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# Sankaku-tori: An Old Western-Japanese Game Played on a Point Set

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**Abstract:** We study a combinatorial game named "sankaku-tori" in Japanese, which means "triangle-taking" in English. It is an old pencil-and-paper game for two players played in Western Japan. The game is played on points on the plane in general position. In each turn, a player adds a line segment to join two points, and the game ends when a triangulation of the point set is completed. The player who completes more triangles than the other wins. In this paper, we formalize this game and investigate three restricted variants of this game. We first investigate a solitaire variant; for a given set of points and line segments with two integers $t$ and $k$, the problem asks if you can obtain $t$ triangles after $k$ moves. We show that this variant is NP-complete in general. The second variant is the standard two player version, but the points are in convex position. In this case, the first player has a nontrivial winning strategy. The last variant is a natural extension of the second one; we have the points in convex position but one point inside. Then, it turns out that the first player has no winning strategy.

**Keywords:** combinatorial game, Sankaku-tori, NP-complete.

## 1. Introduction

"Sankaku-tori" is a classic pencil-and-paper game for two players, traditionally played in Western Japan. Sankaku-tori literally means "triangle taking" in English. The rule is as follows. First, two players put a number of points on a sheet of paper. Then, they join the points alternately by a line segment. Line segments cannot cross each other. When an empty triangle is completed by a move, it scores +1 to the player who draws the (last) line segment. If two empty triangles are completed by the line segment, it scores +2. When no more line segment can be drawn, the game ends, and the player who scores more wins (see **Fig. 1**; in the figure, solid lines and dotted lines are played by the first player $\mathcal{R}$ and the second player $\mathcal{B}$, respectively. Finally, $\mathcal{R}$ wins since she obtains four triangles, while $\mathcal{B}$ obtains two triangles). We study the algorithmic aspects of the sankaku-tori game.

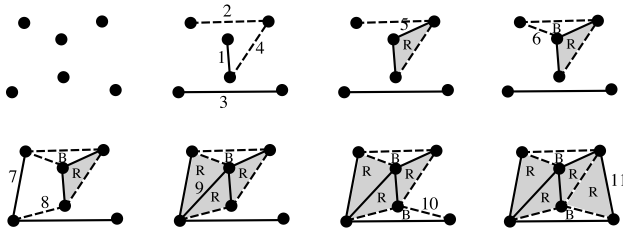

**Fig. 1** Sample play.

The game has a similar flavor to those studied by Aichholzer et al. [1] under the name of "Games on Triangulations." Among variations they studied, the most significant resemblance can be seen in the *monochromatic complete triangulation game*. The only difference between the sankaku-tori game and the monochromatic complete triangulation game is as follows. In the monochromatic complete triangulation game, if a player completes a triangle, then she can continue to draw a line segment. This rule is similarly seen in *Dots and Boxes*, where two players construct a grid instead of a triangulation of the point set. Dots and Boxes has been investigated in the literature (see Refs. [2], [4]), and especially, one book is devoted to the game [3], revealing a rich mathematical structure. Such a rule of the monochromatic complete triangulation game and Dots and Boxes admits us to use related results in combinatorial games such as Kayles and Nimstring. Aichholzer et al. [1] proved that the monochromatic complete triangulation game is a first-player

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win if the number of points is odd, and a tie if it is even. We note that a few problems left by Aichholzer et al. [1] have recently been resolved by Manić et al. [5].

On the other hand, in the sankaku-tori game, even though a player completes a triangle, she should leave the token to the next player. Hence, we cannot directly use the previously known results, and we need to develop new techniques for our game.

We first consider a solitaire version of the sankaku-tori game. Namely, we are given a point set and some line segments connecting pairs of those points, and we want to maximize the number of triangles that can be constructed by drawing $k$ more line segments. We prove that this problem is NP-complete.

Next, we turn to the ordinary two-player version. We consider the case that the points are in convex position. In this case, the first player always has a winning strategy. Then, we extend this case to adding one more point inside of the points in convex position. That is, the points are in convex position but one special point inside. In this case, interestingly, the second player has an advantage. More precisely, the first player has no winning strategy.

## 2. Preliminaries

In this paper, a finite planar point set $S = \{p_1, p_2, \dots, p_n\}$ is always assumed to be in general position, i.e., no three points in $S$ are collinear. A *triangulation* of a finite planar point set $S$ is a decomposition of its convex hull by a maximal set of triangles in such a way that their vertices are precisely the points in $S$. Two players $\mathcal{R}$(ed) and $\mathcal{B}$(lue) play in turns, and we assume that $\mathcal{R}$ is the first player. In terms of games in Ref. [1], our problem is constructing monochromatic triangulation of the point set. That is, the players construct a triangulation of a given point set $S$ [*1].

Starting from no edges, players $\mathcal{R}$ and $\mathcal{B}$ play in turn by drawing one edge in each move. We note that each player draws precisely one edge. This is the difference from the dots-and-boxes-like games. The game ends when a triangulation of the point set is completed. Each triangle belongs to the player who draws the last edge of the triangle [*2]. The player who has more triangles than the other wins.

