Electronic Preprint for Journal of Information Processing Vol.25

Regular Paper

Nash Equilibria in Combinatorial Auctions with Item Bidding by Two Bidders

Hiroyuki Umeda[1,a)]   Takao Asano[1,b)]

Received: November 1, 2016, Accepted: May 16, 2017

Abstract: We discuss Nash equilibria in combinatorial auctions with item bidding. Specifically, we give a characterization for the existence of a Nash equilibrium in such a combinatorial auction when valuations by two bidders satisfy symmetric and subadditive properties. Based on this characterization, we can obtain an algorithm for deciding whether a Nash equilibrium exists in such a combinatorial auction.

Keywords: Nash equilibrium, combinatorial auction, second-price auction, subadditivity, symmetric valuation, price of anarchy

In a *combinatorial auction*, m items $M = \{1, 2, \ldots, m\}$ are offered for sale to n bidders $N = \{1, 2, \ldots, n\}$. Each bidder i has a valuation f_i that assigns a nonnegative real number to every subset S of M. The objective is to find a partition $S_1, S_2, \ldots, S_n$ of M among the bidders such that the *social welfare* $\sum_{i=1}^{n} f_i(S_i)$ is maximized. The combinatorial auction problem is sometimes called the *social welfare problem* when we disregard strategic issues on bidders' selfish concerns. VCG (Vickrey-Clarke-Groves) mechanisms optimize the social welfare in a combinatorial auction with selfish bidders. However, it may take exponential time in m and n. Actually, the social welfare problem is shown to be NP-hard by Lehmann, Lehmann and Nisan, even if every valuation f_i ($i \in N$) satisfies submodularity [12] ($f_i : 2^M \rightarrow \mathbf{R}_+$ is *submodular* if $f_i(S \cup T) + f_i(S \cap T) \leq f_i(S) + f_i(T)$ for all $S, T \subseteq M$ and is *subadditive* if $f_i(S \cup T) \leq f_i(S) + f_i(T)$ for all $S, T \subseteq M$).

Therefore approximation algorithms have also been proposed for the social welfare problem (in a combinatorial auction). Since each valuation f_i is defined by 2^m subsets of M, most proposed approximation algorithms are based on oracle models. Two oracle models, the value queries oracle model and the demand queries oracle model, are commonly used. Furthermore, in most proposed approximation algorithms, each valuation f_i is restricted to satisfy some conditions. Two restrictions, submodularity and subadditivity, are commonly used.

For the submodular social welfare problem (i.e., each valuation is submodular) with the value queries oracle model, the following are known. Lehmann, Lehmann and Nisan proposed a $\frac{1}{2}$-approximation algorithm [12]. Khot et al. showed that this problem cannot be approximated to a factor better than $1 - \frac{1}{e}$ unless $\mathbf{P} = \mathbf{NP}$ [10], where e is the base of the natural logarithm. Vondrák proposed a randomized $(1-\frac{1}{e})$-approximation algorithm [15]. Using the more powerful demand queries oracle model, Dobzinski and Schapira proposed an improved $(1 - \frac{1}{e})$-approximation algorithm for the submodular social welfare problem [6].

For the more general subadditive social welfare problem (where each valuation is subadditive), Dobzinski, Nisan, and Schapira proposed an $\Omega(1/\log m)$-approximation algorithm using the value queries oracle model [5]. Using the more powerful demand queries oracle model, Feige proposed a $\frac{1}{2}$-approximation algorithm for the subadditive social welfare problem and also showed that it is NP-hard to approximate to a factor better than $\frac{1}{2}$ [8]. He also proposed a $(1 - \frac{1}{e})$-approximation algorithm for the fractional subadditive (more general than submodular, but more restricted than subadditive) social welfare problem.

As suggested before, the social welfare problem we overviewed above has a central administrator who has the right to make a decision. The administrator makes a decision by collecting valuations and then performing a centralized computation based on approximation algorithms. Recently, however, market-types of social welfare problems have been actively considered in which there are no central administrators. In these market-types of problems, bidders make decisions based on prices and their own valuations, which involves much less central coordination. Here prices can serve to decentralize the markets, as can be seen in socio-economic activities in the real world. If we replace the role of the central administrator by a particular scheme for pricing items, then allowing bidders to follow their own self-interests based on valuations and prices can lead to good decisions. Thus a market-type of social welfare problem, i.e., a combinatorial auction in this paper, is a game theoretical version of traditional social welfare problem. Bidders have incentives to maximize their own payoffs which are determined based on valuations and prices. Thus, there is a competition for items among bidders in a combinatorial auction. In solutions obtained by algorithms for the traditional social welfare problem, some bidders may feel that they are unfairly treated. Thus, a solution is required so that, in some sense,

1 Chuo University, Bunkyo, Tokyo 112–8551, Japan
a) a16.x6wr@g.chuo-u.ac.jp
b) asano@ise.chuo-u.ac.jp

all bidders are satisfied with items they obtain in the solution. This leads to the concept of Nash equilibria (defined below). For a market-type of social welfare problem (i.e., combinatorial auction), a social optimal solution is very nice in a global sense, however, in some cases, bidders may not be satisfied with the solution from their own selfish concerns of view. Thus, a good Nash equilibrium may be required in a combinatorial auction from the stability point of view. Research on computing a good Nash equilibrium and deciding the existence of Nash equilibria is the most fundamental in a combinatorial auction and there has been much recent research on this topic. The *price of anarchy*, the ratio of the value of a social optimal solution to that of a worst Nash equilibrium, plays a similar role as an approximation ratio in approximation algorithms.

For a partition $S_1, S_2, \ldots, S_n$ of M where each bidder i obtains the items in S_i, the price, denoted by $price(S_i)$, is attached to S_i in a combinatorial auction. The *payoff* of bidder i is defined by $f_i(S_i) - price(S_i)$. Each selfish bidder i wants to maximize his/her own payoff. The combinatorial auctions that are used in practice are different from VCG mechanisms. For example, eBay uses an auction in which m items are sold in m independent second-price auctions. Thus, item bidding, as a combinatorial auction scheme, occurs rather "spontaneously" and this type of auction is called a *combinatorial auction with item bidding* [3]. Thus, a bidder's strategy is the m-dimensional vector of bids that the bidder submits in the different single-item auctions. As mentioned above, each selfish bidder i wants to maximize his/her own payoff. A bid profile of all bidders' bid vectors is a *pure Nash equilibrium* if no bidder wants to change his/her own bid vector assuming that any other bidders keep their own bid vectors.

For a combinatorial auction with item bidding where all bidders' valuations are submodular, Christodoulou, Kovács, and Schapira showed that there is always a pure Nash equilibrium and proposed an algorithm for finding a pure Nash equilibrium which is a $\frac{1}{2}$-approximation to the optimal social welfare in polynomial time in n and m [3]. They also showed that the price of anarchy is at most 2. Bhawalkar and Roughgarden considered a combinatorial auction with item bidding where all bidders' valuations are subadditive and showed that every pure Nash equilibrium has a welfare at least $\frac{1}{2}$ of the social optimal welfare (thus, the price of anarchy is at most 2 if a pure Nash equilibrium exists) under the assumption of no "overbidding" [1]. Furthermore, Bhawalkar and Roughgarden suggested the following open problem: "Identify necessary and sufficient conditions for the existence of a pure Nash equilibrium in a combinatorial auction with item bidding and subadditive valuations."

In this paper, we give a necessary and sufficient condition for the existence of a pure Nash equilibrium in a combinatorial auction with item bidding by two bidders when both valuations are subadditive and *symmetric* (i.e., $f_i(S) = f_i(T)$ for all subsets $S, T \subseteq M$ with $|S| = |T|$) under the assumption of no "overbidding." Symmetric valuations were considered in Refs. [12], [13]. An auction with symmetric valuations is called a *multi-unit auction* and several results have been proposed in multi-unit auctions [2], [9], [11]. The auction for the super-long-term Japanese Goverment Bonds is an example of multi-unit auctions [7].

1. Combinatorial Auctions and Item Bidding

As mentioned before, in a combinatorial auction, we are given a set of n bidders $N = \{1, 2, \ldots, n\}$ and a set of m items $M = \{1, 2, \ldots, m\}$. In this paper, we only consider the case of $n = 2$. Thus, $N = \{1, 2\}$. Each bidder $i \in N$ has a valuation f_i which assigns, for each subset $S \subseteq M$, a nonnegative number $f_i(S)$. We denote a valuation profile of two bidders by $\boldsymbol{f} = (f_1, f_2)$. In a combinatorial auction with item bidding, each bidder $i \in N$ has a nonnegative bid $b_i(j)$ for each item $j \in M$ and i's bid is denoted by

$$b_i = (b_i(1), b_i(2), \ldots, b_i(m)).$$

We denote a bid profile of two bidders by $\boldsymbol{b} = (b_1, b_2)$. We also write b_{-i} for each $i \in N$ which is the bid of the bidder different from bidder i in $\boldsymbol{b} = (b_1, b_2)$.

Feasibility of $\boldsymbol{b} = (b_1, b_2)$ (i.e., "no overbidding") is defined as follows.

Definition 1 Let $\boldsymbol{f} = (f_1, f_2)$ and $\boldsymbol{b} = (b_1, b_2)$ be a valuation profile and a bid profile of two bidders, respectively. For each $i \in N$, if there is a subset $S \subseteq M$ such that $\sum_{j \in S} b_i(j) > f_i(S)$ then b_i is called *overbidding*. Otherwise (i.e., $\sum_{j \in S} b_i(j) \leq f_i(S)$ for all subsets $S \subseteq M$), b_i is called *feasible*. If both b_i ($i \in N$) are feasible, then bid profile $\boldsymbol{b} = (b_1, b_2)$ is called *feasible*.

