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Technical Note

An Algorithm for Hinge Vertex Problem on Circular Trapezoid Graphs

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Abstract: Let $G = (V, E)$ be a simple connected graph. A vertex $u \in V$ is called a hinge vertex if there exist two vertices x and y in G whose distance increases when u is removed. Finding all hinge vertices of a given graph is called the hinge vertex problem. This problem can be applied to improve the stability and robustness of communication network systems. In a number of graph problems, it is known that more efficient sequential or parallel algorithms can be developed by restricting classes of graphs. Circular trapezoid graphs are proper super-classes of trapezoid graphs. In this paper, we propose an $O(n^2)$ time algorithm for the hinge vertex problem of circular trapezoid graphs.

Keywords: design and analysis of algorithms, hinge vertex problem, intersection graphs, circular trapezoid graphs

1. Introduction

Let $G = (V, E)$ be a simple undirected graph with a vertex set V and an edge set E. For a vertex $u \in V$, we denote the subgraph induced by the vertex set $V - \{u\}$ as $G - \{u\}$. The distance $\delta_G(x, y)$ is defined as the length (i.e., the number of edges) of the shortest path between vertices x and y in G. Chang et al. defined $u \in V$ as a *hinge vertex* in G if two vertices $x, y \in V - \{u\}$ exist, such that $\delta_{G-\{u\}}(x, y) > \delta_G(x, y)$ [2]. Hence, a vertex $u \in V$ is a hinge vertex if there exist two vertices x and y in G whose distance increases when u is removed. Note that *articulation vertices* are a special case of hinge vertices in that the removal of an articulation vertex u changes the finite distance of some nonadjacent vertices x and y to infinity. Finding all hinge vertices of a given graph is called the *hinge vertex problem*. For a simple graph G with n vertices, the hinge vertex problem can be solved in $O(n^3)$ time by the results in Ref. [2], e.g., Lemma 1 in this study.

The computation of topological properties is a very important research topic, which influences the design and analysis of distributed networks. For example, the overall cost of communication in a network will increase if a computer that corresponds to a hinge vertex stalls. Therefore, identifying the set of hinge vertices in a graph can help detect critical nodes, which can be useful for constructing more stable communication network systems [6].

Numerous studies of hinge vertices on several *intersection graphs* have been published. For example, Ho et al. [3] presented an $O(n)$ time algorithm for the hinge vertex problem on *permutation graphs*. Moreover, Hsu et al. [8] presented an $O(n)$ time algorithm on *interval graphs*. The class of *trapezoid graphs* properly contains both interval graphs and permutation graphs. Honma and Masuyama [4] and Bera [1] presented $O(n \log n)$ time algorithms for the hinge vertex problem on trapezoid graphs, respectively. Recently, Honma et al. presented an algorithm that runs in $O(n^2)$ time for identifying the *maximum detour hinge vertex* on interval graphs [6] and permutation graphs [7].

Lin [9] introduced a *circular trapezoid graph* (CTG for short), which is a proper superclass of trapezoid graphs and circular-arc graphs. They presented that the maximum weighted independent set can be found in $O(n^2 \log \log n)$ time on a circular trapezoid graph [9]. In this study, we propose an $O(n^2)$ time algorithm for solving the hinge vertex problem on a CTG.

2. Preliminaries

In this section, we propose some useful data structures and interesting properties on CTGs. We show the *circular trapezoid model* (CTM for short) before defining the CTG. The model consists of inner and outer circles C_1 and C_2 with radius $r_1 < r_2$. Each circle is assigned counterclockwise with consecutive integer values $1, 2, \ldots, 2n$, where n is the number of trapezoids. Consider two arcs, A_1 and A_2, on C_1 and C_2, respectively. Points a and b (resp., c and d) are the first points encountered when traversing the arc A_1 (resp., A_2) counterclockwise and clockwise, respectively. A *trapezoid* is the region in circles C_1 and C_2 that lies between two non-crossing chords ac and bd. A trapezoid CT_i is defined by four corner points $[a_i, b_i, c_i, d_i]$. Each trapezoid CT_i is numbered in the ascending order of their corner points b_i's, i.e., $i < j$ if $b_i < b_j$. The geometric representation described above is called the CTM. **Figure 1** (a) illustrates an example of a CTM M with 12 trapezoids. **Table 1** shows the details of M in Fig. 1 (a).