We first note that, for any set $S$ of $n$ points, the number of edges of a triangulation of $S$ is determined by the position of the points. That is, the number of turns of the sankaku-tori game is determined when the position of the points are given, and the total score of two players is a constant. For example, if $n$ points are in convex position, its any triangulation of the point set consists of $2n - 3$ edges, and the resulting triangulation of the point set contains $n - 2$ triangles. Therefore, the number of turns is $2n - 3$, and the total score of both players is $n - 2$.

## 3. NP-completeness

In this section, we consider the solitaire variant by modifying the rule of the original game. We start halfway through the game. That is, we are given a set of $n$ points and $O(n)$ lines joining them. We are also given two integers $k = O(n)$ and $t$. The decision problem asks whether we can obtain $t$ triangles after $k$ moves for the set of points and lines.

**Theorem 1** The solitaire variant of Sankaku-Tori is NP-complete.

The problem is in NP since we can guess $k$ new edges and easily check whether we can obtain $t$ triangles. Later in this section, we reduce POSITIVE PLANAR 1-IN-3-SAT problem [6] to our problem. In POSITIVE PLANAR 1-IN-3-SAT, we are given a 3-CNF formula $\varphi$ with $n$ variables and $m$ clauses, together with a planar embedding of its incidence graph $G(\varphi)$. Each clause of $\varphi$ consists of three positive literals (i.e., variable itself). The incidence graph $G(\varphi)$ of $\varphi$ consists of $n$ vertices $v_{x_i}$ corresponding to the variables $x_i$ and $m$ vertices $v_{C_j}$ corresponding to the clauses $C_j$. There is an edge $(v_{x_i}, v_{C_j})$ if and only if $x_i$ appears in $C_j$. The problem is to decide whether there exists a satisfying assignment to the variables of $\varphi$ such that each clause in $\varphi$ has exactly one literal assigned true. POSITIVE PLANAR 1-IN-3-SAT is NP-complete [6].

**Basic gadgets.** We first introduce two important gadgets; crescent and barrier.

A *crescent* consists of $c$ points $p_1, p_2, \dots, p_c$ for a certain integer $c$. They are in convex position, that is, all points are on its convex hull. The crescent also contains the line segments of convex hull of these points, and a set of line segments that fully triangulates this convex hull (thus the crescent has $2c - 3$ lines). The line segment $p_1p_c$ is relatively long enough so that all the other vertices $p_2, \dots, p_{c-1}$ are on the "same side" from the line $p_1p_c$ (**Fig. 2**). More precisely, we assume that $h$ is so close to 0 that from any point outside of this crescent (except the points on the line including the line segment $p_1p_c$), one can view either only the line segment $p_1p_c$ or all line segments $p_1p_2, p_2p_3, \dots, p_{c-1}p_c$ but $p_1p_c$.

A *barrier* consists of points and lines that inhibits to join any point outside of the region to a (given) point inside of the region surrounded by the barrier. An example of a barrier is given in **Fig. 3** (a). This barrier consists of eight line segments surrounding two crescents, and no point outside of this barrier can be joined to any point of these two crescents.

Our key claim is the following: When we are given the situation illustrated in Fig. 3 (a) and two integers $k = 2c - 1$ and $t = 2c - 2$, a unique solution for obtaining $t$ triangles is join two crescents by the $k$ lines as illustrated in Fig. 3 (b). In other

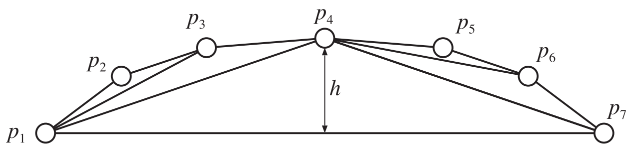



**Fig. 2** A crescent of 7 points.

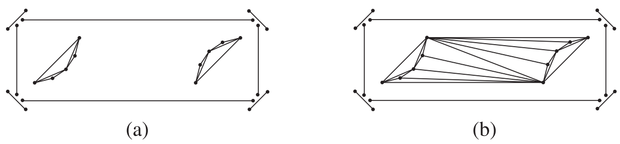



**Fig. 3** Basic gadget: two crescents and barrier enclosing them.

*1 In a real game, two players arbitrarily draw the point set by themselves simultaneously until both agree with.

*2 In a real game, when a player draws the last edge, she writes her initials in the triangle.

words, once we use any line to join two points of the barrier, or one point on a crescent and the other of the barrier, we can no longer obtain $t$ triangles. We deduce this claim by introducing a *loss* which is defined as $i - j$ if we obtain $j$ triangles by drawing $i$ lines. In Fig. 3 (a), we can obtain triangles by drawing lines between two connected components. Note that we have 10 connected components in Fig. 3 (a); two connected components are crescents, and this barrier consists of 8 connected components such that each connected component consists of a line. To connect two connected components, it requires at least one loss. (We are required to draw at least two lines for obtaining one triangle.) Here, $k = 2c - 1$ and $t = 2c - 2$ means the loss should be at most one. Thus, we are required to draw $k$ lines between two connected components, not to connect three or more components. Here, we cannot obtain $t$ or more triangles if we connect two line segments of the barrier, or if we connect a crescent and a line segment of the barrier. Thus the only way is to connect two crescents by drawing $k$ lines; one possible solution is given in Fig. 3 (b).

Now we turn to the construction of the gadget for the reduction from POSITIVE PLANAR 1-IN-3-SAT, which consists of line gadget, variable gadget, and clause gadget.