In a combinatorial auction with item bidding (by two or more bidders) [1], [3], the second price auction is used. Thus, items are allocated as follows. In a bid profile $\boldsymbol{b} = (b_1, b_2)$, if bidder $i \in N$ has bid $b_i(j)$ for $j \in M$ which is higher than the other bidder's bid $b_{-i}(j)$, then item j is allocated to i. That is, if $b_i(j) > b_{-i}(j)$ then bidder i will win and obtain $j \in M$. In this case, the *price* of item $j \in M$, denoted by $price(j)$, is defined by the second highest bid among the bids of all bidders (i.e., the lower bid of two bidders). Thus, $price(j) = b_{-i}(j)$. This implies that bidder $i \in N$ can obtain no item $j \in M$ with $b_i(j) < b_{-i}(j)$.

For item $j \in M$, if both bids for j are the same, then exactly one bidder will win and obtain j. In this case, if i wins, then the price of j will be $price(j) = b_{-i}(j) = b_i(j)$. In this paper, we assume that, for each item $j \in M$, at least one bidder's bid is positive. (We can generalize the arguments in this paper for the case where there can be some items j with $b_i(j) = 0$ for both bidders $i \in N$.)

For a bid profile $\boldsymbol{b} = (b_1, b_2)$ and for each bidder $i \in N$, let $X_i(\boldsymbol{b})$ be the set of items (i wins and) allocated to i. Then $X_i(\boldsymbol{b}) \subseteq \{j \in M \,|\, b_i(j) = \max\{b_1(j), b_2(j)\}\}$ by the argument above. The *payoff* $u_i(X_i(\boldsymbol{b}))$ of bidder $i \in N$ for $X_i(\boldsymbol{b})$ is defined by

$$u_i(X_i(\boldsymbol{b})) = f_i(X_i(\boldsymbol{b})) - \sum_{j \in X_i(\boldsymbol{b})} price(j).$$

Nash equilibrium is defined as follows. For a feasible bid profile $\boldsymbol{b} = (b_1, b_2)$, let $X_i(\boldsymbol{b})$ be the set of items allocated to bidder i. If only bidder 1 changes bid b_1 to b'_1, then the resultant bid profile of both bidders becomes $\boldsymbol{b}'_1 = (b'_1, b_2)$. Similarly, if only bidder 2 changes bid b_2 to b'_2, then the resultant bid profile of both bidders becomes $\boldsymbol{b}'_2 = (b_1, b'_2)$.

For convenience, if only $i \in N$ changes bid b_i to b'_i, the resultant bid profile of both bidders will be written to be $\boldsymbol{b}'_i = (b'_i, b_{-i})$. Furthermore, let $X_i(\boldsymbol{b}'_i)$ be the set of items allocated to i in bid profile $\boldsymbol{b}'_i$. Suppose that, even if bidder i changes bid b_i to arbitrary feasible bid b'_i, the i's payoff $u(X_i(\boldsymbol{b}'_i))$ will not become strictly higher than $u_i(X_i(\boldsymbol{b}))$. In this case, i does not want to change the bid b_i in $\boldsymbol{b} = (b_1, b_2)$. If no bidder $i \in N$ wants to change the bid b_i in the feasible bid profile $\boldsymbol{b} = (b_1, b_2)$, that is, if $u_i(X_i(\boldsymbol{b})) \geq u_i(X_i(\boldsymbol{b}'_i))$ for both bidders $i \in N$ and for all feasible bid profiles $\boldsymbol{b}'_i = (b'_i, b_{-i})$ (and $X_i(\boldsymbol{b}'_i)$) defined above, then $\boldsymbol{b} = (b_1, b_2)$ is called a *pure Nash equilibrium* (*Nash equilibrium* in short).

In this paper, we make the following assumptions on each valuation f_i ($i \in N$):

(i) (normalization) $f_i(\emptyset) = 0$,

(ii) (monotonicity) $0 < f_i(S) \leq f_i(T)$ for all subsets $S, T \subseteq M$ with $\emptyset \neq S \subset T$,

(iii) (subadditivity) $f_i(S \cup T) \leq f_i(S) + f_i(T)$ for all subsets $S, T \subseteq M$, and

(iv) (symmetry) $f_i(S) = f_i(T)$ for all subsets $S, T \subseteq M$ with $|S| = |T|$.

Thus, we can define $v_i : \{0, 1, 2, \ldots, m\} \to \mathbf{R}_+$ by $v_i(|S|) = f_i(S)$ for any subset $S \subseteq M$. Then v_i is well defined by symmetry of f_i in the assumption above. Using this symmetric valuation v_i, we can write (i), (ii) and (iii) in the assumption above and the payoff as follows.

Assumption 1 For each $i \in N$, v_i in $\boldsymbol{v} = (v_1, v_2)$ satisfies the following:

1. (Normalization) $v_i(0) = 0$.
2. (Monotonicity) $0 < v_i(k) \leq v_i(k')$ for all k, k' with $1 \leq k < k' \leq m$.
3. (Subadditivity) $v_i(\min\{k + k', m\}) \leq v_i(k) + v_i(k')$ for all k, k' with $1 \leq k, k' \leq m$.

Definition 2 Let $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{b} = (b_1, b_2)$ be a valuation profile and a bid profile of two bidders, respectively. The payoff $u_i(X_i(\boldsymbol{b}))$ of bidder $i \in N$ is then defined by

$$u_i(X_i(\boldsymbol{b})) = v_i(|X_i(\boldsymbol{b})|) - \sum_{j \in X_i(\boldsymbol{b})} price(j). \tag{1}$$

Since we will give a characterization of the existence of Nash equilibria under the assumption of no "overbidding", we first consider the feasibility of a bid profile.

Definition 3 For each bidder $i \in N$, let v_i in $\boldsymbol{v} = (v_1, v_2)$ be a valuation satisfying Assumption 1 and let w_i be a function with $w_i(0) = 0$ and, for each $k_i \in \{1, 2, \ldots, m\}$,

$$w_i(k_i) = k_i \min\left\{v_i(1), \frac{v_i(2)}{2}, \ldots, \frac{v_i(k_i - 1)}{k_i - 1}, \frac{v_i(k_i)}{k_i}\right\}. \tag{2}$$

Then each w_i ($i \in N$) has the following properties.

First we have

$$\begin{aligned} &w_i(0) = v_i(0), \quad w_i(1) = v_i(1), \\ &w_i(k_i) \leq v_i(k_i) \quad (k_i = 2, 3, \ldots, m), \end{aligned} \tag{3}$$

since $w_i(0) = v_i(0) = 0$ and, for all $k_i \in \{1, 2, \ldots, m\}$,

$$\frac{w_i(k_i)}{k_i} = \min\left\{v_i(1), \frac{v_i(2)}{2}, \ldots, \frac{v_i(k_i - 1)}{k_i - 1}, \frac{v_i(k_i)}{k_i}\right\} \leq \frac{v_i(k_i)}{k_i}$$

by the definition of $w_i(k_i)$. Similarly, we have

$$w_i(k_i) = k_i \min\left\{\frac{w_i(k_i - 1)}{k_i - 1}, \frac{v_i(k_i)}{k_i}\right\} \quad (k_i = 2, 3, \ldots, m), \tag{4}$$

and

$$w_i(1) \geq \frac{w_i(2)}{2} \geq \cdots \geq \frac{w_i(m)}{m}, \tag{5}$$

since

$$\begin{aligned} \frac{w_i(k_i)}{k_i} &= \min\left\{v_i(1), \frac{v_i(2)}{2}, \ldots, \frac{v_i(k_i - 1)}{k_i - 1}, \frac{v_i(k_i)}{k_i}\right\} \\ &= \min\left\{\min\left\{v_i(1), \frac{v_i(2)}{2}, \ldots, \frac{v_i(k_i - 1)}{k_i - 1}\right\}, \frac{v_i(k_i)}{k_i}\right\} \\ &= \min\left\{\frac{w_i(k_i - 1)}{k_i - 1}, \frac{v_i(k_i)}{k_i}\right\} \leq \frac{w_i(k_i - 1)}{k_i - 1} \end{aligned}$$

for all $k_i \in \{2, 3, \ldots, m\}$. Furthermore, if $w_i(k_i) < v_i(k_i)$ then, by Eq. (4), we have

$$w_i(k_i) = \frac{k_i}{k_i - 1} w_i(k_i - 1) \quad (k_i = 2, 3, \ldots, m). \tag{6}$$

Finally, we have

$$w_i(1) \leq w_i(2) \leq \cdots \leq w_i(m), \tag{7}$$

since $w_i(k_i) = \frac{k_i}{k_i - 1} w_i(k_i - 1)$ or $w_i(k_i) = v_i(k_i)$ for each $k_i \in \{2, 3, \ldots, m\}$, and

$$w_i(k_i) = w_i(k_i - 1) + \frac{1}{k_i - 1} w_i(k_i - 1) \geq w_i(k_i - 1),$$

or

$$w_i(k_i) = v_i(k_i) \geq v_i(k_i - 1) \geq w_i(k_i - 1)$$

by the monotonicity of v_i and $w_i(k_i - 1) \leq v_i(k_i - 1)$ in Eq. (3).

Throughout this paper, we use the following assumption.

Assumption 2 For each $i \in N$, v_i in $\boldsymbol{v} = (v_1, v_2)$ satisfies Assumption 1 and w_i in $\boldsymbol{w} = (w_1, w_2)$ is the function defined in Definition 3.

Then we have the following theorem, which will play a central role in the proof of the main result in this paper.

