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# Achievement Games on a One-dimensional Board

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**Abstract:** Achievement games are usually played and studied for polynominos which are commonly called animals, and it is known whether each animal is a winner or a loser, except for one animal called Snaky. In this paper, we study achievement games for unconnected shapes which we call creatures. We mostly discuss them on a one-dimensional board, because they exhibit interesting behavior even in this simple setting. We determined whether a creature is a winner or a loser for those composed of no more than three stones. Furthermore, we proved there is an arbitrary large paving winner.

**Keywords:** achievement games, Harary's generalized tic-tac-toe, paving winner, one-dimensional board

## 1. Introduction

Achievement games for animals have been studied by many people, following the pioneering work by Frank Harary [1]. An animal is a polynomino on a lattice board modulo translation, rotation, and reflection. This game is played by two players A and B. Player A places a black stone and B places a white stone on a lattice board alternately, and a player wins if the player makes a copy of a given animal $L$ with her stones.

An animal $L$ for which player A has a winning strategy is called a *winner*, and otherwise is called a *loser*. Therefore, if $L$ is a loser then B's strategy is to form an $L$ first or prevent A from ever forming $L$. However, note that B's strategy cannot simply be to construct an instance of $L$, because such a strategy could be used by A one move earlier.

The animals in **Fig. 1**, **Fig. 2**, **Fig. 3** are already known to be winners. It is not known whether the animal in **Fig. 4** is a winner or a loser.

One can show that an animal is a winner by generating a strategy which can be expressed as a finite tree of movements of B followed by a corresponding movement by A. On the other hand, in order to show that an animal is a loser, one has to define an infinite strategy. A paving strategy is a simple method to define such an infinite strategy.

In this strategy, at the beginning of the game, player B partitions the cells into pairs. Such a partition is usually called a paving. During the game, player B responds to player A by placing a stone in the cell paired with the cell where A played.

Here is an example of a paving strategy.

**Example 1.1.** The animal in **Fig. 5** is a loser.

*Proof.* Player B makes a paving as in **Fig. 6** and moves as explained above. Player A needs to put her stone on both components of one of the pairs to form the animal, but this is prevented by player B. □

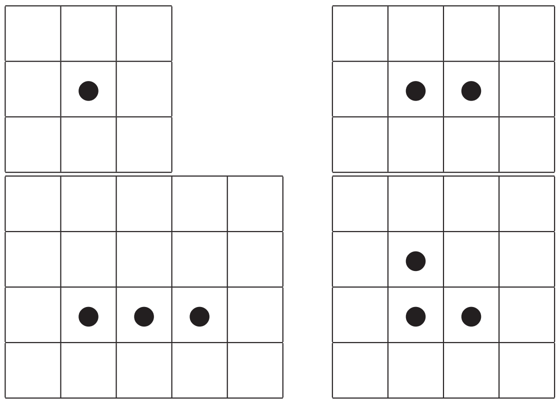

**Fig. 1** Winners composed of no more than three stones.

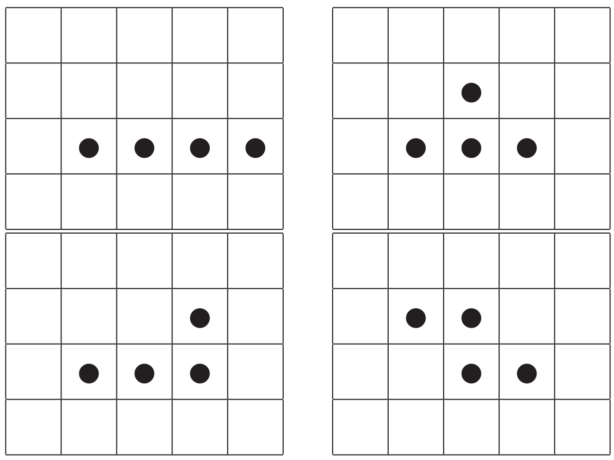

**Fig. 2** Winners composed of four stones.

As this example shows, one can show that an animal is a loser by finding a proper paving of the lattice. We can generalize the argument in Example 1.1 into the following proposition.