**Line gadget.** For a given integer $\ell$, a line gadget of length $\ell$ consists of $\ell + 1$ crescents and its barrier with $4(\ell + 1)$ lines surrounding the crescents (**Fig. 4** (a)). In the gadget, the barrier plays a role of an obstacle to crescents, and hence each crescent can view at most two other crescents. We call each visible crescent *neighbor*, and two crescents are *adjacent* if they are neighbors with each other. By drawing $2c - 1$ lines between two adjacent crescents, we can obtain $2c - 2$ triangles. The barrier prevents us from drawing a line between nonadjacent crescents.

Suppose that we are required to obtain $i(2c - 2)$ triangles by drawing $i(2c - 1)$ lines ($0 < i \leq \lceil \ell/2 \rceil$). By an argument similar to one in the basic gadgets, loss should be at most $i$. We cannot obtain $i(2c - 2)$ triangles, if we connect two lines in the barrier, if we connect a crescent and a line in the barrier, or if we connect three or more consecutive crescents along the barrier. The only way to obtain $i(2c - 2)$ triangles is to connect $i$ pairs of two consecutive crescents in the barrier. For example, Fig. 4 (a) is a line gadget of length 4, and we can obtain $i(2c - 2)$ triangles only if we connect two pairs of crescents by $2(2c - 1)$ lines as shown in Fig. 4 (b) or (c).

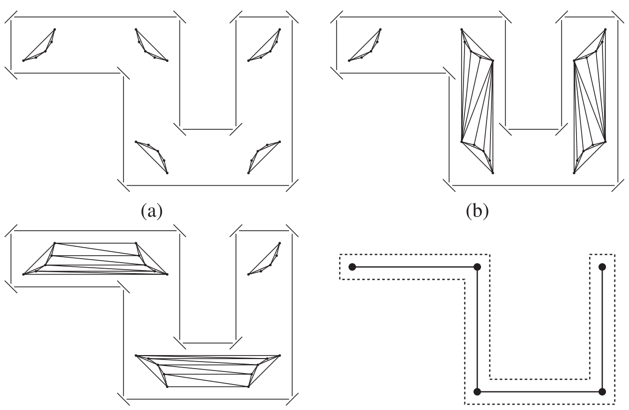



**Fig. 4** Line gadget.

To simplify, we abbreviate Fig. 4 (a) as in Fig. 4 (d). The points in Fig. 4 (d) denote crescents, and the (solid) edges denote the adjacency between the crescents. The dotted rectilinear polygon in Fig. 4 (d) denotes the barrier: Each line segment of the dotted polygon corresponds to a line, and we have a short line at each corner of the dotted polygon. Since each crescent can be connected with at most one adjacent crescent, the connected pairs of crescents in Fig. 4 (a) correspond to a matching of the graph in Fig. 4 (d).

Line gadgets have flexibility on their shapes: We can extend or shorten the distance between any two adjacent crescents. We can select a direction at each bend of the gadget. We can also set any angles at the bends within 90 degrees.

**Variable gadget.** As illustrated in **Fig. 5** (a), we arrange $c_i$ line gadgets of length $2\ell + 1$ ($c_i = 3$ and $2\ell + 1 = 9$ in the figure). As in Fig. 4 (d), the dotted polygons denote barriers. The uppermost line gadget crosses the remaining $c_i - 1$ line gadgets. We have $8(c_i - 1)$ crossing points, each of which requires eight additional lines as barriers. Since a non-crossing line gadget re-

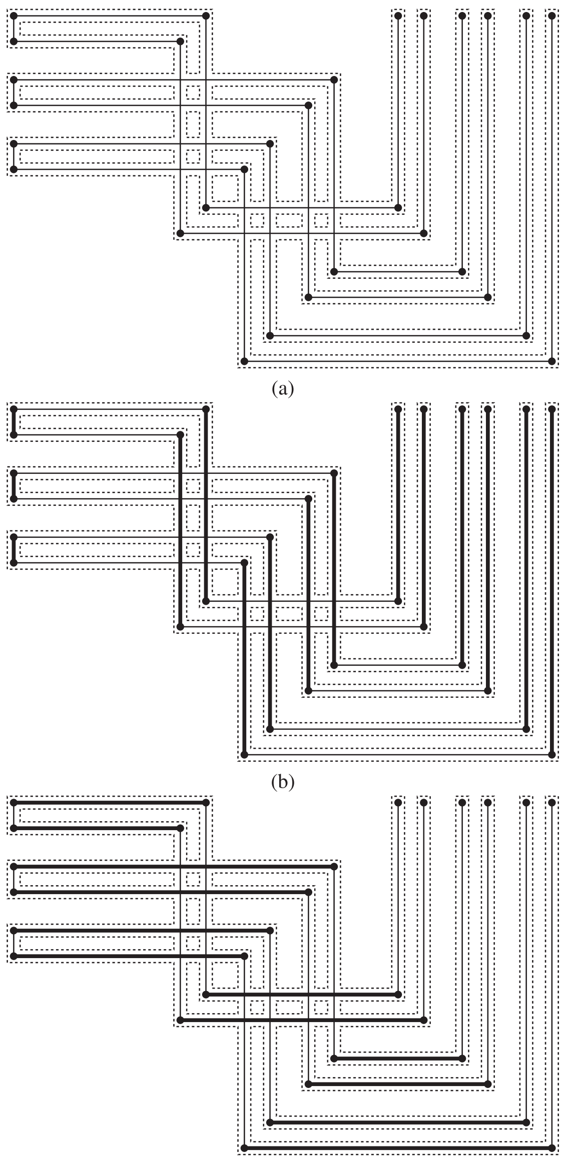



**Fig. 5** Variable gadget.

quires $4(2\ell + 2)$ lines of barriers, a variable gadget with $c_i$ line gadgets requires $4(2\ell+2)c_i+8\cdot 8(c_i-1) = O(\ell c_i)$ lines of barriers.