Theorem 1 For any bid profile $\boldsymbol{b} = (b_1, b_2)$ and for each bidder $i \in N$, let the elements of each $b_i = (b_i(1), b_i(2), \ldots, b_i(m))$ be ordered in nondecreasing order by using a permutation π_i on $M = \{1, 2, \ldots, m\}$ as follows:

$$b_i(\pi_i(1)) \leq b_i(\pi_i(2)) \leq \cdots \leq b_i(\pi_i(m)). \tag{8}$$

Then bidder i's bid $b_i = (b_i(1), b_i(2), \ldots, b_i(m))$ is feasible if and only if

$$\sum_{j=m-k_i+1}^{m} b_i(\pi_i(j)) \leq w_i(k_i) \quad (k_i = 1, 2, \ldots, m), \tag{9}$$

that is, the sum of largest k_i bids in $b_i = (b_i(1), b_i(2), \ldots, b_i(m))$ is at most $w_i(k_i)$ for every $k_i \in \{1, 2, \ldots, m\}$.

Thus, the bid profile $\boldsymbol{b} = (b_1, b_2)$ is feasible if and only if Eq. (9) holds for every $i \in N$ and every $k_i \in \{1, 2, \ldots, m\}$.

Proof: (Sufficiency) Suppose Eq. (9) holds for $i \in N$ and every $k_i \in \{1, 2, \ldots, m\}$. For any subset $S_i \subseteq M$, let $k_i = |S_i|$. Then, in bid vector $b_i = (b_i(1), b_i(2), \ldots, b_i(m))$, the sum of the k_i bids of items in S_i is at most the sum of the largest k_i bids, i.e.,

$$\sum_{j \in S_i} b_i(j) \leq \sum_{j=m-k_i+1}^{m} b_i(\pi_i(j))$$

by Eq. (8). Thus, by Eqs. (9) and (3) (i.e., $w_i(k_i) \le v_i(k_i)$), we have

$$\sum_{j\in S_i} b_i(j) \le \sum_{j=m-k_i+1}^{m} b_i(\pi_i(j)) \le w_i(k_i) \le v_i(k_i)$$

and $b_i = (b_i(1), b_i(2), \dots, b_i(m))$ is feasible by Definition 1.

(Necessity) Suppose that, for $i \in N$, bid vector $b_i = (b_i(1), b_i(2), \dots, b_i(m))$ is feasible. Then, for any subset $S_i \subseteq M$ with $k_i = |S_i| \in \{1, 2, \dots, m\}$,

$$\sum_{j\in S_i} b_i(j) \le v_i(k_i).$$

Let $S_i = \{\pi_i(m-k_i+1), \pi_i(m-k_i+2), \dots, \pi_i(m)\}$. Then we have

$$\sum_{j=m-k_i+1}^{m} b_i(\pi_i(j)) \le v_i(k_i) \tag{10}$$

for all $k_i \in \{1, 2, \dots, m\}$ and

$$\frac{1}{k_i}\sum_{j=m-k_i+1}^{m} b_i(\pi_i(j)) \le \frac{v_i(k_i)}{k_i}. \tag{11}$$

Furthermore, since, in $b_i = (b_i(1), b_i(2), \dots, b_i(m))$, for each $k'_i \in \{2, 3, \dots, k_i\}$, the average of the largest k'_i bids is at most the average of the largest $k'_i - 1$ bids, we have

$$\frac{1}{k'_i}\sum_{j=m-k'_i+1}^{m} b_i(\pi_i(j)) \le \frac{1}{k'_i-1}\sum_{j=m-k'_i+2}^{m} b_i(\pi_i(j))$$

by Eq. (8). Thus, we have

$$\begin{aligned}\frac{1}{k_i}\sum_{j=m-k_i+1}^{m} b_i(\pi_i(j)) &\le \frac{1}{k_i-1}\sum_{j=m-k_i+2}^{m} b_i(\pi_i(j))\\ &\le \cdots\\ &\le \frac{1}{2}\sum_{j=m-1}^{m} b_i(\pi_i(j))\\ &\le b_i(\pi_i(m)).\end{aligned}$$

By combining this with Eq. (11), we have

$$\frac{1}{k_i}\sum_{j=m-k_i+1}^{m} b_i(\pi_i(j)) \le \frac{1}{k'_i}\sum_{j=m-k'_i+1}^{m} b_i(\pi_i(j)) \le \frac{v_i(k'_i)}{k'_i} \tag{12}$$

for any $k'_i = 1, 2, \dots, k_i$. Thus, by combining this with the definition of w_i in Eq. (2), we have

$$\frac{1}{k_i}\sum_{j=m-k_i+1}^{m} b_i(\pi_i(j)) \le \min\left\{v_i(1), \frac{v_i(2)}{2}, \dots, \frac{v_i(k_i)}{k_i}\right\} = \frac{w_i(k_i)}{k_i}$$

and Eq. (9) for every $k_i \in \{1, 2, \dots, m\}$. □

By Theorem 1, we have the following corollary.

Corollary 1 A bid profile $\boldsymbol{b} = (b_1, b_2)$ of two bidders can be determined as to whether or not it is feasible in $O(m)$ time, if the elements of b_i for both $i \in N$ are sorted as in Eq. (8) in advance.

2. Existence of Nash Equilibria

In this section, we first give some technical terms and lemmas for explaining the main result in this paper, and then give its proof.

Definition 4 Let $P = (M_1, M_2)$ be a partition of M into two subsets, i.e., $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = M$. For each $i \in N$, let $d_i = (d_i(1), d_i(2), \dots, d_i(m))$ be defined by

$$d_i(j) = \begin{cases} \frac{w_i(|M_i|)}{|M_i|} & (\text{if } j \in M_i),\\ 0 & (\text{otherwise}).\end{cases} \tag{13}$$

Then we have the following lemma and the main result.

Lemma 1 The bid profile $\boldsymbol{d} = (d_1, d_2)$ defined by Eq. (13) is feasible and $X_i(\boldsymbol{d}) = M_i$ for each $i \in N$ (i.e., the set of items (i wins and) allocated to i in $\boldsymbol{d} = (d_1, d_2)$ is M_i).

Proof: Clearly, $X_i(\boldsymbol{d}) = M_i$, since $P = (M_1, M_2)$ is the partition of M into two subsets and $d_{-i}(j) = 0$ for each $j \in M_i$ by Eq. (13).

We will prove that $\boldsymbol{d} = (d_1, d_2)$ is feasible. To clarify the argument, we will give a proof for $i = 1$. By symmetry, a proof for $i = 2$ is also obtained.

Let $k_1 = |M_1|$ and let the elements of bid vector d_1 be ordered by using some permutation σ_1 on M as follows:

$$d_1(\sigma_1(1)) \le d_1(\sigma_1(2)) \le \cdots \le d_1(\sigma_1(m)). \tag{14}$$

Then,

$$\begin{aligned}&d_1(\sigma_1(1)) = d_1(\sigma_1(2)) = \cdots = d_1(\sigma_1(m-k_1)) = 0,\\ &d_1(\sigma_1(m-k_1+1)) = \cdots = d_1(\sigma_1(m)) = \tfrac{w_1(k_1)}{k_1},\end{aligned} \tag{15}$$

and, for every k' with $1 \le k' \le k_1$, we have

$$\sum_{j=m-k_1+1}^{m-k_1+k'} d_1(\sigma_1(j)) = \frac{k'}{k_1} w_1(k_1). \tag{16}$$

The feasibility of d_1 can be obtained as follows. Since $w_1(1) \ge \frac{w_1(2)}{2} \ge \cdots \ge \frac{w_1(m)}{m}$ by Eq. (5), we have

$$\sum_{j=m-k''+1}^{m} d_1(\sigma_1(j)) = \frac{k''}{k_1} w_1(k_1) \le w_1(k'') \quad \text{for each } k'' \le k_1.$$

On the other hand, if $k'' > k_1$, then, since $d_1(\sigma_1(m-k''+1)) = d_1(\sigma_1(m-k''+2)) = \cdots = d_1(\sigma_1(m-k_1)) = 0$, we have

$$\sum_{j=m-k''+1}^{m} d_1(\sigma_1(j)) = \sum_{j=m-k_1+1}^{m} d_1(\sigma_1(j)) = w_1(k_1) \le w_1(k''),$$

where the last inequality is obtained from Eq. (7). Thus, d_1 is feasible by Theorem 1. □

Theorem 2 A valuation profile $\boldsymbol{v} = (v_1, v_2)$ satisfying Assumption 1 has a Nash equilibrium if and only if there is a partition $P = (M_1, M_2)$ of M into two subsets such that the feasible bid profile $\boldsymbol{d} = (d_1, d_2)$ of two bidders defined by Eq. (13) is a Nash equilibrium.

Before giving a proof of Theorem 2, we give simple examples.

Example 1. Let $N = \{1, 2\}$, $M = \{1, 2, 3\}$ and let

$$\begin{aligned}&v_1(0) = 0, \quad v_1(1) = v_1(2) = 3, \quad v_1(3) = 6,\\ &v_2(0) = 0, \quad v_2(1) = v_2(2) = 2, \quad v_2(3) = 4.\end{aligned} \tag{17}$$

Then each v_i ($i \in N$) satisfies Assumption 1, and

$$\begin{aligned}&w_1(0) = 0, \quad w_1(1) = 3, \quad \tfrac{w_1(2)}{2} = 1.5, \quad \tfrac{w_1(3)}{3} = 1.5,\\ &w_2(0) = 0, \quad w_2(1) = 2, \quad \tfrac{w_2(2)}{2} = 1, \quad \tfrac{w_2(3)}{3} = 1.\end{aligned}$$

In this case, by Theorem 2, there is no Nash equilibrium which can be shown as follows.

