

Balanced Line Separators of Unit Disk Graphs*

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Abstract: We prove a geometric version of the graph separator theorem for the unit disk intersection graph: for any set of n unit disks in the plane there exists a line ℓ such that ℓ intersects at most $O(\sqrt{(m+n)\log n})$ disks and each of the halfplanes determined by ℓ contains at most $2n/3$ unit disks from the set, where m is the number of intersecting pairs of disks. We also show that an axis-parallel line intersecting $O(\sqrt{m+n})$ disks exists, but each halfplane may contain up to $4n/5$ disks. We give an almost tight lower bound (up to sublogarithmic factors) for our approach, and also show that no line-separator of sublinear size in n exists when we look at disks of arbitrary radii, even when $m = 0$. Proofs are constructive and suggest simple algorithms that run in linear time. Experimental evaluation has also been conducted, which shows that for random instances our method outperforms the method by Fox and Pach (whose separator has size $O(\sqrt{m})$).

Balanced separators in graphs are a fundamental tool and used in many divide-and-conquer-type algorithms as well as for proving theorems by induction. Given an undirected graph $G = (V, E)$ with V as its vertex set and E as its edge set, and a non-negative real number $\alpha \in [1/2, 1]$, we say that a subset $S \subseteq V$ is an α -separator if the vertex set of $G \setminus S$ can be partitioned into two sets A and B , each of size at most $\alpha|V|$ such that there is no edge between A and B . The parameter α determines how balanced the two sets A and B are in terms of size. For a balanced separator to be useful we want both the size $|S|$ of the separator and $\alpha \geq 1/2$ to be small.

Much work has been done to prove the existence of separators with certain properties in general sparse graphs. For example, the well-known Lipton–Tarjan planar separator theorem [14] states that for any n -vertex planar graph, there exists a $2/3$ -separator of size $O(\sqrt{n})$. Similar theorems have been proven for bounded-genus graphs [11], minor-free

graphs [3], low-density graphs, and graphs with polynomial expansion [12], [19].

These separator results apply to graph classes that do not contain complete graphs of arbitrary size, and each graph in the classes contains only $O(n)$ edges, where n is the number of vertices. Since any α -separator of a complete graph has $\Omega(n)$ vertices, the study of separators for graph classes that contain complete graphs seems useless. However, it is not clear how small a separator can be with respect to the number of edges for possibly dense graphs.

Our focus of interest is possibly dense geometric graphs, which often encode additional geometric information other than adjacency. Even though one can use the separator tools in geometric graphs, often the geometric information is lost in the process. As such, a portion of the literature has focused on the search of balanced separators that also preserve the geometric properties of the geometric graph.

Among several others, we highlight the work of Miller *et al.* [17], and Smith and Wormald [20]. They considered intersection graphs of n balls in \mathbb{R}^d and proved that if every point in d -dimensional space is covered by at most k of the given balls, then there exists a $(d+1)/(d+2)$ -separator of size $O(k^{1/d}n^{1-1/d})$ (and such a separator can be found in deterministic linear time [6]). More interestingly, the separator itself and the two sets it creates have very nice properties; they show that there exists a $(d-1)$ -dimensional sphere that intersects at most $O(k^{1/d}n^{1-1/d})$ balls and contains at most $(d+1)n/(d+2)$ balls in its interior and at most $(d+1)n/(d+2)$ balls in its exterior. In this case, the sphere acts as the separator (properly speaking, the balls that intersect the sphere), whereas the two sets A and B are the balls that are inside and outside the separator sphere, respec-

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tively. Note that the graph induced by the set A consists of the intersection graph of the balls inside the separator (similarly, B for the balls outside the separator and S for the balls intersecting the sphere).