A unique non-crossing maximum matching of Fig. 5 (a) is, as illustrated in the bold lines in Fig. 5 (b), achieved by taking $\ell + 1$ matching edges in each line gadget. In case we are not allowed to use crescents at both ends of the line gadgets, a unique non-crossing maximum matching for the remaining crescents is, as illustrated in the bold lines in Fig. 5 (c), achieved by taking $\ell$ matching edges in each line gadget. As the rule of the Sankaku-tori prohibits crossing lines, matching edges for Fig. 5 (a) cannot cross each other. Thanks to this property, as in Fig. 5 (b) and (c), we can synchronize the selection of vertical or horizontal matching edges among all line gadgets in a variable gadget.

**Clause gadget.** As illustrated in **Fig. 6** (a), we share the crescents at the ends of three line gadgets of length $2\ell+1$. Since each clause has three literals, we use the line gadgets corresponding to them. A clause gadget has $6\ell + 2$ crescents. The crucial part is on the top of the clause gadget, whose details are illustrated in Fig. 6 (b). We note here that the number of barriers in a clause gadget is the same as that of (non-sharing) three line gadgets.

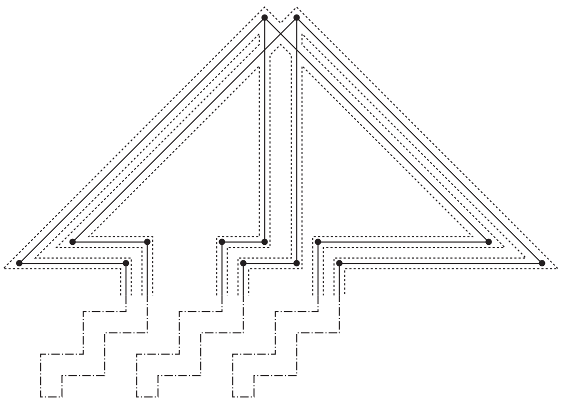
In case one of the three line gadgets takes both of the top two crescents to realize the matching in Fig. 5 (b), the other two line gadgets cannot use the top crescents. This means that the maximum matching for the other two is the matching in Fig. 5 (c). In this case, the number of matching edges is $3\ell + 1$, which is a perfect matching on $6\ell + 2$ crescents.

If no line gadgets take either of the top two crescents, we cannot obtain a perfect matching. If two line gadgets take one of the top two respectively, each of them has unmatched crescents, which means we cannot obtain a perfect matching. In this case, we can observe that we have no profit; when one edge is missing from a matching as shown in **Fig. 7** (a), we cannot obtain any other profit from the other part. On the other hand, even we flip some edge into one of the top crescents as shown in Fig. 7 (b), we have no profit from this gadget. Thus, in total, we will miss one matching edge if we have inconsistency at the clause gadget. Thus, we can obtain a perfect matching if and only if one of the three line gadgets takes the top two crescents, and the other two do not take any.

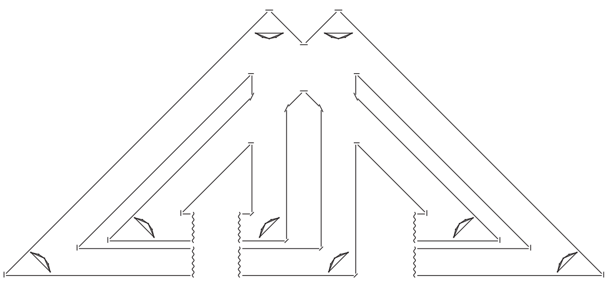
**Reduction.** Let $c_i$ denote the number of occurrences of literal $x_i$ in $\varphi$ ($i = 1, 2, \ldots, n$). As illustrated in Fig. 5 (a), a variable gadget for $x_i$ has $c_i$ line gadgets of length $2\ell + 1$. Since we have $3m$ literals in $\varphi$, $n$ variable gadgets have $3m$ line gadgets in total. Since line gadgets have flexibility on their shape, line gadgets are arranged along the planar embedding of $G(\varphi)$.

Each clause has variable gadgets as illustrated in Fig. 6 (a), which consists of three line gadgets corresponding to the three literals in the clause. Since $\varphi$ has $m$ clauses, we have $m$ clause gadgets with $(6\ell + 2)m$ crescents and $O(\ell m)$ barriers. Two magic numbers are set to $k = (3\ell+1)m(2c-1)$ and $t = (3\ell+1)m(2c-2)$.