By symmetry, we can assume there are only four distinct partitions $P^{(k)} = (M_1^{(k)}, M_2^{(k)})$ of M ($k = 0, 1, 2, 3$), where

$M_1^{(k)} = \{j \in M \mid j \le k\}$ and $M_2^{(k)} = M - M_1^{(k)}$. Thus, $M_1^{(0)} = \emptyset$, $M_1^{(1)} = \{1\}$, $M_1^{(2)} = \{1,2\}$, $M_1^{(3)} = \{1,2,3\}$. Corresponding to the partition $P^{(k)} = (M_1^{(k)}, M_2^{(k)})$ of M, the feasible bid profiles $\boldsymbol{d}^{(k)} = (d_1^{(k)}, d_2^{(k)})$ defined by Eq. (13) are

$$\begin{array}{ll} d_1^{(0)} = (0,0,0), & d_2^{(0)} = (1,1,1), \\ d_1^{(1)} = (3,0,0), & d_2^{(1)} = (0,1,1), \\ d_1^{(2)} = (1.5,1.5,0), & d_2^{(2)} = (0,0,2), \\ d_1^{(3)} = (1.5,1.5,1.5), & d_2^{(3)} = (0,0,0). \end{array}$$

Thus, $X_i(\boldsymbol{d}^{(k)}) = M_i^{(k)}$ for all $k = 0,1,2,3$ and $i = 1,2$. Now let bidder 1 change bid $d_1^{(k)}$ to $d_1'^{(k)}$ for $k = 0,1,2$ as follows:

$$d_1'^{(0)} = (3,0,0), \quad d_1'^{(1)} = (0.8,1.1,1.1), \quad d_1'^{(2)} = (0.4,0.4,2.2).$$

It can then be easily seen that bidder 1 can improve his payoff in the feasible bid profile $\boldsymbol{d'}^{(k)} = (d_1'^{(k)}, d_2^{(k)})$. Actually, $X_1(\boldsymbol{d'}^{(k)})$ and the payoff $u_1(X_1(\boldsymbol{d'}^{(k)}))$ for $k = 0,1,2$ become as follows.

$$X_1(\boldsymbol{d'}^{(0)}) = \{1\},\ u_1(X_1(\boldsymbol{d'}^{(0)})) = 3 - 1 > u_1(X_1(\boldsymbol{d}^{(0)})) = 0,$$

and, for $k = 1,2$,

$$X_1(\boldsymbol{d'}^{(k)}) = \{1,2,3\},\ \ u_1(X_1(\boldsymbol{d'}^{(k)})) = 6 - 2 > u_1(X_1(\boldsymbol{d}^{(k)})) = 3.$$

Similarly, for $k = 3$, if bidder 2 changes bid $d_2^{(3)}$ to $d_2'^{(3)} = (0,0,2)$ then bidder 2 can improve her payoff in the feasible bid profile $\boldsymbol{d}_2'^{(3)} = (d_1^{(3)}, d_2'^{(3)})$ from 0 to 0.5. Actually,

$$X_2(\boldsymbol{d'}^{(3)}) = \{3\},\ \ u_2(X_2(\boldsymbol{d'}^{(3)})) = 2 - 1.5 > u_2(X_2(\boldsymbol{d}^{(3)})) = 0.$$

By Theorem 2, the valuation profile $\boldsymbol{v} = (v_1, v_2)$ in Eq. (17) has no Nash equilibrium. □

Example 2. Let $N = \{1,2\}$, $M = \{1,2,3,4,5\}$ and

$$v_i(0) = 0, \quad v_i(1) = v_i(2) = v_i(3) = 3, \quad v_i(4) = v_i(5) = 6$$

for each $i \in N$. Then each v_i ($i \in N$) satisfies Assumption 1 and

$$w_i(0) = 0,\ \ w_i(1) = 3,\ \ \tfrac{w_i(2)}{2} = 1.5,\ \ \tfrac{w_i(3)}{3} = \tfrac{w_i(4)}{4} = \tfrac{w_i(5)}{5} = 1.$$

As in Example 1, for $k = 3$ with $M_1^{(3)} = \{1,2,3\}$ and $M_2^{(3)} = \{4,5\}$, $\boldsymbol{d}^{(3)} = (d_1^{(3)}, d_2^{(3)})$ defined by Eq. (13) is

$$d_1^{(3)} = (1,1,1,0,0), \qquad d_2^{(3)} = (0,0,0,1.5,1.5).$$

The feasible bid profile $\boldsymbol{d}^{(3)} = (d_1^{(3)}, d_2^{(3)})$ with $X_1^{(3)}(\boldsymbol{d}^{(3)}) = M_1^{(3)} = \{1,2,3\}$ and $X_2^{(3)}(\boldsymbol{d}^{(3)}) = M_2^{(3)} = \{4,5\}$ is not a Nash equilibrium: if bidder 2 changes bid $d_2^{(3)}$ to $d_2'^{(3)} = (0,1.2,1.2,0.3,0.3)$ then

$$X_2(\boldsymbol{d'}^{(3)}) = \{2,3,4,5\},\ \ u_2(X_2(\boldsymbol{d'}^{(3)})) = 6 - 2 > u_2(X_2(\boldsymbol{d}^{(3)})) = 3$$

and she can improve her payoff in the feasible bid profile $\boldsymbol{d}_2'^{(3)} = (d_1^{(3)}, d_2'^{(3)})$ from 3 to 4.

However, $\boldsymbol{d}^{(1)} = (d_1^{(1)}, d_2^{(1)})$ with

$$\begin{array}{ll} d_1^{(1)} = (3,0,0,0,0), & d_2^{(1)} = (0,1,1,1,1) \\ (M_1^{(1)} = \{1\}, & M_2^{(1)} = \{2,3,4,5\}) \\ (u_1(X_1(\boldsymbol{d}^{(1)})) = u_1(M_1^{(1)}) = 3 & u_2(X_2(\boldsymbol{d}^{(1)})) = u_2(M_2^{(1)}) = 6) \end{array}$$

and $\boldsymbol{d}^{(4)} = (d_1^{(4)}, d_2^{(4)})$ with

$$\begin{array}{ll} d_1^{(4)} = (1,1,1,1,0), & d_2^{(4)} = (0,0,0,0,3) \\ (M_1^{(4)} = \{1,2,3,4\}, & M_2^{(4)} = \{5\}) \\ (u_1(X_1(\boldsymbol{d}^{(4)})) = u_1(M_1^{(4)}) = 6 & u_2(X_2(\boldsymbol{d}^{(4)})) = u_2(M_2^{(4)}) = 3) \end{array}$$

are both Nash equilibria. □

We give an outline of the proof of Theorem 2 using the following notation.

Definition 5 For a bid profile $\boldsymbol{b} = (b_1, b_2)$, let $Y_i = X_i(\boldsymbol{b})$ be the set of items allocated to bidder i and let $y_i = |Y_i|$ ($i = 1,2$). Then, clearly $P = (Y_1, Y_2)$ is a partition of M into two subsets, i.e., $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 = M$ and $y_1 + y_2 = m$. For each $i \in N$, let $c_i = (c_i(1), c_i(2), \ldots, c_i(m))$ be defined by

$$c_i(j) = \begin{cases} \frac{w_i(y_i)}{y_i} & (\text{if } j \in Y_i), \\ 0 & (\text{otherwise}). \end{cases} \tag{18}$$

If we let $M_i = Y_i$ then $c_i = (c_i(1), c_i(2), \ldots, c_i(m))$ is the bid d_i defined by Eq. (13) in Definition 4. Thus, we have the following lemma (we will give its proof in Section 4).

Lemma 2 In a valuation profile $\boldsymbol{v} = (v_1, v_2)$, if a feasible bid profile $\boldsymbol{b} = (b_1, b_2)$ is a Nash equilibrium, then $\boldsymbol{c} = (c_1, c_2)$ defined by Eq. (18) is also a Nash equilibrium.

Using this lemma, we can easily prove Theorem 2 as follows.

Proof of Theorem 2: (Necessity) If there is a feasible bid profile $\boldsymbol{b} = (b_1, b_2)$ which is a Nash equilibrium, then, by Lemma 2, $\boldsymbol{c} = (c_1, c_2)$ defined by Eq. (18) is also a Nash equilibrium. Thus, by setting $M_i = Y_i$ and $d_i = c_i$ for each $i \in N$, we have a desired partition of M into two subsets and the necessity for Theorem 2 is proved.

(Sufficiency) If there is a partition $P = (M_1, M_2)$ of M into two subsets such that the feasible bid profile $\boldsymbol{d} = (d_1, d_2)$ of two bidders defined by Eq. (13) is a Nash equilibrium, then it is clearly a Nash equilibrium in the valuation profile $\boldsymbol{v} = (v_1, v_2)$. □

3. Basic Properties of a Feasible Bid Profile *b*

To prove Lemma 2, we need the concept of prestability and stability. For a bid vector $b = (b(1), b(2), \ldots, b(m))$, let $b(j \leftrightarrow j')$ be the bid vector obtained from b by swapping $b(j)$ and $b(j')$. For example, if $b = (b(1), b(2), b(3))$ then $b(1 \leftrightarrow 3) = (b(3), b(2), b(1))$.

Definition 6 Let $\boldsymbol{b} = (b_1, b_2)$ be a feasible bid profile. For $i \in N$, let $X_i(\boldsymbol{b})$ be the set of items allocated to bidder i. Then b_i is called *prestable* in $\boldsymbol{b} = (b_1, b_2)$, if

$$u_i(X_i(\boldsymbol{b})) \ge u_i(X_i(\boldsymbol{b}_i'))$$

for all feasible bid profiles $\boldsymbol{b}_i' = (b_i', b_{-i})$ with $b_i' = b_i(j \leftrightarrow j')$ ($1 \le j \ne j' \le m$) and $|X_i(\boldsymbol{b}_i')| = |X_i(\boldsymbol{b})|$. Otherwise, b_i is called *unprestable* in $\boldsymbol{b} = (b_1, b_2)$. If both b_i ($i \in N$) are prestable in $\boldsymbol{b} = (b_1, b_2)$, then $\boldsymbol{b} = (b_1, b_2)$ is called *prestable*.

