2m+2$ or $2m+1$ when $\mathcal{G}_d = 2m+1$ or $2m+2$, respectively.

Table 5,6 illustrate the case that $2 > d > 1$ and a is even.

References

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