If $\varphi$ is satisfiable in the sense of POSITIVE 1-IN-3-SAT, by the following strategy, we can obtain $t$ triangles. For literals assigned true, the corresponding line gadgets take the two crescents at both ends of the gadgets and achieve the maximum matching as illustrated in Fig. 5 (b). For literals assigned false, as illustrated in Fig. 5 (c), the corresponding line gadgets do not take the two

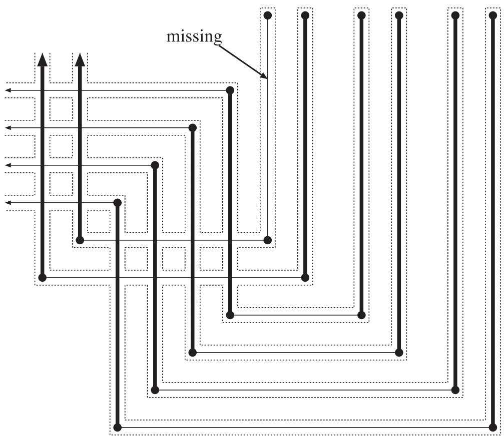


(a)

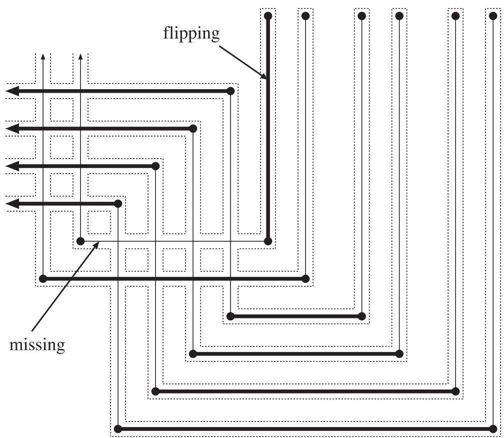


(b)

**Fig. 6** Clause gadget.




(a)




(b)

**Fig. 7** Inconsistency gives no profit.

crescents at both ends of the gadgets and achieve the maximum matching for other crescents. The formula $\varphi$ has $m$ literals assigned true, and they satisfy all clauses in the sense of POSITIVE 1-IN-3-SAT. This means that, for every clause gadget, exactly one line gadget takes the top two crescents, and no two line gadgets take the same crescents at the same time. From the argument above, each clause gadget has one line gadget with $\ell + 1$ matching edges and two line gadgets with $\ell$ matching edges. Thus, we have $(3\ell + 1)m$ matching edges in total, which means we can obtain $t = (3\ell+1)m(2c-2)$ triangles by drawing $k = (3\ell+1)m(2c-1)$ lines. The opposite direction is clear from the above discussion.

As mentioned before, line gadgets have flexibility on their shapes. Using this fact, it is easy to see that all gadgets can be joined appropriately by polynomial number of line gadgets. Thus this is a polynomial time reduction.

Therefore, we complete the proof of Theorem 1.

## 4. Convex Position

In this section, the main theorem is the following.

**Theorem 2** Let $p_1, \ldots, p_n$ be a point set $S$ in convex position. Then the first player $\mathcal{R}$ has a winning strategy.

To prove the theorem, we describe a winning strategy for $\mathcal{R}$ in Lemma 3. Once the first player $\mathcal{R}$ draws a line $p_ip_j$ in the first move, we have two intervals $I_1 = [p_i, p_{i+1}, \ldots, p_{j-1}, p_j]$ and $I_2 = [p_j, p_{j+1}, \ldots, p_{i-i}, p_i]$. Then any point $p$ in $I_1$ can be joined to the other point $q$ if and only if $q$ is in $I_1$ when the points are in convex position. That is, each line segment separates an interval of the points into two independent intervals. The winning strategy is an inductive one that consists of three substrategies. We note that the strategy in Lemma 3 is applied simultaneously in each interval. For example, suppose that $\mathcal{R}$ has two strategies $S_1$ and $S_2$ on intervals $I_1 = [p_i, p_{i+1}, \ldots, p_{j-1}, p_j]$ and $I_2 = [p_j, p_{j+1}, \ldots, p_{i-i}, p_i]$, respectively. If $\mathcal{B}$ joins two points in $I_1$, $\mathcal{R}$ uses $S_1$ on the interval $I_1$, and then, if $\mathcal{B}$ joins two points in $I_2$, $\mathcal{R}$ now uses $S_2$ on the interval $I_2$, and so on. Since the points are in convex position, they can apply their strategies independently in each interval.

**Lemma 3** Suppose that, at a certain point of the game, $\mathcal{B}$ has to move and there are some intervals in Cases 1, 2, and 3 in **Figs. 8**, **9**, and **10**, respectively. (The dotted line indicates there are some (possibly zero) points along this line.) Then, after two moves, $\mathcal{R}$ can replicate the same configuration without losing points. Moreover, if the number of vertices in an interval is odd, at the end it is possible for $\mathcal{R}$ to get one more point.

Proof. We prove the lemma by an induction for the number of turns of the game. As mentioned in Preliminaries, if we have $n$ points in convex position, the number of turns is exactly $2n - 3$. In the figures, dotted lines illustrate the isolated points. In base cases, dotted lines mean that no points are there. We can check the claims in Lemma 3 in base cases by simple case analysis. Now we turn to general cases.