Note that, by the definition, a prestable bid profile $\boldsymbol{b} = (b_1, b_2)$ is always feasible and that, if b_i is unprestable in a feasible bid profile $\boldsymbol{b} = (b_1, b_2)$, then $\boldsymbol{b} = (b_1, b_2)$ is not a Nash equilibrium. Furthermore, as mentioned in Corollary 1, we can determine whether a given bid profile $\boldsymbol{b} = (b_1, b_2)$ is feasible or not in $O(m)$ time (excluding $O(m \log m)$ time for sorting). We can also determine whether a given feasible bid profile $\boldsymbol{b} = (b_1, b_2)$ is prestable or not in $O(m^2)$ time based on Definition 6.

Furthermore, we have the following lemma (see Appendix for its proof).

Lemma 3 For a prestable bid profile $\boldsymbol{b} = (b_1, b_2)$, let $Y_i = X_i(\boldsymbol{b})$ be the set of items allocated to bidder $i \in N$ and let $y_i = |Y_i|$. Then we can always choose a permutation π_{-i} on

$M = \{1, 2, \ldots, m\}$ appropriately such that

$$b_{-i}(\pi_{-i}(1)) \leq b_{-i}(\pi_{-i}(2)) \leq \cdots \leq b_{-i}(\pi_{-i}(m)), \quad \text{and} \tag{19}$$

$$Y_i = \{\pi_{-i}(1), \pi_{-i}(2), \ldots, \pi_{-i}(y_i)\}. \tag{20}$$

Based on these observations, from now on, we will consider only a prestable bid profile $\boldsymbol{b} = (b_1, b_2)$, and assume that, for each $i \in N$, the set of items $Y_i = X_i(\boldsymbol{b})$ allocated to bidder i and a permutation π_{-i} satisfy Eqs. (19) and (20).

Definition 7 Let $\boldsymbol{b} = (b_1, b_2)$ be a prestable bid profile satisfying Eqs. (19) and (20), where $Y_i = X_i(\boldsymbol{b})$ is the set of items allocated to bidder $i \in N$ and $y_i = |Y_i|$ (thus, $P = (Y_1, Y_2)$ is a partition of M into two subsets and $y_1 + y_2 = m$). For $i \in N$, if

$$v_i(y_i + k) - v_i(y_i) \leq \sum_{j=1}^{k} b_{-i}(\pi_{-i}(y_i + j)) \tag{21}$$

for all k with $1 \leq k \leq m - y_i$ and

$$v_i(y_i - k') \leq v_i(y_i) - \sum_{j=0}^{k'-1} b_{-i}(\pi_{-i}(y_i - j)) \tag{22}$$

for all k' with $1 \leq k' \leq y_i$, then b_{-i} is called *stable* in $\boldsymbol{b} = (b_1, b_2)$, and otherwise it is called *unstable*. If both b_1 and b_2 are stable in $\boldsymbol{b} = (b_1, b_2)$, then $\boldsymbol{b} = (b_1, b_2)$ is called *stable*.

Note that, if a prestable bid profile $\boldsymbol{b} = (b_1, b_2)$ is stable, then even if bidder i changes b_i to b'_i which may or may not be feasible, the payoff of bidder i will not increase in (b'_i, b_{-i}), which can be shown by Eqs. (21), (22) and the definition of the payoff of bidder i. Thus, if a prestable bid profile $\boldsymbol{b} = (b_1, b_2)$ is stable, then it is a Nash equilibrium. The converse is also true and we have the following theorem (see Appendix for its proof).

Theorem 3 A prestable bid profile $\boldsymbol{b} = (b_1, b_2)$ of two bidders with $(X_1(\boldsymbol{b}), X_2(\boldsymbol{b}))$ satisfying Eqs. (19) and (20), where $X_i(\boldsymbol{b})$ is the set of items allocated to bidder i with $|X_i(\boldsymbol{b})| = y_i$ $(i \in N)$, is a Nash equilibrium if and only if $\boldsymbol{b} = (b_1, b_2)$ is stable.

By Theorem 3 (and Definition 7), we can determine whether a prestable bid profile $\boldsymbol{b} = (b_1, b_2)$ is a Nash equilibrium or not in $O(m)$ time. Furthermore, by combining this with Theorem 2, we have the following corollary.

Corollary 2 We can determine whether a valuation profile $\boldsymbol{v} = (v_1, v_2)$ satisfying Assumption 1 has a Nash equilibrium or not in $O(m^2)$ time and, if it has, we can find such a Nash equilibrium in $O(m^2)$ time.

From this theorem, we can also obtain the proof of Lemma 2 without much difficulty.

4. Proof of Lemma 2

Finally, we study properties of $\boldsymbol{c} = (c_1, c_2)$ defined by Eq. (18) and complete the proof of Lemma 2. Note that, for each $i \in N$, $Y_i = X_i(\boldsymbol{b})$ and $y_i = |Y_i|$. For each $i \in N$, let $X_i(\boldsymbol{c})$ be the set of items allocated to bidder i in $\boldsymbol{c} = (c_1, c_2)$. Thus, we have

$$X_i(\boldsymbol{c}) = Y_i, \quad |X_i(\boldsymbol{c})| = y_i \quad (i \in N), \tag{23}$$

$$y_1 + y_2 = m, \tag{24}$$

by the definition of $\boldsymbol{c} = (c_1, c_2)$ in Eq. (18). We order the items not contained in $X_i(\boldsymbol{c})$ in nondecreasing order in b_{-i}. Thus, we can assume that the items in $X_{-i}(\boldsymbol{c}) = Y_{-i} = M - X_i(\boldsymbol{c}) = \{j_1^{(-i)}, j_2^{(-i)}, \ldots, j_{m-y_i}^{(-i)}\}$ are ordered as follows:

$$b_{-i}(j_1^{(-i)}) \leq b_{-i}(j_2^{(-i)}) \leq \cdots \leq b_{-i}(j_{m-y_i}^{(-i)}). \tag{25}$$

Similarly, we consider c_{-i} and order the items in $Y_{-i} = \{j_1^{(-i)}, j_2^{(-i)}, \ldots, j_{m-y_i}^{(-i)}\}$ in nondecreasing order in c_{-i} by using a permutation σ_{-i} as follows:

$$c_{-i}(\sigma_{-i}(j_1^{(-i)})) \leq c_{-i}(\sigma_{-i}(j_2^{(-i)})) \leq \cdots \leq c_{-i}(\sigma_{-i}(j_{m-y_i}^{(-i)})). \tag{26}$$

Then the following lemma holds (see Appendix for its proof).

Lemma 4 Let $i \in N$ and $k_i \leq m - y_i$ be a nonnegative integer. Then

$$\sum_{h=1}^{k_i} b_{-i}(j_h^{(-i)}) \leq \sum_{h=1}^{k_i} c_{-i}(\sigma_{-i}(j_h^{(-i)})), \tag{27}$$

i.e., the sum of the k_i smallest bids for the items in $Y_{-i} = \{j_1^{(-i)}, j_2^{(-i)}, \ldots, j_{m-y_i}^{(-i)}\}$ in b_{-i} is at most the sum of the k_i smallest bids for the items in Y_{-i} in c_{-i}.

By using this lemma and Theorem 3, we can obtain the proof of Lemma 2.

Proof of Lemma 2: Suppose to the contrary that, $\boldsymbol{c} = (c_1, c_2)$ is not a Nash equilibrium even though $\boldsymbol{b} = (b_1, b_2)$ is a Nash equilibrium. Then there would be a bidder $i \in N$ such that if bidder i changes the bid then in the resulting bid profile bidder i will obtain a greater payoff. Thus, by symmetry, we can assume $i = 1$ and bidder 1 changes c_1 to c'_1 so that his payoff $u_1(X_1(\boldsymbol{c}'))$ of $X_1(\boldsymbol{c}')$ of items allocated to him in the bid profile $\boldsymbol{c}' = (c'_1, c_2)$ is greater than his payoff $u_1(X_1(c_1))$ in the bid profile $\boldsymbol{c} = (c_1, c_2)$. Thus, we have

$$\begin{aligned} u_1(X_1(\boldsymbol{c}')) &= v_1(|X_1(\boldsymbol{c}')|) - \sum_{j \in X_1(\boldsymbol{c}')} c_2(j) \\ &> u_1(X_1(\boldsymbol{c})) = v_1(y_1). \end{aligned} \tag{28}$$

We will show below that this leads to a contradiction.

We can assume $X_1(\boldsymbol{c}') \supseteq X_1(\boldsymbol{c})$. Actually, for every $j \in X_1(\boldsymbol{c})$, we have $c_2(j) = 0$ by the definition of $\boldsymbol{c}$, and, by the monotonicity of v_1, we can modify $c'_1(j)$ so that $X_1(\boldsymbol{c}')$ may include j without decreasing the value of $u_1(X_1(\boldsymbol{c}'))$ (by decreasing a bid for some item in $X_1(\boldsymbol{c}') - X_1(\boldsymbol{c})$ if necessary). Now let $Y'_2 = X_1(\boldsymbol{c}') - X_1(\boldsymbol{c}) \subseteq Y_2$ and $k_1 = |Y'_2|$. Then we can write

$$u_1(X_1(\boldsymbol{c}')) = v_1(|X_1(\boldsymbol{c}')|) - \sum_{j \in Y'_2} c_2(j). \tag{29}$$

Since $X_1(\boldsymbol{c}') = X_1(\boldsymbol{c}) \cup Y'_2$ and $|X_1(\boldsymbol{c}')| = |X_1(c_1)| + |Y'_2| = y_1 + k_1$, by applying Lemma 4, we have

$$\begin{aligned} u_1(X_1(\boldsymbol{c}')) &= v_1(|X_1(\boldsymbol{c}')|) - \sum_{j \in Y'_2} c_2(j) \\ &\leq v_1(|X_1(\boldsymbol{c}')|) - \sum_{h=1}^{k_1} b_2(j_h^{(2)}) \\ &= v_1(y_1 + k_1) - \sum_{h=1}^{k_1} b_2(j_h^{(2)}) \end{aligned}$$

by Eq. (29). Moreover, since $\boldsymbol{b} = (b_1, b_2)$ is a Nash equilibrium, we have

$$v_1(y_1+k_1) - v_1(y_1) \leq \sum_{h=1}^{k_1} b_2(j_h^{(2)})$$

by Definition 7 and Theorem 3. By combining these, we have $u_1(X_1(\boldsymbol{c}')) \leq v_1(y_1)$. However, this contradicts

$$u_1(X_1(\boldsymbol{c}')) > u_1(X_1(\boldsymbol{c})) = v_1(y_1)$$

in Eq. (28). Thus, $\boldsymbol{c} = (c_1, c_2)$ is a Nash equilibrium. □

5. Concluding Remarks

In this paper, we have given a necessary and sufficient condition for a valuation profile $\boldsymbol{v} = (v_1, v_2)$ satisfying Assumption 1 to have a Nash equilibrium in Theorem 2. We give some remarks below.