An undirected graph G is a CTG if it can be represented by the following CTM M: each vertex of the graph corresponds to a trapezoid, and two vertices are adjacent in G if and only if their corresponding trapezoids intersect. Figure 1 (b) illustrates the CTG G corresponding to M shown in Fig. 1 (a). In this example, $\delta_G(3, 7) = 2$ and $\delta_{G-\{5\}}(3, 7) = 4$, therefore, vertex 5 is a

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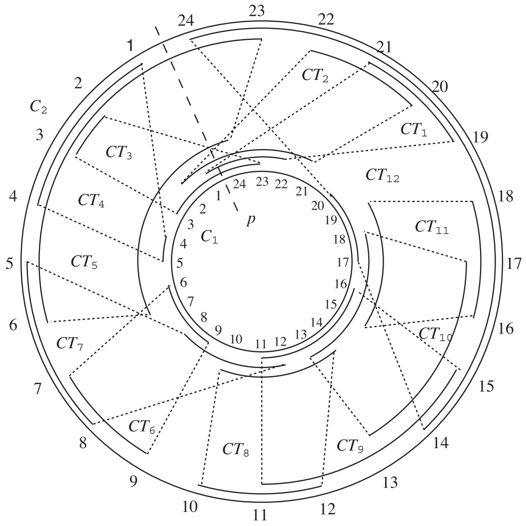
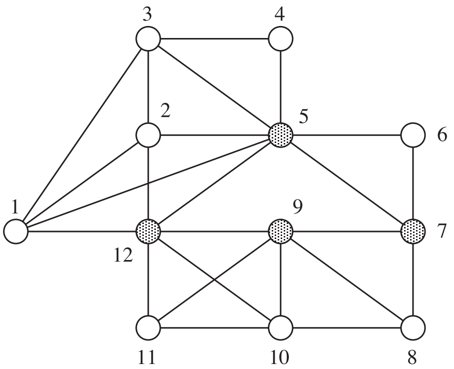


(a) CTM M



(b) CTG G

Fig. 1 Circular trapezoid model M and graph G.

Table 1 Details of CTM M.

i	1	2	3	4	5	6	7	8	9	10	11	12
a_i	22	21	23	4	24	6	8	10	11	13	15	17
b_i	1	2	3	5	7	9	12	14	16	18	19	20
c_i	19	20	2	1	23	7	5	10	11	13	16	14
d_i	21	22	3	4	6	9	8	12	15	17	18	24

hinge vertex for 3 and 7. All hinge vertices in G are 5, 7, 9 and 12.

In the following, we introduce an *extended circular trapezoid model* (ECTM for short) constructed from a CTM. Let n be the number of trapezoids in CTM M. Consider a fictitious line p that connects the points placed between 1 and $2n$ of C_1 and C_2. An ECTM EM is obtained by opening a CTM M along a fictitious line p. The ECTM EM consists of two horizontal parallel lines called *top* and *bottom* channel, respectively. To avoid confusion, we denote a trapezoid in CTM and ECTM by CT_i and T_i, respectively. For each T_i, $1 \leqslant i \leqslant n$, copies T_{i+n} are created by shifting $2n$ to the right. A procedure for constructing ECTM EM from CTM M in $O(n)$ time is presented in Ref. [5]. **Figure 2** illustrates an ECTM EM constructed from the CTM M shown in Fig. 1 (a).

The following properties can be derived in a straightforward manner from the processes of constructing an ECTM [5].

(1) T_i and T_{i+n} in ECTM EM correspond to the vertex i in CTG G.

(2) A vertex i is adjacent to j in G if and only if T_i and T_j, or T_j and T_{i+n} intersect in EM. □

3. Useful Lemmas for Hinge Vertex Problem

We introduce some notations that will be used in our algorithm. Let EM be an ECTM constructed from CTM M. We define $mt(i)$, $smt(i)$, $mb(i)$, and $smb(i)$ as follows. Here, the set (including i) of all trapezoids that intersect T_i in EM is denoted by $N_T[i]$.

- $mt(i) = k$ such that $b_k = \max\{\, b_j \mid j \in N[i]\}$,
- $smt(i) = k$ such that $b_k = \max\{\, b_j \mid j \in (N[i] - mt(i) \cup \{i\})\}$,
- $mb(i) = k$ such that $d_k = \max\{\, d_j \mid j \in N[i]\}$,
- $smb(i) = k$ such that $d_k = \max\{\, d_j \mid j \in (N[i] - mb(i) \cup \{i\})\}$.