(Case 1) Player $\mathcal{B}$ has two choices. If $\mathcal{B}$ joins $p_i$ and $p_j$ with $1 < i < j < n$, $\mathcal{R}$ joins $p_1$ and $p_j$ and obtain (Case 2). Therefore, without loss of generality, we assume that $\mathcal{B}$ joins $p_1$ and $p_i$ with $1 < i < n$. In this case, $\mathcal{R}$ can join $p_i$ and $p_n$, and obtain the triangle $p_1p_ip_n$. Moreover, (Case 1) applies to both intervals


**Fig. 8** Case 1.

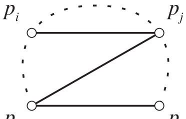



**Fig. 9** Case 2.

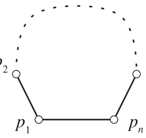



**Fig. 10** Case 3.

$[p_1..p_i]$ and $[p_i..p_n]$. Therefore, by induction, $\mathcal{R}$ wins in this case because $\mathcal{R}$ already obtains +1 by the triangle $p_1p_ip_n$.

(Case 2) The same analysis of (Case 1) can be applied in the interval $[p_i..p_j]$. Therefore, by inductive hypothesis, $\mathcal{B}$ cannot take an advantage in this interval. Without loss of generality, we can assume that $\mathcal{B}$ plays in interval $[p_1..p_i]$. Essentially, $\mathcal{B}$ has four choices.

(Subcase 2-1) If $\mathcal{B}$ joins $p_1$ and $p_i$, $\mathcal{R}$ joins $p_j$ and $p_n$, and they have three intervals in (Case 1). Then it is easy to check that the claim holds.

(Subcase 2-2) If $\mathcal{B}$ picks $p_{i'}$ with $1 < i' < i$ and joins it to either $p_1$ or $p_i$, $\mathcal{R}$ again joins $p_j$ and $p_n$. Then we have two intervals $[p_i..p_j]$ and $[p_j..p_n]$ in (Case 1). If $\mathcal{B}$ joins $p_1$ and $p_{i'}$, we have an interval $[p_1..p_{i'}]$ in (Case 1), and the other interval $[p_{i'}..p_i]$ in (Case 3). The other case ($\mathcal{B}$ joins $p_{i'}$ and $p_i$) is symmetric. In any case, by inductive hypothesis, the claim holds.

(Subcase 2-3) If $\mathcal{B}$ joins $p_j$ and $p_{i'}$ for some $1 < i' < i$, $\mathcal{R}$ joins $p_{i'}$ to $p_i$. Then $\mathcal{R}$ obtains the triangle $p_ip_jp_{i'}$, and two intervals $[p_{i'}..p_i]$ and $[p_i..p_j]$ are in (Case 1), and two intervals $[p_1..p_{i'}]$ and $[p_j..p_n]$ together essentially in the same case as (Case 2). Therefore, $\mathcal{R}$ wins in this case.

(Subcase 2-4) The last case is that $\mathcal{B}$ picks up two points $p_{i'}$ and $p_{i''}$ with $1 < i' < i'' < i$ and join them by an edge. Then $\mathcal{R}$ joins $p_{i'}$ to $p_j$, and obtain two intervals $[p_1..p_{i'}]$ and $[p_j..p_n]$ together in (Case 2), an interval $[p_{i''}..p_i]$ with an edge $(p_{i''}, p_{i'})$ in (Case 3), and two intervals $[p_{i'}..p_{i''}]$ and $[p_i..p_j]$ in (Case 1). Therefore, we have the claim in this case again.

(Case 3) Now we have three subcases.

(Subcase 3-1) $\mathcal{B}$ joins two points in $\{p_1, p_2, p_{n-1}, p_n\}$. If $\mathcal{B}$ joins $p_2$ and $p_{n-1}$, $\mathcal{R}$ joins $p_1$ and $p_{n-1}$, and obtain two triangles ($p_1p_2p_{n-1}$ and $p_1p_{n-1}p_n$), and they end up in Case 1. On the other hand, if $\mathcal{B}$ joins $p_1$ and $p_{n-1}$, $\mathcal{R}$ joins $p_2$ and $p_{n-1}$ and obtains (Case 1). The other cases are symmetric. Thus we have the claim.

(Subcase 3-2) $\mathcal{B}$ joins one point in $\{p_1, p_2, p_{n-1}, p_n\}$ and another one $p_i$ with $2 < i < n-1$. If $\mathcal{B}$ joins $p_1$ and $p_i$, $\mathcal{R}$ joins $p_i$ and $p_2$ and obtain the triangle $p_1p_2p_i$. Then they also have an interval $[p_2..p_i]$ in (Case 1) and $[p_i..p_n]$ with $p_1$ in (Case 3) again.

Thus we have the claim. If $\mathcal{B}$ joins $p_2$ and $p_i$, $\mathcal{R}$ now joins $p_i$ and $p_1$ and get the same situation. The other two cases are symmetric.

(Subcase 3-3) $\mathcal{B}$ joins two points $p_i$ and $p_j$ with $2 < i < j < n-1$. In the case, $\mathcal{R}$ joins $p_i$ and $p_n$. Then both of the interval $[p_1..p_i]$ with $p_n$ and the interval $[p_i..p_n]$ are independently in (Case 3). Therefore, we again use the induction.

By the induction for the number of points, we have the lemma. □

Now we prove Theorem 2:

*Proof* (of Theorem 2). When $n = 2k+1$ for some $k > 1$, $\mathcal{R}$ joins $p_1$ and $p_k$. Then two intervals $[p_1..p_k]$ and $[p_k..p_n]$ are both in (Case 1) in Lemma 3. Moreover, one of two intervals consists of odd number of points. Thus $\mathcal{R}$ obtains at least one more triangle than $\mathcal{B}$.