Note that, if all valuations v_i are submodular and symmetric then it is easily shown that we can obtain a Nash equilibrium which also maximizes the social welfare (thus, it is optimal) in polynomial time in n and m, however, the price of anarchy remains 2 [14]. This implies that the price of anarchy cannot be improved even if we restrict valuations to be symmetric.

The results in this paper can be generalized to the case of $n \geq 3$. That is, if n is fixed, we can decide in polynomial time whether a combinatorial auction has a Nash equilibrium or not if all valuations v_i are subadditive and symmetric [14]. However, if n is not fixed, our algorithm becomes exponential in n. Thus, we pose the following questions: is there a polynomial time algorithm to decide whether the model of the combinatorial auction in this paper with general $n \geq 3$ has a Nash equilibrium or not? Is it possible to relax the constraint of symmetry in a valuation and to obtain a similar result which might lead to an answer to the open question posed by Bhawalkar and Roughgarden in Ref. [1].

A recent paper by Dobzinski, Fu, and Kleinberg [4] revealed that exponential communication is required in order to find a pure no-overbidding Nash equilibrium in combinatorial auctions with subadditive bidders, even if such an equilibrium is known to exist. However, this does not settle the open question posed by Bhawalkar and Roughgarden. Note also that, this does not imply that any algorithm for deciding whether there is a pure no-overbidding equilibrium in combinatorial auctions with subadditive bidders requires exponential time.

Acknowledgments The authors would like to express their sincere gratitude to anonymous referees for giving valuable comments which helped improving this article. For this work, the second author was partly supported by Grants-in-Aid for Exploratory Research (15K11988), Chuo University Personal Research Grant, and Chuo University Grant for Special Research.

References

[1] Bhawalkar, K. and Roughgarden, T.: Welfare guarantees for combinatorial auctions with item bidding, *Proc. 22nd Annual ACM-SIAM Symposium on Discrete Algorithms*, pp.700–709 (2011).
[2] Blume, A., Heidhues, P., Lafky, J., Münster, J. and Zhang, M.: All Nash Equilibria of the Multi-Unit Vickrey Auction, *SFB/TR 15 Discussion Paper 116* (2006).
[3] Chrisodoulou, G., Kovács, A. and Schapira, M.: Bayesian combinatorial auctions, *Proc. 35th ICALP*, pp.820–832 (2008).
[4] Dobzinski, S., Fu, H. and Kleinberg, R.: On the complexity of computing an equilibrium in combinatorial auctions, *Proc. 26th Annual ACM-SIAM Symposium on Discrete Algorithms*, pp.110–122 (2015).
[5] Dobzinski, S., Nisan, N. and Schapira, M.: Approximation algorithms for combinatorial auctions with complement-free bidders, *Proc. 37th Annual ACM Symposium on Theory of Computing*, pp.610–618 (2005).
[6] Dobzinski, S. and Schapira, M.: An improved approximation algorithm for combinatorial auctions with submodular bidders, *Proc. 17th Annual ACM-SIAM Symposium on Discrete Algorithms*, pp.1064–1073 (2006).
[7] Debt Management Report 2011, The Government Debt Management and the State of Public Debts, Financial Bureau, Ministry of Finance, Japan, pp.34–46, available from ⟨http://www.mof.go.jp/english/jgbs/publication/debt_management_report/2011/saimu.pdf⟩.
[8] Feige, U.: On maximizing welfare where utility functions are subadditive, *Proc. 38th Annual ACM Symposium on Theory of Computing*, pp.41–50 (2006). (see also *SIAM J. Computing*, Vol.39, pp.122–142 (2009)).
[9] de Keijzer, B., Markakis, E., Schäfer, G. and Telelis, O.: Inefficiency of standard multi-unit auctions, *Proc. 21st Annual European Symposium on Algorithms*, pp.385–396 (2013).
[10] Khot, S., Lipton, R., Markakis, E. and Mehta, A.: Inapproximability results for combinatorial auctions with submodular utility functions, *Proc. WINE 2005, Lecture Notes in Computer Science 3828*, pp.92–101 (2005).
[11] Kwasnica, A.M. and Sherstyuk, K.: Multi-unit auctions, *Journal of Economic Surveys*, Vol.27, No.3, pp.461–490 (2013).
[12] Lehmann, B., Lehmann, D. and Nisan, N.: Combinatorial auctions with decreasing marginal utilities, *Proc. 3rd Annual ACM Symposium on Electronic Commerce*, pp.18–28 (2001).
[13] Nisan, N.: Bidding and allocation in combinatorial auctions, *Proc. 2nd Annual ACM Symposium on Electronic Commerce*, pp.1–12 (2000).
[14] Umeda, H. and Asano, T.: *Unpublished note, Department of Information and System Engineering*, Chuo University (2017).
[15] Vondrák, J.: Optimal approximation for the submodular welfare problem in the value oracle model, *Proc. 40th Annual ACM Symposium on Theory of Computing*, pp.67–74 (2008).

Appendix

A.1 Proof of Lemma 3

Suppose that $Y_i \neq \{\pi_{-i}(1), \pi_{-i}(2), \ldots, \pi_{-i}(y_i)\}$. Then for some $j \in \{1, 2, \ldots, y_i\}$ and $j' \in \{y_i+1, y_i+2, \ldots, m\}$, we have $\pi_{-i}(j) \notin Y_i$ and $\pi_{-i}(j') \in Y_i$. Thus,

$$j < j', \quad b_{-i}(\pi_{-i}(j)) \leq b_{-i}(\pi_{-i}(j')),$$
$$b_i(\pi_{-i}(j)) \leq b_{-i}(\pi_{-i}(j)), \quad \text{and} \quad b_{-i}(\pi_{-i}(j')) \leq b_i(\pi_{-i}(j')).$$

Then $b_{-i}(\pi_{-i}(j)) = b_{-i}(\pi_{-i}(j'))$ holds, which will be shown below. Now (let $b_{-i}(\pi_{-i}(j)) = b_{-i}(\pi_{-i}(j'))$ and) let $\pi_{-i}(j \leftrightarrow j')$ be the permutation obtained from π_{-i} by swapping $\pi_{-i}(j)$ and $\pi_{-i}(j')$. Thus,

$$\pi_{-i}(j \leftrightarrow j')(j'') = \begin{cases} \pi_{-i}(j'') & (j'' \neq j, j'), \\ \pi_{-i}(j) & (j'' = j'), \\ \pi_{-i}(j') & (j'' = j). \end{cases}$$

Then, by updating $\pi_{-i} = \pi_{-i}(j \leftrightarrow j')$, we obtain $\pi_{-i}(j) \in Y_i$ and $\pi_{-i}(j') \notin Y_i$. Note that, by this process, only π_{-i} changes, but none of b_1, b_2, Y_1, Y_2, and π_i changes, and Eq. (19) always holds. By repeating this process, we can finally obtain a permutation π_{-i} satisfying Eqs. (19) and (20).

Now we prove $b_{-i}(\pi_{-i}(j)) = b_{-i}(\pi_{-i}(j'))$. Suppose to the contrary that $b_{-i}(\pi_{-i}(j)) < b_{-i}(\pi_{-i}(j'))$. Let $b'_i = b_i(\pi_{-i}(j) \leftrightarrow \pi_{-i}(j'))$ be obtained from b_i by swapping $b_i(\pi_{-i}(j))$ and $b_i(\pi_{-i}(j'))$. Then $\boldsymbol{b}'_i = (b'_i, b_{-i})$ is a feasible bid profile and

$$b'_i(\pi_{-i}(j')) \leq b_{-i}(\pi_{-i}(j)) < b_{-i}(\pi_{-i}(j')) \leq b'_i(\pi_{-i}(j)).$$

Thus, $X_i(\boldsymbol{b}'_i) = X_i(\boldsymbol{b}) - \{\pi_{-i}(j')\} \cup \{\pi_{-i}(j)\}$ and $|X_i(\boldsymbol{b}'_i)| = |X_i(\boldsymbol{b})|$, and we have

$$u_i(X_i(\boldsymbol{b}'_i)) = u_i(X_i(\boldsymbol{b})) + b_{-i}(\pi_{-i}(j')) - b_{-i}(\pi_{-i}(j)) > u_i(X_i(\boldsymbol{b})).$$

However, this is a contradiction, because b_i is prestable. Thus, we have $b_{-i}(\pi_{-i}(j)) = b_{-i}(\pi_{-i}(j'))$.