**Proposition 1.1.** *Let L be an animal. If there is a paving P such that every instance of L contains both elements of a pair in P, then L is a loser.*

If a pattern is a loser, then any animal bigger than that pattern

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is also a loser. We write this fact as a lemma.

**Lemma 1.1.** *Let L be an animal. If an animal that is obtained by removing a stone from L is a loser, then L is also a loser.*

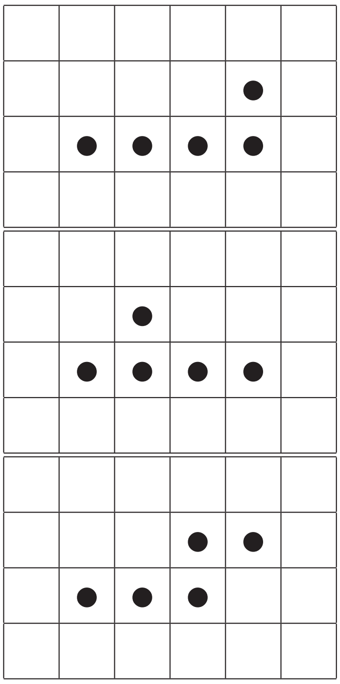

**Fig. 3** Winners composed of five stones.

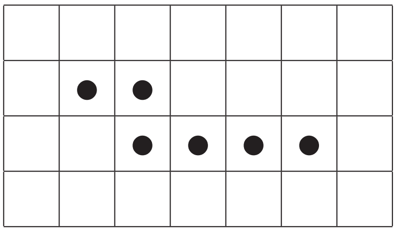

**Fig. 4** Snaky.

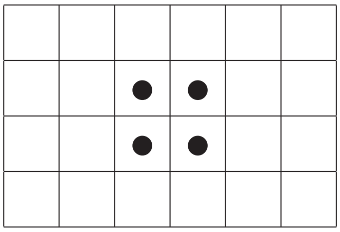

**Fig. 5** A loser composed of four stones.

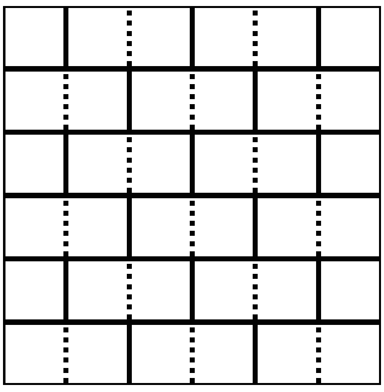

**Fig. 6** An example of paving strategy.

Some of the animals composed of five or six stones were proved to be losers by paving strategies.

Using Proposition 1.1 and Lemma 1.1, all animals composed of six or more stones are proven to be losers with the exception of Snaky, which is shown in Fig. 4 [1], [2]. Therefore, Snaky is the last unsolved animal and the subject of much research, such as Refs. [3], [4], [5].

In this paper, we study achievement games with unconnected shapes instead of animals. We call such a finite collection of squares in a lattice modulo translation, rotation and reflection, a creature. Note that Proposition 1.1 and Lemma 1.1 also hold for creatures. In particular, we mostly study one-dimensional achievement games of creatures.

One-dimensional achievement games are trivial if we consider animals, because only one animal of each size exists. However, as we will show, one-dimensional achievement games with creatures as target shapes are not at all trivial.

## 2. Achievement Games on a One-dimensional Board

### 2.1 Creature $kx$

In this section, we consider a one-dimensional board, which is a sequence of cells that is infinite in both directions. On this board we can consider many creatures, and we describe them as sequences of distances between their stones. We define that the distance between two stones is $n$ if there are $n-1$ cells between them. For example, the creature in **Fig. 7** is written (1, 3). Note that $(n_1, n_2, ..., n_{k-1}, n_k) = (n_k, n_{k-1}, ..., n_2, n_1)$ due to reflection invariance.