In the following, we define $Yt(i)$ and $Yb(i)$ as follows.

$$Yt(i) = \begin{cases} \{\, j \mid b_{smt(i)} < a_j < b_{mt(i)}, d_{smb(i)} < c_j\} : \ mt(i) = mb(i), \\ \{\, j \mid b_{smt(i)} < a_j < b_{mt(i)}, d_{mb(i)} < c_j\} \ : \ otherwise. \end{cases}$$

$$Yb(i) = \begin{cases} \{\, j \mid d_{smb(i)} < c_j < d_{mb(i)}, b_{smt(i)} < a_j\} : \ mt(i) = mb(i), \\ \{\, j \mid d_{smb(i)} < c_j < d_{mb(i)}, b_{mt(i)} < a_j\} \ : \ otherwise. \end{cases}$$

For the example shown in Fig. 2, for the vertex 5, we have $mt(5) = 7$, $smt(5) = 6$, $mb(5) = 6$, $smb(5) = 7$, $Yt(5) = \{8, 9\}$, and $Yb(5) = \emptyset$. **Table 2** shows details of $mt(i)$, $smt(i)$, $mb(i)$, $smb(i)$, $Yt(i)$, and $Yb(i)$ for EM shown in Fig. 2.

We present some lemmas of hinge vertices on CTGs, which are useful for efficiently identifying the hinge vertices. Lemma 1 is proposed by Chang et al. [2] characterizes the hinge vertices of a simple graph.

Lemma 1 For a simple graph G_s, a vertex u is a hinge vertex of G_s if and only if there exist two nonadjacent vertices x, y such that u is the only vertex adjacent to both x and y in G_s. □

We can easily obtain the following Lemma 2 from Lemma 1.

Lemma 2 For a CTG G, a vertex u is a hinge vertex of G if and only if there exist two trapezoids CT_x and CT_y such that CT_x and CT_y do not intersect, and CT_u is the only trapezoid intersecting both CT_x and CT_y in a CTM M. □

For the example shown in Fig. 1, CT_5 and CT_8 do not intersect and CT_7 is the only trapezoid intersecting both CT_5 and CT_8 in M. Therefore, vertex 7 is a hinge vertex for 5 and 8 in the corresponding CTG G.

The following Lemma 3 provides the necessary and sufficient condition for hinge vertices in a trapezoid graph presented by Honma and Masuyama [4].

Lemma 3 A vertex u is a hinge vertex of a trapezoid graph if and only if there exist two vertices x, y satisfying either of the following conditions.

(1) $u = mt(x)$ and $y \in Yt(x)$,

(2) $u = mb(x)$ and $y \in Yb(x)$. □

The following Lemma 4 provides the necessary and sufficient condition for hinge vertices in a CTG.

Lemma 4 Let EM be an ECTM constructed from CTM M. A vertex u is a hinge vertex of a CTG G if and only if there exist two vertices x, y satisfying either of the following conditions.

(1) $u = mt(x)$, $y \in Yt(x)$, $b_{mt(y)} < a_{x+n}$, and $d_{mb(y)} < c_{x+n}$ in ETCM EM,

(2) $u = mb(x)$, $y \in Yb(x)$, $b_{mt(y)} < a_{x+n}$, and $d_{mb(y)} < c_{x+n}$ in ETCM EM.

(Proof) We only prove this lemma for Condition (1). Condition (2) can be handled in a similar manner.

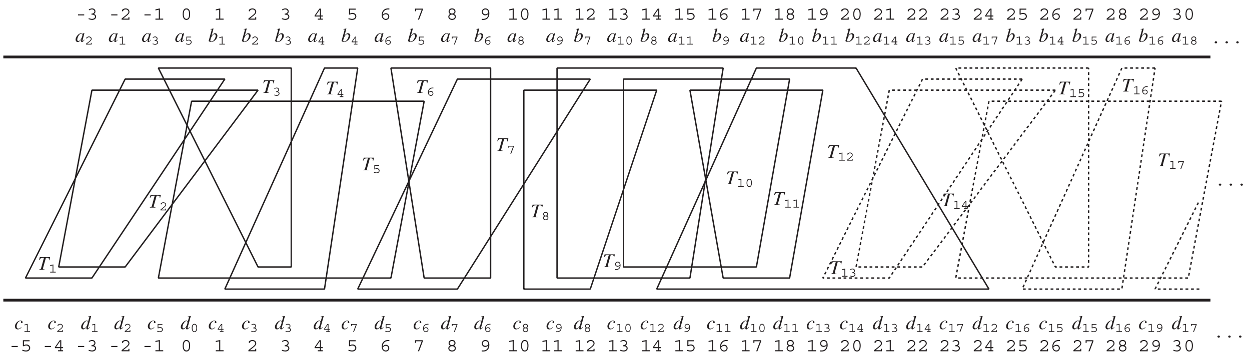


Fig. 2 Extended circular trapezoid model EM.