Now let $i = 1$, and let π_2 be an identical permutation (i.e., $\pi_2(j) = j$ for all $j = 1, 2, \ldots, m$) by changing labels of items if necessary. Then by Eqs. (19) and (20), we have

$$b_2(1) \le b_2(2) \le \cdots \le b_2(m) \quad \text{and} \quad Y_1 = \{1, 2, \ldots, y_1\}, \tag{A.1}$$

and

$$\begin{aligned} &b_1(\pi_1(1)) \le b_1(\pi_1(2)) \le \cdots \le b_1(\pi_1(m)) \quad \text{and} \\ &Y_2 = \{\pi_1(1), \pi_1(2), \ldots, \pi_1(y_2)\} \ \ (y_2 = m - y_1) \end{aligned} \tag{A.2}$$

by choosing π_1 appropriately. Such a permutation π_1 is obtained by the same argument above. □

A.2 Proof of Lemma 4

To clarify the argument, we will give a proof for $i = 1$. Thus, $b_i = b_1$, $c_i = c_1$, $\pi_i = \pi_1$, $\sigma_i = \sigma_1$, $b_{-i} = b_2$, $c_{-i} = c_2$, $\pi_{-i} = \pi_2$, and $\sigma_{-i} = \sigma_2$. By symmetry, a proof for $i = 2$ is also obtained and we will omit it.

Let Y'_2 be the set of k_1 items of $Y_2 = X_2(\boldsymbol{c}) = M - X_1(\boldsymbol{c})$ corresponding to k_1 smallest bids in c_2, i.e.,

$$Y'_2 = \{\sigma_2(j_1^{(2)}), \sigma_2(j_2^{(2)}), \ldots, \sigma_2(j_{k_1}^{(2)})\} \subseteq Y_2. \tag{A.3}$$

Note that $c_2(j) = \frac{w_2(y_2)}{y_2}$ for each $j \in Y_2 = X_2(\boldsymbol{c})$. Thus,

$$\sum_{j \in Y'_2} c_2(j) = k_1 \frac{w_2(y_2)}{y_2} \tag{A.4}$$

Since $Y_2 = X_2(\boldsymbol{b})$ and b_2 is feasible, the sum of the smallest k_1 bids of Y_2 in b_2 is at most $k_1 \frac{w_2(y_2)}{y_2}$. In fact, this can be obtained as follows. If the sum of the smallest k_1 bids of Y_2 in b_2 were greater than $k_1 \frac{w_2(y_2)}{y_2}$, then the k_1th smallest bid of Y_2 would be greater than $\frac{w_2(y_2)}{y_2}$ (and each larger bid of Y_2 would also be greater than $\frac{w_2(y_2)}{y_2}$) and we would have $\sum_{j \in Y_2} b_2(j) > y_2 \frac{w_2(y_2)}{y_2} = w_2(y_2)$, a contradiction for the feasibility of b_2.

Thus, we have the sum of the smallest k_1 bids of Y_2 in b_2 is at most

$$\sum_{j \in Y'_2} c_2(j) = k_1 \frac{w_2(y_2)}{y_2}$$

and Eq. (27) for $i = 1$. □

A.3 Proof of Theorem 3

We assume Eq. (19), i.e.,

$$b_{-i}(\pi_{-i}(1)) \le b_{-i}(\pi_{-i}(2)) \le \cdots \le b_{-i}(\pi_{-i}(m)).$$

To prove Theorem 3, we need some notation.

Definition 8 For a feasible bid profile $\boldsymbol{b} = (b_1, b_2)$, let $g_i(b_{-i})$ denote the maximum number of items which bidder $i \in N$ can obtain by choosing his feasible bid vector b'_i appropriately. That is, he can choose a feasible bid vector b'_i so that he can obtain $g_i(b_{-i})$ items, however, he cannot obtain $g_i(b_{-i}) + 1$ items for any feasible bid vector b''_i.

In the feasible bid profile $\boldsymbol{b} = (b_1, b_2)$, we will say, $g_i \le g_i(b_{-i})$ items are *allocatable* to bidder i, but $h_i \ge g_i(b_{-i}) + 1$ items are *unallocatable* to bidder i using this notation $g_i(b_{-i})$. Then we have the following lemma.

Lemma 5 Suppose that, in a feasible bid profile $\boldsymbol{b} = (b_1, b_2)$, for each $i \in N$, $g_i(b_{-i})$ items are allocatable to bidder i, but $g_i(b_{-i}) + 1$ items are not. Then there is an integer h_i with $1 \le h_i \le g_i(b_{-i}) + 1$ such that

$$\sum_{j=0}^{h_i - 1} b_{-i}(\pi_{-i}(g_i(b_{-i}) + 1 - j)) > w_i(h_i).$$

Let g'_i be the smallest such integer h_i. Then

$$1 \le g'_i \le g_i(b_{-i}) + 1, \tag{A.5}$$

$$w_i(g'_i) = v_i(g'_i), \tag{A.6}$$

$$\sum_{j=0}^{g'_i - 1} b_{-i}(\pi_{-i}(g_i(b_{-i}) + 1 - j)) > w_i(g'_i), \quad \text{and} \tag{A.7}$$

$$\sum_{j=0}^{k-1} b_{-i}(\pi_{-i}(g_i(b_{-i}) + 1 - j)) \le w_i(k) \quad \text{for all } k \le g'_i - 1. \tag{A.8}$$

Proof: To clarify the argument, we will give a proof for $i = 1$. Thus, $b_i = b_1$, $\pi_i = \pi_1$, $b_{-i} = b_2$ and $\pi_{-i} = \pi_2$. By symmetry, a proof for $i = 2$ is also obtained. Furthermore, we can assume that π_2 is an identical permutation and $\pi_2(j) = j$ for all $j = 1, 2, \ldots, m$ by changing labels of items if necessary. Thus, Eq. (19) can be written as follows:

$$b_2(1) \le b_2(2) \le \cdots \le b_2(m). \tag{A.9}$$

Suppose that there were no such h_1 with $1 \le h_1 \le g_1(b_2) + 1$. Then

$$\sum_{j=0}^{h_1 - 1} b_2(g_1(b_2) + 1 - j) \le w_1(h_1), \tag{A.10}$$

for each $h_1 = 1, 2, \ldots, g_1(b_2) + 1$. Let b'_1 and π'_1 be defined by

$$b'_1(j) = \begin{cases} b_2(j) & (j = 1, 2, \ldots, g_1(b_2) + 1) \\ 0 & (j = g_1(b_2) + 2, g_1(b_2) + 3, \ldots, m), \end{cases}$$

$$\pi'_1(j) = \begin{cases} j + g_1(b_2) + 1 & (j = 1, 2, \ldots, m - g_1(b_2) - 1), \\ j - (m - g_1(b_2)) + 1 & (j = m - g_1(b_2), \ldots, m). \end{cases}$$

Then we have

$$\begin{aligned} &b'_1(\pi'_1(1)) \le b'_1(\pi'_1(2)) \le \cdots \le b'_1(\pi'_1(m)), \\ &b'_1(\pi'_1(m - j)) = b_2(g_1(b_2) + 1 - j) \quad (j = 0, 1, \ldots, g_1(b_2)), \\ &b'_1(\pi'_1(m - j)) = 0 \quad (j = g_1(b_2) + 1, g_1(b_2) + 2, \ldots, m - 1). \end{aligned}$$

Thus,

$$\sum_{j=0}^{h_1 - 1} b'_1(\pi'_1(m - j)) = \sum_{j=0}^{h_1 - 1} b_2(g_1(b_2) + 1 - j) \le w_1(h_1)$$

for all h_1 with $1 \le h_1 \le g_1(b_2) + 1$ (and

$$\sum_{j=0}^{h_1 - 1} b'_1(\pi'_1(m - j)) = \sum_{j=0}^{g_1(b_2)} b_2(g_1(b_2) + 1 - j) \le w_1(g_1(b_2) + 1)$$

for all h_1 with $g_1(b_2) + 2 \le h_1 \le m$) would hold, and b'_1 would be a feasible bid of bidder i by Theorem 1. Thus, $g_1(b_2)+1$ items would be allocatable to bidder 1 in a feasible bid profile $\boldsymbol{b}' = (b_1', b_2)$. However, this is a contradiction, since $g_1(b_2)+1$ items are not allocatable to bidder 1. Thus, such an integer h_1 exists, and $1 \le g_1' \le g_1(b_2)+1$, Eqs. (A.7) and (A.8) hold.

Next, we prove $w_1(g_1') = v_1(g_1')$. If $g_1' = 1$, then it is clear that $w_1(g_1') = w_1(1) = v_1(1) = v_1(g_1')$ by the definition of w_1. Thus, from now on we can assume $g_1' \ge 2$. Suppose that $w_1(g_1') \ne v_1(g_1')$. Then we have $w_1(g_1') < v_1(g_1')$ by Eq. (3). Thus, by Eq. (6), we have

$$w_1(g_1') = g_1' \frac{w_1(g_1'-1)}{g_1'-1}. \tag{A.11}$$

On the other hand, by the choice of g_1', we have

$$w_1(g_1'-1) \ge \sum_{j=0}^{g_1'-2} b_2(g_1(b_2)+1-j).$$

Furthermore, by Inequality (A.9) and Eq. (A.11), we have

$$\begin{aligned}\frac{w_1(g_1')}{g_1'} = \frac{w_1(g_1'-1)}{g_1'-1} &\ge \frac{\sum_{j=0}^{g_1'-2} b_2(g_1(b_2)+1-j)}{g_1'-1}\\ &\ge b_2(g_1(b_2)-g_1'+3)\\ &\ge b_2(g_1(b_2)-g_1'+2).\end{aligned}$$

Thus, we have

$$\begin{aligned}w_1(g_1') &= g_1'\frac{w_1(g_1'-1)}{g_1'-1} = (g_1'-1)\frac{w_1(g_1'-1)}{g_1'-1} + \frac{w_1(g_1'-1)}{g_1'-1}\\ &\ge \left(\sum_{j=0}^{g_1'-2} b_2(g_1(b_2)+1-j)\right) + b_2(g_1(b_2)-g_1'+2)\\ &= \sum_{j=0}^{g_1'-1} b_2(g_1(b_2)+1-j).\end{aligned}$$

However, this is a contradiction by the choice of g_1' since Eq. (A.7) holds, as shown above. Note that, here in this argument, Eq. (A.7) is

$$\sum_{j=0}^{g_1'-1} b_2(g_1(b_2)+1-j) > w_1(g_1').$$

Thus, we have $w_1(g_1') = v_1(g_1')$. □

Now, we are ready to prove Theorem 3.

