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Refutability and Reliability for Inductive Inference of Recursive Real-Valued Functions

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Inductive inference gives us a theoretical model of concept learning from examples. In this paper, we study *refutably* and *reliably* inductive inference of *recursive real-valued functions*. First we introduce the new criteria RealRefEx for refutable inference and RealRelEx for reliable inference. Then, we compare these two criteria with RealEx for identification in the limit, RealFin for learning finitely and RealNum! for learning by enumeration that have been already introduced in the previous works, and investigate their interaction. In particular, we show that RealRefEx and RealRelEx are closed under union, as similar as the criteria RefEx and RelEx for inductive inference of recursive functions.

1. Introduction

It is desirable for scientists to automatically learn real-valued functions from given observed data. Note first that, in our scientific activities, it is impossible to observe the exact value of a real number, but possible to observe only its approximations. Nevertheless it is necessary for Theoretical Computer Science to deal with real numbers. Several formulations for *computable* real numbers [6),15),21),22)] are known and deeply studied.

In this paper, we pay our attention to *recursive real-valued functions* [10),11)]. Since input data of recursive real-valued functions are not discrete but numerical as similar as scientific experiments or observations, they inevitably involve some range of errors. Then, such numerical data are represented by pairs of rational numbers approximating an exact value and an error bound, which are related to interval numbers [1),18)], that is, closed intervals containing the exact value.

On the other hand, there are also many models that provide us the theoretical foundation of concept learning from given data. In this paper, we adopt *inductive inference*, which comes from the famous Gold's paper [9)] in the 1960's and is currently one of the most important research topics in the field of Algorithmic/Computational Learning Theory. In inductive inference, we formulate a learning machine as an algorithm that sometimes outputs a hypothesis from given data. Then, whether or not a learning process is successful is determined by a sequence of hypotheses as outputs under several *criteria*.

Historically, the famous criteria Ex [9)], Fin [9)] and Num [4),5)] have been introduced for inductive inference of *recursive functions*. They correspond to *identification in the limit*, *learning finitely* and *learning by enumeration*, respectively. More formally, the criterion Ex means that the sequence of all hypotheses that a learning machine outputs converges to just one correct hypothesis *in the limit*. The criterion Fin means that a learning machine outputs a correct hypothesis just once within *finite time* and halts. The criterion Num means that a learning machine enumerates hypotheses until a hypothesis is found that agrees with all the data received so far.

By Ex, Fin and Num, we denote the class of all sets of recursive functions that are inferable in the limit, the class of all inferable sets of recursive functions that are finitely inferable and all subsets of recursively enumerable sets of recursive functions, respectively[☆]. As for the relationship between Ex, Fin and

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☆ Note that Num! denotes the class of all recursively enumerable sets of recursive functions. In this paper, we will extend Num! to RealNum! for inductive inference of recursive real-valued functions. The criterion Num! is contained in Num for inductive inference of recursive functions.

Num, it is known that Fin $\subsetneq$ Ex, Num $\subsetneq$ Ex, Num$\setminus$Fin $\neq \emptyset$ and Fin$\setminus$Num $\neq \emptyset$[8),9),13),14),24)].

In 1990's, Hirowatari and Arikawa[10),11)] have first formulated theory of inductive inference of *recursive real-valued functions* and then their co-authors[2),12)] have deeply studied the theory. In particular, Hirowatari and Arikawa[11)] have extended the criteria Ex, Fin and Num! to RealEx, RealFin and RealNum! for inductive inference of recursive real-valued functions, and shown that RealFin $\subsetneq$ RealEx, RealFin $\cap$ RealNum! $\neq \emptyset$ and RealNum! $\setminus$ RealEx $\neq \emptyset$.

In this paper, we develop *inductive inference of recursive real-valued functions*, by paying our attention to both *refutability* and *reliability.*

In our scientific activities, first we must select a hypothesis space from which we propose theories or hypotheses. If the hypothesis is not in the space from the observed data, we stop searching for the hypothesis space and *refute* it. On the other hand, in such a case, most learning machines will continue forever to search the space for a new hypothesis, because they cannot know the time when to stop such an ineffective searching.

Mukouchi and Arikawa[19)] have first formulated and developed *refutably* inductive inference of formal languages and formal systems. In their framework, the machine will discover a hypothesis which is producing the sequence if it is in the space, otherwise it will *refute* the whole space and stop. Hence, when the space is refuted, we may give another space to the machine and try to make such a discovery in the new space. After their introduction, various researchers[13),16),20)] have been developed refutable inference/learning.

On the other hand, in the criterion Ex for inductive inference of recursive functions in the limit, a learning machine may converge to an incorrect hypothesis, after receiving data of a recursive function which is not learned by the machine. In order to avoid such phenomenon, Minicozzi[17)] and L. Blum and M. Blum[7)] have introduce the *reliability* requiring that whenever a learning machine converges to a hypothesis from given data of a recursive function, it always identifies the function. We call such a learning machine *reliable.* The reliability realizes the requirement that a *reliable* scientist never fails to signal the inaccuracy of a previous incorrect hypothesis. The signal is given by eventually changing the previous hypothesis, or by producing no hypothesis at all on a later input[14)]. In other words, a reliable learning machine is regarded as a model of ideal scientists. We denote the criterion for reliably inductive inference of recursive functions in the limit by RelEx.