When $n = 2k$ for some $k > 1$, $\mathcal{R}$ joins $p_1$ and $p_3$. Then two intervals $[p_1..p_3]$ and $[p_3..p_n]$ are both in (Case 1), and they are of odd length. Thus $\mathcal{R}$ obtains at least two more triangles than $\mathcal{B}$.

In any case, $\mathcal{R}$ always wins. □

## 5. Convex Position with One More Point

In this section, we extend the case in Theorem 2 to convex cases with one more point. That is, the point set $S$ consists of $n$ points $p_1, \dots, p_n$ in convex position, and one more point $q$ inside of the convex hull of $p_1, \dots, p_n$. In this case, interestingly, the second player $\mathcal{B}$ has an advantage, which is contrary to Theorem 2.

**Example 4** When $S = \{p_1, p_2, p_3, q\}$, it is easy to see that $\mathcal{B}$ has a winning strategy, who can take 2 points, while $\mathcal{R}$ takes only one. On the other hand, if $S = \{p_1, p_2, p_3, p_4, q\}$, $\mathcal{R}$ can end in a tie when it first joins $q$ and $p_1$, but $\mathcal{R}$ has no winning strategy (by an exhaustive search).

We generalize it and obtain the following theorem:

**Theorem 5** Let $S$ be the set of points $\{p_1, \dots, p_n, q\}$. We assume that $p_1, \dots, p_n$ are in convex position, and $q$ be a point inside of the convex hull of $S \setminus \{q\}$. Then the first player $\mathcal{R}$ does not have a winning strategy.

Let $q$ be the *central point* in this point set. In order to deal with the central point, we introduce *closest lines* as follows: First, we draw all line segments joining $p_i$ and $p_j$ with $i \neq j$. Then we have unique convex polygon $Q$ that includes $q$. The line segment $p_ip_j$ is said to be a *closest line to q* if it contains an edge of the convex polygon including $q$. For a closest line $p_ip_j$ to $q$, we have the following proposition:

**Proposition 6** Assume that $p_ip_j$ is a closest line to $q$ and the convex polygon formed by $p_i, p_{i+1}, \dots, p_{j-1}, p_j$ includes $q$. Then both triangles $p_ip_jp_{i+1}$ and $p_ip_jp_{j-1}$ include $q$. In other words, we cannot draw any line segment $p_{i'}p_{j'}$ between the point $q$ and the line segment $p_ip_j$.

To show the main theorem, we first consider two simple cases:

**Lemma 7** If $\mathcal{R}$ draws a line segment $qp_i$ as the first move for any $i$, $\mathcal{R}$ cannot win. If $\mathcal{R}$ draws one of the closest lines as the first move, then $\mathcal{R}$ cannot win.

Proof. (1) Without loss of generality, we assume that $\mathcal{R}$ draws a line segment $qp_1$ at the first move. The line $qp_1$ intersects one of edges of the $Q$, which is formed by the closest lines. Therefore, there exists a closest line $p_ip_j$ that does not intersect $qp_1$. The second player choose any closest $p_ip_j$ that does not intersect $qp_1$. This case is similar to Case 2 in Fig. 9 except (potential) triangle $p_iqp_j$ (see **Fig. 11**). If we have two lines $p_iq$ and $p_jq$, this is the same as Case 2.

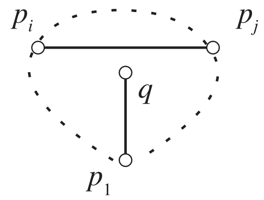


**Fig. 11** Case 2e.

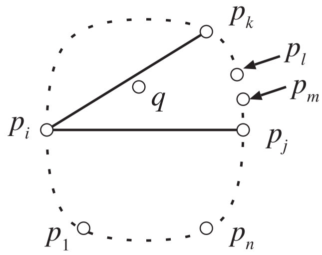


**Fig. 12** Case 4.

When $\mathcal{R}$ chooses one of $p_iq$ and $p_jq$, then $\mathcal{B}$ takes the other one. Taking care of the triangles formed by $p_iqp_{i'}$ with $1 \le i' \le i-1$, $p_jqp_{j'}$ with $j+1 \le j' \le n$, and $p_iqp_j$, we can observe that $\mathcal{R}$ cannot win in this case. Now we consider $\mathcal{R}$ does not choose any of $p_iq$ and $p_jq$ and $\mathcal{B}$ has to choose one of them. This time, $\mathcal{B}$ takes any of them, and $\mathcal{R}$ eventually takes the other. But this is not a winning strategy for $\mathcal{R}$.

(2) In this case, since $p_1, \dots, p_n$ are in convex position, $\mathcal{B}$ can take one point $p_k$ such that $p_kq$ does not intersect the closest line $\mathcal{R}$ chosen. Then we can apply the same argument as in (1). □

By Lemma 7, it is sufficient to consider the case that the first player $\mathcal{R}$ joins two points $p_ip_j$ that is not a closest line to $q$. Without loss of generality, we assume that $1 \le i < j \le n$ and the point $q$ is included in the convex polygon formed by $p_1, p_2, \dots, p_{j-1}, p_j$. Then the following lemma covers the last case.