Proof of Theorem 3

Since we have already shown the sufficiency of Theorem 3, we have only to show the necessity: If $\boldsymbol{b} = (b_1, b_2)$ is a Nash equilibrium then it is stable.

Suppose to the contrary that $\boldsymbol{b} = (b_1, b_2)$ is not stable even if it is Nash equilibrium (and thus, prestable). Then, for some $i \in N$ and some k, k' ($1 \le k \le m - y_i$, $1 \le k' \le y_i$) with $y_1 + y_2 = m$, Eqs. (21) or (22) would not hold.

To clarify the argument, we will give a proof for $i = 1$. Thus, $b_i = b_1$, $\pi_i = \pi_1$, $b_{-i} = b_2$ and $\pi_{-i} = \pi_2$. By symmetry, a proof for $i = 2$ is also obtained and we will omit it. Furthermore, we can assume that π_2 is an identical permutation (i.e., $\pi_2(j) = j$ for all $j = 1, 2, \ldots, m$) by changing labels of items if necessary. Thus, Eqs. (19) and (20), can be written by

$$b_2(1) \le b_2(2) \le \cdots \le b_2(m) \text{ and } Y_1 = \{1, 2, \ldots, y_1\} \tag{A.12}$$

and we have

$$v_1(y_1+k) - v_1(y_1) > \sum_{j=1}^{k} b_2(y_1+j) \tag{A.13}$$

for some k with $1 \le k \le m - y_1$, or

$$v_1(y_1-k') > v_1(y_1) - \sum_{j=0}^{k'-1} b_2(y_1-j) \tag{A.14}$$

for some k' with $1 \le k' \le y_1$. Now we can assume that, in a prestable bid profile $\boldsymbol{b} = (b_1, b_2)$, $g_1(b_2)$ items are allocatable to bidder 1, but $g_1(b_2)+1$ items are not. Thus, $y_1 \le g_1(b_2)$. Since $\boldsymbol{b} = (b_1, b_2)$ is a Nash equilibrium, $Y_1 = X_1(\boldsymbol{b}) = \{1, 2, \ldots, y_1\}$, and

$$\begin{aligned}u_1(Y_1) &= v_1(y_1) - \sum_{j\in Y_1} b_2(j)\\ &\ge u_1(X_1(\boldsymbol{b}')) = v_1(|X_1(\boldsymbol{b}')|) - \sum_{j\in X_1(\boldsymbol{b}')} b_2(j)\end{aligned}$$

for any feasible $\boldsymbol{b}' = (b_1', b_2)$, we can assume

$$v_1(y_1+k) - v_1(y_1) \le \sum_{j=1}^{k} b_2(y_1+j) \tag{A.15}$$

for all $1 \le k \le g_1(b_2) - y_1$ and

$$v_1(y_1-k') + \sum_{j=0}^{k'-1} b_2(y_1-j) \le v_1(y_1) \tag{A.16}$$

for all $1 \le k' \le y_1$. Thus, Eq. (A.14) never holds. Similarly, Eq. (A.13) does not hold for any k with $1 \le k \le g_1(b_2) - y_1$. Therefore, Eq. (A.13) holds for some k with $g_1(b_2) - y_1 + 1 \le k \le m - y_1$ (and this implies $g_1(b_2) < m$).

Let k^* be the smallest integer among such ks. Thus, we have

$$g_1(b_2) - y_1 + 1 \le k^* \le m - y_1, \tag{A.17}$$

$$v_1(y_1+k^*) - v_1(y_1) > \sum_{j=1}^{k^*} b_2(y_1+j), \tag{A.18}$$

$$v_1(y_1+k) - v_1(y_1) \le \sum_{j=1}^{k} b_2(y_1+j) \tag{A.19}$$

for all $0 \le k \le k^*-1$. The last two inequalities imply

$$v_1(y_1+k^*) - v_1(y_1+k) > \sum_{j=k+1}^{k^*} b_2(y_1+j) \tag{A.20}$$

for all $0 \le k \le k^*-1$. Similarly, by inequalities (A.16) and (A.18), we have

$$v_1(y_1+k^*) - v_1(y_1-k') > \sum_{j=1}^{k'} b_2(y_1-k'+j) + \sum_{j=1}^{k^*} b_2(y_1+j) \tag{A.21}$$

for all $1 \le k' \le y_1$. This is equivalent to

$$v_1(y_1+k^*) - v_1(k') > \sum_{j=k'+1}^{y_1+k^*} b_2(j) \tag{A.22}$$

for all $0 \le k' \le y_1 - 1$. Combining inequalities (A.20) and (A.22), we have

$$v_1(y_1 + k^*) - v_1(k') > \sum_{j=k'+1}^{y_1+k^*} b_2(j) \tag{A.23}$$

for all $0 \le k' \le y_1 + k^* - 1$.

On the other hand, in prestable bid profile $\boldsymbol{b} = (b_1, b_2)$, $g_1(b_2)$ items are allocatable to bidder 1 but $g_1(b_2) + 1$ items are not allocatable to bidder 1. Thus, we can consider g'_1 defined in Lemma 5. That is, g'_1 is the smallest k'' such that $1 \le k'' \le g_1(b_2) + 1$ and $b_2(g_1(b_2) - k'' + 2) + b_2(g_1(b_2) - k'' + 3) + \cdots + b_2(g_1(b_2) + 1) > w_1(k'')$. Thus, we have

$$1 \le g'_1 \le g_1(b_2) + 1, \tag{A.24}$$

$$w_1(g'_1) = v_1(g'_1), \tag{A.25}$$

$$\sum_{j=1}^{g'_1} b_2(g_1(b_2) + 1 - g'_1 + j) > w_1(g'_1), \quad \text{and} \tag{A.26}$$

$$\sum_{j=1}^{k} b_2(g_1(b_2) + 1 - g'_1 + j) \le w_1(k) \text{ for all } 1 \le k \le g'_1 - 1. \tag{A.27}$$

Since $g_1(b_2)$ items are allocatable to bidder 1, we also have

$$\sum_{j=1}^{k} b_2(g_1(b_2) - k + j) \le w_1(k) \quad \text{for all } 1 \le k \le g_1(b_2). \tag{A.28}$$

Now subtract g'_1 from $y_1 + k^*$ several times, say $q \ge 1$ times, so that $y_1 + k^* - qg'_1$ will be in the interval $[g_1(b_2) + 1 - g'_1, g_1(b_2)]$ of g'_1 integers. Let $k' = y_1 + k^* - qg'_1$. Then, we have

$$g_1(b_2) + 1 - g'_1 \le k' = y_1 + k^* - qg'_1 \le g_1(b_2). \tag{A.29}$$

Since $\sum_{j=1}^{g'_1} b_2(g_1(b_2) + 1 - g'_1 + j) > w_1(g'_1)$ by Eq. (A.26), $g_1(b_2) + 1 - g'_1 \le k' = y_1 + k^* - qg'_1$ by Eq. (A.29), and $y_1 + k^* \ge g_1(b_2) + 1$ by Eq. (A.17), we have

$$\begin{aligned} qw_1(g'_1) &< q \sum_{j=1}^{g'_1} b_2(g_1(b_2) + 1 - g'_1 + j) \\ &\le q \sum_{j=1}^{g'_1} b_2(y_1 + k^* - qg'_1 + j) \\ &\le \sum_{j=1}^{qg'_1} b_2(y_1 + k^* - qg'_1 + j) \\ &= \sum_{j=1}^{qg'_1} b_2(k' + j) = \sum_{j=k'+1}^{y_1+k^*} b_2(j) \end{aligned}$$

by Eq. (A.12) and $0 \le k' \le y_1 + k^* - 1$. Thus, by inequality (A.23), we have

$$qw_1(g'_1) < \sum_{j=k'+1}^{y_1+k^*} b_2(j) < v_1(y_1 + k^*) - v_1(k'). \tag{A.30}$$

Furthermore, since v_1 is subadditive and $y_1 + k^* - k' = qg'_1$, we have

$$v_1(y_1 + k^*) - v_1(k') \le v_1(y_1 + k^* - k') = v_1(qg'_1) \le qv_1(g'_1),$$

$$qw_1(g'_1) < \sum_{j=k'+1}^{y_1+k^*} b_2(j) < v_1(y_1 + k^*) - v_1(k') \le qv_1(g'_1) \tag{A.31}$$

and

$$w_1(g'_1) < v_1(g'_1). \tag{A.32}$$

However, this contradicts $w_1(g'_1) = v_1(g'_1)$ in Eq. (A.25).

Thus, for $i = 1$, both Eqs. (21) and (22) hold. By symmetry, both Eqs. (21) and (22) also hold for $i = 2$. Thus, $\boldsymbol{b} = (b_1, b_2)$ is stable. □

Hiroyuki Umeda was born in 1989. He received his M.E. degree from Chuo University in 2014. He joined the Information Processing Society of Japan in 2012. He is currently a Ph.D. student at Chuo University. His research interest is discrete algorithms in Economics and Computation. He is a member of IPSJ.

Takao Asano was born in 1949. He received his M.E. and D.E. from Tohoku University in 1974 and 1977, respectively. He became a research associate at Tohoku University in 1977, a lecturer at the University of Tokyo in 1980, an associate professor at Sophia University in 1985, and a professor at Chuo University in 1992. His current research interest is discrete algorithms. He received the IPSJ Yamashita SIG Research Award in 2000. He is a member (and Fellow) of IPSJ and a member of IEICE, JSIAM, ORSJ, ACM, IEEE-CS and SIAM.