Table 2 Details of ECTM EM shown in Fig. 2.

i	1	2	3	4	5	6	7	8	9	10	11	12
a	−2	−3	−1	4	0	6	8	10	11	13	15	17
b	1	2	3	5	7	9	12	14	16	18	19	20
c	−5	−4	2	1	−1	7	5	10	11	13	16	14
d	−3	−2	3	4	6	9	8	12	15	17	18	24
mt	5	5	5	5	7	7	9	10	12	12	12	17
smt	3	3	4	4	6	6	8	9	11	11	11	14
mb	5	5	5	5	6	6	9	10	12	12	12	17
smb	3	3	4	4	7	6	8	9	11	11	11	12
Yt	{6}	{6}	{6}	{6}	{8, 9}	{8, 9}	{11}	∅	∅	∅	∅	{16, 18}
Yb	{7}	{7}	{7}	{7}	∅	∅	{12}	∅	{13, 14, 17}	{13, 14, 17}	{13, 14, 17}	{16, 19}

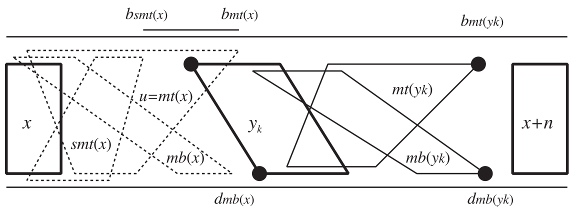
i	13	14	15	16	17	18	19	20	21	22	23	24
a	22	21	23	28	24	30	32	34	35	37	39	41
b	25	26	27	29	31	33	36	38	40	42	43	44
c	19	20	26	25	23	31	29	34	35	37	40	38
d	21	22	27	28	30	33	32	36	39	41	42	48
mt	17	17	17	17	19	19	21	22	24	24	24	29
mb	17	17	17	17	18	18	21	22	24	24	24	29

We first prove the necessity. By Lemma 2, if a vertex u is a hinge vertex of a CTG G, there exist two trapezoids CT_x and CT_y such that CT_x and CT_y do not intersect and, CT_u is the only trapezoid intersecting both CT_x and CT_y in CTM M. From the properties of ECTM, T_i and T_{i+n} correspond to the vertex i in G, and a vertex i is adjacent to j in G if and only if T_i and T_j, or T_j and T_{i+n} intersect in EM. Therefore, if a vertex u is a hinge vertex of a CTG G, there exist two trapezoids T_x and T_y such that T_x and T_y do not intersect, and T_u is the only trapezoid intersecting both T_x and T_y, and neither $T_{mt(y)}$ nor $T_{mb(y)}$ intersect T_{x+n} in ECTM EM.

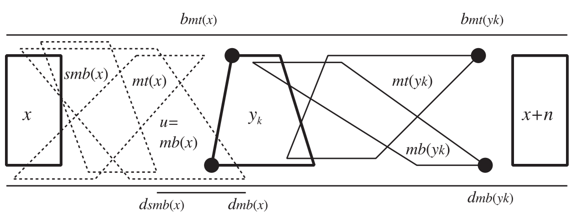
By Lemma 3 (1), if T_x and T_y do not intersect, and T_u is the only trapezoid intersecting both T_x and T_y, we have $u = mt(x)$ and $y \in Yt(x)$. Moreover, if neither $T_{mt(y)}$ nor $T_{mb(y)}$ intersect T_{x+n}, then $b_{mt(y)} < a_{x+n}$ and $d_{mb(y)} < c_{x+n}$. Thus, Condition (1) holds (**Fig. 3** (a)).