In addition, for a creature $x = (x_1, x_2, ..., x_n)$ and a positive integer $k$, we define $kx$ to be the creature $(kx_1, kx_2, ..., kx_n)$.

**Lemma 2.1.** *For all positive integers k and for all creatures x, kx is a winner if and only if x is a winner.*

*Proof.* The original board can be partitioned into k-separated boards so that cells $n$ and $n + k$ have adjacent positions in the same separated board. From the definition of $kx$, if $kx$ is formed, all stones of $kx$ are on one of the separated boards.

Assume that $x$ is a winner. The strategy of player A is then to form $x$ on one of the separated boards. This means that player A can form $kx$ on the original board. On the other hand, if $x$ is a loser, then the strategy of player B is to prevent A from forming $x$ in each separated board. This means that player B can prevent A from forming $kx$ on the original board. □

### 2.2 Creatures Composed of Three Stones

We will show some results on creatures composed of a small number of stones. First, if a creature has no more than two stones, it is clearly a winner.

Suppose that $x = (n, m)$ is a creature with three stones. We say that $x$ has type N-N if $n = m$, and $x$ has type N-M if $n \neq m$.

**Proposition 2.1.** *If the type of x is N-N, then x is a loser.*

*Proof.* Applying Proposition 1.1 to the paving in **Fig. 8** shows

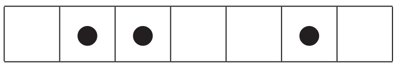

**Fig. 7** Creature (1, 3).

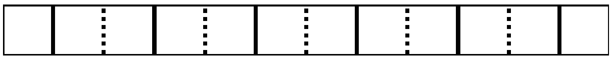

**Fig. 8** Paving strategy for $(1, 1)$.

that $x = (1, 1)$ is a loser. Therefore, by Lemma 2.1, $(n, n)$ is a loser for $n \geq 1$. □

**Proposition 2.2.** *If the type of x is N-M, then x is a winner.*

*Proof.* We fix one cell as the origin and input coordinates to the cells. Without loss of generality, assume that player A places her first stone on cell zero, and player B places her first stone on a cell which has a positive number.

There are two cases for $x = (n, m)$:

(i) Player B places her first stone on cell $m$. Then player A will place her second stone on cell $n$ and she can place her third stone on cell $n + m$ or on cell $-m$.

(ii) Player B places her first stone on another cell. Then player A will place her second stone on cell $-n$ and she can place her third stone on cell $-m - n$ or on cell $m$. □

### 2.3 Creatures Composed of Four Stones

Next, we study creatures composed of four stones. If one can obtain a creature of type N-N by removing one stone from a creature, then it is also a loser by Proposition 2.1 and Lemma 1.1. Otherwise, one of the following two cases holds: (i) all the distances between pairs of stones in $x$ are different, or (ii) the distances of the two stones in the middle from both sides are the same. We say the former has type DD (different distances), and the later has type N-M-N.

**Theorem 2.1.** *For every odd number n and every even number m, creature $x = (n, m, n)$ is a loser.*

*Proof.* Since $n + m$ is an odd number, make a pavement P that consists of pairs of cell $2k$ and cell $2k + n + m$ for each integer $k$. From Proposition 1.1, we can show that $x$ is a loser by showing that any instance of $x$ would contain both elements of one of the pairs in P.

There are two cases:

(i) The leftmost stone of the instance is on cell $2k$. So the stones of the instance are on cells $2k$, $2k + n$, $2k + n + m$ and $2k + 2n + m$. In this case, cell $2k$ and cell $2k + n + m$ form a pair in the paving P.