We emphasize that, even though the size of the separator is larger than the one from Lipton–Tarjan for planar graphs (specially for high values of d), the main advantage is that the three subgraphs it creates are geometric graphs of the same family (intersection graphs of balls in \mathbb{R}^d). The bound on the separator size does not hold up well when k is large, even for $d = 2$: if \sqrt{n} disks overlap at a single point and the other disks form a path we have $k = \sqrt{n}$ and $m = \Theta(n)$, where m is the number of edges in the intersection graph. Hence, the separator has size $O(\sqrt{kn}) = O(m^{3/4})$.

Fox and Pach [7] gave another separator result that follows the same spirit: the intersection graph of a set of Jordan curves in the plane has a 2/3-separator of size $O(\sqrt{m})$ if every pair of curves intersects at a constant number of points.^{*1} A set of disks in \mathbb{R}^2 satisfies this condition, and thus the theorem applies to disk graphs. Their proof can be turned into a polynomial-time algorithm. However, we need to construct the arrangement of disks, which takes $O(n^2 2^{\alpha(n)})$ time, where $\alpha(n)$ is the inverse Ackermann function [5], and in practice an efficient implementation is non-trivial.

From a geometric perspective these two results show that, given a set of unit disks in the plane, we can always find a closed curve in the plane (a circle [17], [20] and a Jordan curve [7], respectively) to partition the set. The disks intersected by the curve are those in the separator, and the two disjoint sets are the disks inside and outside the curve, respectively.

Results

In this paper we continue the idea of geometric separators and show that a balanced separator always exists, even if we constrain the separator to be a line (see Fig. 1). Given a set of n unit disks with m pairs of intersecting disks, we show that a line 2/3-separator of size $O(\sqrt{(m+n)\log n})$ can be found in expected $O(n)$ time, and that an axis-parallel line 4/5-separator of size $O(\sqrt{m+n})$ can be found in deterministic $O(n)$ time.

Comparing our results with the previous work, our algorithm matches or improves in four ways, see also Table 1. (i) simplicity of the shape: circle [17], [20] vs. Jordan curve [7] vs. our line, (ii) balance of the sets A and B : 3/4 [17], [20] vs. 2/3 for both [7] and us, (iii) size of the separator: $O(m^{3/4})$ [17], [20] vs. $O(\sqrt{m})$ [7] vs. our $\tilde{O}(\sqrt{m})$.^{*2} Finally, (iv) our algorithms are easy to implement and asymptotically faster: $O(n)$ [17], [20] vs. $\tilde{O}(n^2)$ [7] vs. our $O(n)$.

^{*1} Without restriction on the number of intersection points for every pair of curves, the bound of $O(\sqrt{m \log m})$ can be achieved [16].

^{*2} The $\tilde{O}(\cdot)$ notation suppresses sublogarithmic factors. In particular, we note that our separator is slightly larger than the

We emphasize that our results focus on *unit* disk graphs, while the other results hold for disk graphs of arbitrary radii, too. Indeed, if we want to separate disks of arbitrary radii with a line, we show that the separator’s size may be as large as $\Omega(n)$. We also prove that for unit disks our algorithm may fail to find a line 2/3-separator of size better than $O(\sqrt{m \log(n/\sqrt{m})})$ in the worst case. In this sense, the size of our separators is asymptotically almost tight.

Experimental results are also presented. We evaluate the performance of our algorithm, compare it with the method by Fox and Pach [7] in terms of the size of the produced separators for random instances, and conclude that our algorithm outperforms theirs for the intersection graphs of unit disks.

Working with a line separator for intersecting disks has some difficulty. If we chose to separate pairwise disjoint geometric objects by a Jordan curve, then we could employ a volume argument for the interior of the curve. However, we cannot use a volume argument for line separators since the line does not determine a bounded region.

Other Related Work

In a different context, line separators of *pairwise disjoint* unit disks have also been studied. Since the disks are pairwise disjoint, the intersection graph is trivially empty and can be easily separated. Instead, the focus is now to find a closed curve that intersects few disks, such that the two connected components it defines contain roughly the same number of disks.