We prove the sufficiency. By Lemma 3 (1), if $u = mt(x)$ and $y \in Yt(x)$ in EM, then T_x and T_y do not intersect, and T_u is the only trapezoid intersecting both T_x and T_y. Assume that $b_{mt(y)} < a_{x+n}$ and $d_{mb(y)} < c_{x+n}$, and there exist some trapezoid T_z intersecting both T_y and T_{x+n}. It means that $b_{mt(y)} < b_z$ or $d_{mb(y)} < d_z$, contradicting the definitions of $mt(y)$ and $mb(y)$. Thus, if $b_{mt(y)} < a_{x+n}$ and $d_{mb(y)} < c_{x+n}$, then neither $T_{mt(y)}$ nor $T_{mb(y)}$ intersect T_{x+n} (Fig. 3 (a)). □

We show how to identify a hinge vertex by applying Condition (1) of Lemma 4. For x and $y \in Yt(x)$, we check whether $b_{mt(y)} < a_{x+n}$ and $d_{mb(y)} < c_{x+n}$. For example, for $x = 5$ and $y = 8$ ($Yt(5) = \{8, 9\}$), we have $b_{mt(8)} = b_{10} = 18 < a_{5+n} = a_{17} = 24$ and $d_{mb(8)} = d_{10} = 17 < c_{5+n} = c_{17} = 23$. Hence, vertex $mt(5) = 7$ is a hinge vertex for 5 and 8.



(a) $u = mt(x)$, $y \in Yt(x)$, $b_{mt(y)} < a_{x+n}$, and $d_{mb(y)} < c_{x+n}$. u is a hinge vertex for x and y.



(b) $u = mb(x)$, $y \in Yb(x)$, $b_{mt(y)} < a_{x+n}$, and $d_{mb(y)} < c_{x+n}$ u is a hinge vertex for x and y.

Fig. 3 Illustration of Lemma 4.

4. Algorithm IHV and its Analysis

In this section, we present Algorithm IHV for identifying all hinge vertices of a CTG G. Algorithm IHV takes a CTM M as an input. We formally describe Algorithm IHV and analyze its

Algorithm 1: Identify Hinge Vertices (IHV)

Input: Each trapezoid's corner points a_i, b_i, c_i, d_i for n circular trapezoids in a CTM M.
Output: A set of all hinge vertices HV in the CTG G.

(Step 1)
Construct an ECTM EM from M ;
(Step 2)
Compute $mt(i)$ and $mb(i)$ for $1 \leqslant i \leqslant 2n$;
Compute $smt(i)$ and $smb(i)$ for $1 \leqslant i \leqslant n$;
(Step 3)
Compute $Yt(i)$ and $Yb(i)$ for $1 \leqslant i \leqslant n$;
(Step 4)
Set $HV := \emptyset$;
/* Condition (1) of Lemma 6 */
for $1 \leqslant i \leqslant n$ **do**
  for $j \in Yt(i)$ **do**
    if $b_{mt(j)} < a_{i+n}$ *and* $d_{mb(j)} < c_{x+n}$ **then**
      $HV := HV \cup \{\text{Normalize}(mt(i))\}$;
/* Condition (2) of Lemma 6 */
for $1 \leqslant i \leqslant n$ **do**
  for $j \in Yb(i)$ **do**
    if $b_{mt(j)} < a_{i+n}$ *and* $d_{mb(j)} < c_{x+n}$ **then**
      $HV := HV \cup \{\text{Normalize}(mb(i))\}$;

Function Normalize(v){
if $v > n$ **then return** $v - n$;
else return v;
}

inherent complexity as follows.

Steps 1 to 3 are preparatory steps for identifying all hinge vertices of G. In Step 1, we construct an ECTM EM that can be executed in $O(n)$ time [5]. In Step 2, $mt(i)$, $mb(i)$, $smt(i)$, and $smb(i)$ are computed. This step can be done in $O(n)$ time using prefix computation [1], [4]. Step 3 computes $Yt(i)$ and $Yb(i)$ for $1 \leqslant i \leqslant n$. This step runs in $O(n^2)$ time because the size of $\sum_{i=1}^{n} |Yt(i)|$ is proportional to n^2. In Step 4, we find all hinge vertices by applying Lemma 4 that can be executed in $O(n^2)$ time. Thus, we obtain the following theorem.

Theorem 1 Algorithm IHV identifies all hinge vertices of a CTG G in $O(n^2)$ time by taking its CTM M as an input. □

5. Conclusion

In this study, we proposed Algorithm IHV, which operates in $O(n^2)$ time to identify all hinge vertices on a CTG. Identifying all hinge vertices requires an $O(n^3)$ time by a simple method. Therefore, our algorithm outperforms the simple method. Algorithm IHV partly uses the algorithms of Honma et al. [4]. Reducing the complexity of the algorithm and extending the results to other graphs will be addressed in future research.

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