(ii) The leftmost stone of the instance is on cell $2k + 1$. So the stones of the instance are on cells $2k + 1$, $2k + n + 1 = 2(k + \frac{n+1}{2})$, $2k + n + m + 1$ and $2k + 2n + m + 1 = 2(k + \frac{n+1}{2}) + n + m$. In this case, cell $2(k + \frac{n+1}{2})$ and cell $2(k + \frac{n+1}{2}) + n + m$ form a pair in the paving P. □

Next, we consider a creature $x = (n, m, l)$ of type DD. Since distances between pairs of stones in $x$ are different, we have $n \neq m$, $m \neq l$, $l \neq n$, $n + m \neq l$ and $n \neq m + l$.

We say that $x$ is a *paving winner* if no paving strategy can prevent player A from forming $x$. Harborth and Seemann showed that Snaky is a paving winner [3].

From Proposition 1.1, we can say the following about paving winners:

**Proposition 2.3.** *Let L be a creature. If for every paving P, there exists an instance of L such that it does not contain both elements of a pair in P, then L is a paving winner.*

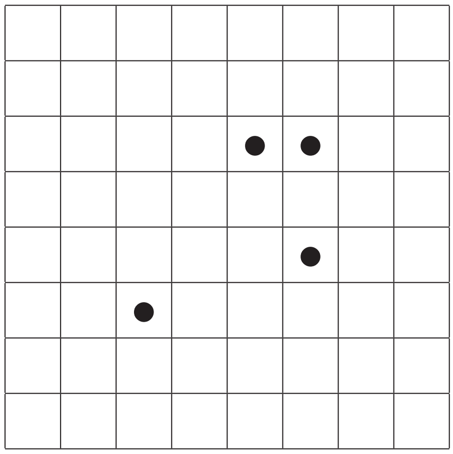

**Fig. 9** A two-dimensional paving winner.

**Theorem 2.2.** *If all the distances in a creature $x = (n_1, n_2, ..., n_{k-1}, n_k)$ are different, then x is a paving winner.*

*Proof.* Let P be a paving and let D be a sequence of cells of length $2(n_1 + n_2 + ... + n_k + 1)$.

D contains at most $(n_1 + n_2 + ... + n_k + 1)$ components of P. On the other hand, because all the distances between two stones are different, each pair is contained in at most two potential instances of $x$.

Therefore the number of potential instances of $x$ contained in D that are prevented by P is at most $2(n_1 + n_2 + ... + n_k + 1)$. On the other hand, there are $2(n_1 + n_2 + ... + n_k + 2)$ possible instances of $x$ which are contained in D. Therefore, there exist instances of $x$ that are not prevented by P. □

**Corollary 2.1.** *For all positive integers k, $(1, 2, 4, ..., 2^k)$ is a paving winner.*

It is an open question whether $(1, 2, 4, ..., 2^k)$ is a winner for $k \geq 2$. If $(1, 2, 4, ..., 2^k)$ is a winner for all $k$, then this means that there is an arbitrarily large winner. On the other hand, if $(1, 2, 4, ..., 2^k)$ is a loser for some $k$, then any proof of this will not be obtained through the paving strategy.

### 2.4 Expansion to a Normal Board

We can use a method similar to Theorem 2.2 to show that some two-dimensional creatures are paving winners. For example, consider the creature $x$ in **Fig. 9**.

Note that all the distances of pairs of stones of $x$ are different.

Let D be a $8 \times 8$ grid of cells. Since D is contained in a $4 \times 4$ grid, the number of possible instances of $x$ in D is $(8 - 4 + 1) \times (8 - 4 + 1) \times 8 = 200$ (considering translation, rotation, and reflection). If two instances share a pair, then they are the same two stones of D. By considering its reflection and 180 degree rotation around the middle of the two stones, there are at most four potential instances of $x$ shared by a pair of cells. Therefore, a paving strategy prevents at most $\frac{8 \times 8}{2} \times 4 = 128$ instances of $x$, since there are at most $\frac{8 \times 8}{2}$ pairs in D.

Thus, this creature is also a paving winner.

## 3. Conclusion

In this article, we show some results of achievement games on a one-dimensional board. Our main results are complete solutions for every creature composed of three or less stones, and partial solutions for creatures composed of four stones. In particular, we show there are arbitrarily large paving winners.

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