**Lemma 8** Assume that $\mathcal{R}$ joins two vertices $p_ip_j$ which is not a closest line to $q$ at the first move. Then it is not a winning strategy for $\mathcal{R}$.

Proof. Since $q$ is the central point in convex shape formed by $p_1, \dots, p_n$, we can assume that we have at least one point $p_k$ with $i < k < j$ (**Fig. 12**). We select $p_k$ so that the triangle $p_ip_kp_j$ contains $q$ but $p_ip_{k+1}p_j$ does not (we may have $k+1 = j$). (Note that the line segment itself may not be the closest line to $q$ in the original points.) Then $\mathcal{B}$ draws the line $p_ip_k$. Now we have two Case 1s in two intervals $[i..k]$ and $[j, j+1, \dots, n, 1, 2, \dots, i]$.

Let $p_\ell$ and $p_m$ be two points between $p_k$ and $p_j$. That is, $k < \ell < m < j$, and they may not exist. We now describe the strategy of $\mathcal{B}$ depending on what $\mathcal{R}$ chooses by pairing; see **Table 1**. In the table, each pair describes one move by $\mathcal{R}$ and the corresponding move by $\mathcal{B}$. For example, if $\mathcal{R}$ draws the line $qp_j$, $\mathcal{B}$ draws the line $qp_i$ in the next move. Then we have Case 5 (described later) and $\mathcal{B}$ obtains one point by the triangle $qp_jp_i$. The other entries are similar. We note that these pairs are not disjoint. For example, when $\mathcal{R}$ draws the line $p_iq$, then $\mathcal{B}$ has three choices

**Table 1** Strategy for $\mathcal{B}$ for Case 4.

| Pairs | $\{qp_j, qp_i\}$ | $\{qp_k, qp_i\}$ | $\{qp_\ell, qp_i\}$ | $\{p_j p_\ell, p_i p_\ell\}$ | $\{p_k p_\ell, p_j p_\ell\}$ | $\{p_\ell p_m, p_i p_\ell\}$ |
|---|---|---|---|---|---|---|
| Case(s) of $\mathcal{B}$/Score | Case 5/+1 | Case 5/+1 | Cases 3 & 5/0 | Cases 1 & 4/+1 | Cases 1 & 1 & 6/0 | Cases 1 & 3 & 4/0 |

**Table 2** Strategy for $\mathcal{B}$ for Case 5.

| Pairs | $\{qp_2, p_n p_2\}$ | $\{qp_i, p_n p_i\}$ | $\{p_2 p_i, qp_2\}$ | $\{p_1 p_i, p_2 p_i\}$ | $\{p_i p_j, qp_i\}$ |
|---|---|---|---|---|---|
| Case(s) of $\mathcal{B}$/Score | Case 1/+1 − 1 = 0 or Case 1/+2 | Cases 1 & (3 or 5)/+1 | Cases 1 & 3/+1 | Cases 1 & 5/+1 | Cases 1 & 3 & (3 or 5)/0 |

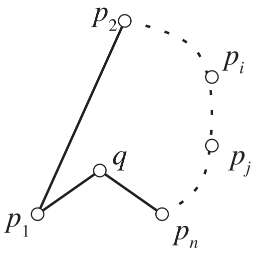



**Fig. 13** Case 5.

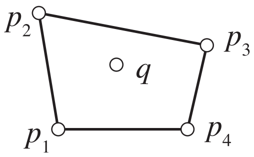



**Fig. 14** Case 6.

$qp_j$, $qp_k$, and $qp_\ell$, and $\mathcal{B}$ can choose any.

Hereafter, we show that $\mathcal{R}$ cannot win in any case. In each case below, the points are relabeled to simplify.

(Case 5): In this case, we have some points between $p_2$ and $p_n$, and the line segment $p_2 p_n$ does not intersect with $qp_n$ and $p_1 q$. In this case, **Table 2** gives the strategy for $\mathcal{B}$.

(Case 6): In this case, we have a convex quadrilateral with central point $q$. As in Example 4, it is easy to see that $\mathcal{R}$ has no winning strategy in this case.

In each case, the number of points is decreasing, and $\mathcal{R}$ has no winning strategy, which completes the proof. □

## 6. Conclusion

In this paper, we formalized a combinatorial game that is an old pencil-and-paper game for two players played in Western Japan. This game has a similar flavor to "Games on Triangulations" investigated by Aichholzer et al. [1]. We have only showed the computational complexity in a few restricted cases of the game. Along the line in Ref. [1], we have a lot of unsolved variants in our game. For example, the hardness of a two-player variant of this game in general position is not settled. By Theorem 2, we show that the first player has a winning strategy if all points are in convex position. The strategies for the points in convex position with (fixed) $k > 1$ points inside of the points in convex position is one natural extension. If we follow the same line as the analysis of the case where $k = 1$, the proofs will be quite complicated. In Theorem 5, we showed that the first player does not have a winning strategy. In some positions, the second player has a winning strategy, and in some positions the game ends in tie. We note that there is an unsettled problem here; for any given position in Theorem 5, we only say that the first player does not have a winning strategy. That is, we cannot decide if this position will make in tie or the second player has a winning strategy. To solve this problem, we need more careful analysis or new ideas.

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