Alon *et al.* [2] proved that for a given set \mathcal{D} of n pairwise disjoint unit disks,^{*3} there exists a slope a such that every line with slope a intersects $O(\sqrt{n \log n})$ unit disks of \mathcal{D} . In particular, the halving line of that slope will be a nice separator (each halfplane will have at most $n/2 - O(\sqrt{n \log n})$ disks fully contained in). Their proof is probabilistic, which can be turned into an expected $O(n)$ -time randomized algorithm [15]. A deterministic $O(n)$ -time algorithm was afterwards given by Hoffmann *et al.* [13], who also showed how to find a line ℓ that intersects at most $O(\sqrt{n/(1-2\alpha)})$ unit disks and each halfplane contains at most $(1-\alpha)n$ disks (for any $0 < \alpha < 1/2$). Löffler and Mulzer [15] proved that there exists an axis-parallel line ℓ such that ℓ intersects $O(\sqrt{n})$ disks, and each halfplane contains at most $9n/10$ unit disks. For comparison purposes, these three results are also shown in Table 1.

A significant amount of research has focused on the search for balanced line separators of unit disk graphs in the plane, but unlike the ones mentioned before no guarantee is given on the shape of the separator. Yan *et al.* [21] studied a separator of unit disk graphs for designing a low-delay compact routing labeling scheme for ad-hoc networks modeled by unit disk graphs. Their separator is a 2/3-separator, but has no

Fox-Pach separator.

^{*3} The result extends to pairwise disjoint fat objects that are convex and of similar area (see Theorem 4.1 of [2]). For the sake of conciseness we only talk about unit disks.

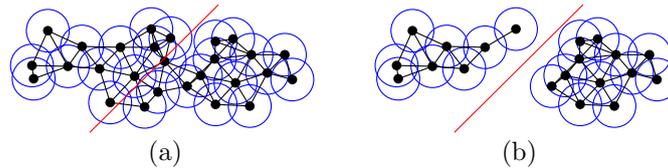


Fig. 1 An example of a line separator of a unit disk graph. (a) A family of unit disks (blue) and a line (red). (b) Removing the disks intersected by the red line leaves a disconnected graph.

Table 1 Comparison of our results with other geometric separator results.

result	separator shape	balance	separator size	run-time	object type
[17], [20]	circle	3/4	$O(m^{3/4})$	$O(n)$	arbitrary disks
[7]	Jordan curve	2/3	$O(\sqrt{m})$	$\tilde{O}(n^2)$	pseudodisks
This paper	line	2/3	$\tilde{O}(\sqrt{m})$	$O(n)$	unit disks
This paper	axis-parallel line	4/5	$O(\sqrt{m})$	$O(n)$	unit disks
[2], [13], [15]	line	1/2	$O(\sqrt{n \log n})$	$O(n)$	disjoint unit disks
[13]	line	$1 - \alpha$	$O(\sqrt{n(1 - 2\alpha)})$	$O(n)$	disjoint unit disks
[15]	axis-parallel line	9/10	$O(\sqrt{n})$	$O(n)$	disjoint unit disks

size guarantee. Fu and Wang [9] studied the case where a unit disk graph is a $\sqrt{n} \times \sqrt{n}$ grid generated from a regular grid, and proved that there exists a line $2/3$ -separator of size at most $1.129\sqrt{n}$. They used the obtained separator to give the first subexponential-time algorithm for the protein folding problem in the HP model. The bound of $1.129\sqrt{n}$ was afterwards improved by Fu *et al.* [8] to $1.02074\sqrt{n}$. We note that it is not known whether a minimum-size $2/3$ -separator of a unit disk graph can be computed in polynomial time, although the problem is known to be NP-hard for graphs of maximum degree three [4], 3-regular graphs [18], and planar graphs [10]. Finally, Alber and Fiala [1] studied the existence of separators for disk intersection graphs, but ask for additional constraints to the set of disks (such as requiring the disks to be at least λ units apart, or bounding the ratio between the radii of the smallest and the largest disks).

References

[1] Jochen Alber and Jiri Fiala. Geometric separation and exact solutions for the parameterized independent set problem on disk graphs. *J. Algorithms*, 52(2):134–151, 2004.

[2] Noga Alon, Meir Katchalski, and William R. Pulleyblank. Cutting disjoint disks by straight lines. *Discrete & Computational Geometry*, 4:239–243, 1989.

[3] Noga Alon, Paul Seymour, and Robin Thomas. A separator theorem for nonplanar graphs. *J. Amer. Math. Soc.*, 3:801–808, 1990.

[4] Thang Nguyen Bui and Curt Jones. Finding good approximate vertex and edge partitions is NP-hard. *Inf. Process. Lett.*, 42(3):153–159, 1992.

[5] Herbert Edelsbrunner, Leonidas J. Guibas, János Pach, Richard Pollack, Raimund Seidel, and Micha Sharir. Arrangements of curves in the plane—topology, combinatorics and algorithms. *Theor. Comput. Sci.*, 92(2):319–336, 1992.

[6] David Eppstein, Gary L. Miller, and Shang-Hua Teng. A deterministic linear time algorithm for geometric separators and its applications. *Fundam. Inform.*, 22(4):309–329, 1995.

[7] Jacob Fox and János Pach. Separator theorems and Turán-type results for planar intersection graphs. *Advances in Mathematics*, 219(3):1070–1080, 2008.

[8] Bin Fu, Sorinel Adrian Oprisan, and Lizhe Xu. Multi-directional width-bounded geometric separator and protein folding. *Int. J. Comput. Geometry Appl.*, 18(5):389–413, 2008.

[9] Bin Fu and Wei Wang. Geometric separators and their applications to protein folding in the HP-model. *SIAM J. Comput.*, 37(4):1014–1029, 2007.

[10] Junichiro Fukuyama. NP-completeness of the planar separator problems. *J. Graph Algorithms Appl.*, 10(2):317–328, 2006.

[11] John R. Gilbert, Joan P. Hutchinson, and Robert Endre Tarjan. A separator theorem for graphs of bounded genus. *J. Algorithms*, 5(3):391–407, 1984.

[12] Sarel Har-Peled and Kent Quanrud. Approximation algorithms for polynomial-expansion and low-density graphs. In *Proc. of the 23rd Annual European Symposium*, pages 717–728. Springer, 2015.

[13] Michael Hoffmann, Vincent Kusters, and Tillmann Miltzow. Halving balls in deterministic linear time. In *Proc. of the 22th Annual European Symposium*, pages 566–578. Springer, 2014.

[14] Richard J. Lipton and Robert Endre Tarjan. A separator theorem for planar graphs. *SIAM Journal on Applied Mathematics*, 36(2):177–189, 1979.

[15] Maarten Löffler and Wolfgang Mulzer. Unions of onions: Preprocessing imprecise points for fast onion decomposition. *JoCG*, 5(1):1–13, 2014.

[16] Jiri Matousek. Near-optimal separators in string graphs. *Combinatorics, Probability & Computing*, 23(1):135–139, 2014.

[17] Gary L. Miller, Shang-Hua Teng, William P. Thurston, and Stephen A. Vavasis. Separators for sphere-packings and nearest neighbor graphs. *J. ACM*, 44(1):1–29, 1997.

[18] Rudolf Müller and Dorothea Wagner. α -Vertex separator is NP-hard even for 3-regular graphs. *Computing*, 46(4):343–353, 1991.

[19] Jaroslav Nešetřil and Patrice Ossona de Mendez. Grad and classes with bounded expansion II. Algorithmic aspects. *Eur. J. Comb.*, 29(3):777–791, 2008.

[20] Warren D. Smith and Nicholas C. Wormald. Geometric separator theorems & applications. In *Proc. of the 39th Annual Symposium on Foundations of Computer Science*, pages 232–243, 1998.

[21] Chenyu Yan, Yang Xiang, and Feodor F. Dragan. Compact and low delay routing labeling scheme for unit disk graphs. *Comput. Geom.*, 45(7):305–325, 2012.