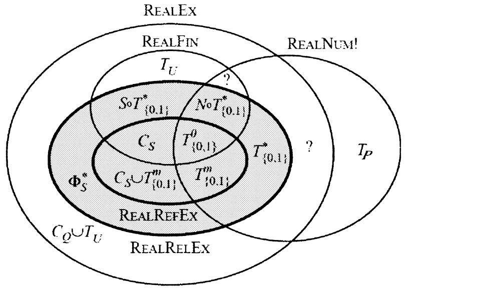

Fig. 1 The relationship between criteria RealRefEx, RealRelEx, RealEx, RealFin and RealNum!. For $\mathcal{T}^m_{\{0,1\}}$ and $\mathcal{C}_S \cup \mathcal{T}^m_{\{0,1\}}$, m is a positive natural number.

Recently, Jain, et al.[13)] have deeply studied *refutably* inductive inference of *recursive functions*, together with *reliably* inductive inference and others. They have introduced the new criterion RefEx for refutably inductive inference of recursive functions in the limit, and compared it with criteria RelEx, Ex, Fin and Num. Hence, the interaction for the criteria[7),13),17)] has been shown that RefEx $\subsetneq$ RelEx $\subsetneq$ Ex, RefEx $\setminus$ Num $\neq \emptyset$, Fin $\setminus$ RelEx $\neq \emptyset$, RelEx $\setminus$ Fin $\neq \emptyset$, and Num $\subsetneq$ RelEx. Furthermore, RefEx and RelEx are closed under union[7),13),17)].

Hence, in this paper, we investigate *refutably* and *reliably* inductive inference of *recursive real-valued functions.* First, we introduce the new criteria RealRefEx and RealRelEx for *refutably* and *reliably* inductive inference of recursive real-valued functions in the limit, respectively. Then, we show the interaction of our criteria RealRefEx, RealRelEx, RealEx, RealFin and RealNum! described in **Fig. 1**. In particular, we show that RealRefEx and RealRelEx are closed under union as similar as RefEx and RelEx.

2. Recursive Real-Valued Functions

In this section, we prepare some notions for *recursive real-valued functions* according to papers[11),12)].

Let N, Q and R be the sets of all natural numbers, rational numbers and real numbers, respectively. By N^+ and Q^+ we denote the sets of all positive natural numbers and positive rational numbers, respectively.

Definition 1 Let f and g be functions from N to Q and Q^+, respectively, and x a real number. A pair $\langle f, g\rangle$ is an *approximate expression* of x if f and g satisfy the following conditions:
(1) $\lim_{n\to\infty} g(n) = 0$.
(2) $|f(n) - x| \leq g(n)$ for each $n \in N$.
A real number x is *recursive* if there exists an approximate expression $\langle f, g\rangle$ of x such that f and g are recursive functions.

Note that $f(n)$ and $g(n)$ represent an approximate value of a real number and an error bound at point n, respectively.

In order to formulate recursive real-valued functions, we introduce the concepts of *rationalized domains* and *rationalized functions*.

Definition 2 A *rationalized domain* of $S \subseteq R$, denoted by Dom_S, is a subset of $Q \times Q^+$ which satisfies the following conditions:
(1) *Every interval in Dom_S is contained in S.* For each $\langle p, \alpha\rangle \in Dom_S$, it holds that $[p - \alpha, p + \alpha] \subseteq S$.
(2) *Dom_S covers the whole S.* For each $x \in S$, there exists an element $\langle p, \alpha\rangle \in Dom_S$ such that $x \in [p - \alpha, p + \alpha]$. Especially, if $x \in S$ is an interior point, then there exists an element $\langle p, \alpha\rangle \in Dom_S$ such that $x \in (p - \alpha, p + \alpha)$.
(3) *Dom_S is closed under subintervals.* For each $\langle p, \alpha\rangle \in Dom_S$ and $\langle q, \beta\rangle \in Q \times Q^+$ such that $[q - \beta, q + \beta] \subseteq [p - \alpha, p + \alpha]$, it holds that $\langle q, \beta\rangle \in Dom_S$.

Definition 3 Let $h : S \to R$ $(S \subseteq R)$ be a real-valued function, and Dom_S a rationalized domain of S. A *rationalized function* of h, denoted by $\mathcal{A}_h$, is a computable function from Dom_S to $Q \times Q^+$ which satisfies the following condition:

> For each $x \in S$ and approximate expression $\langle f, g\rangle$ of x, there exists an approximate expression $\langle f_0, g_0\rangle$ of $h(x)$ such that $\mathcal{A}_h(\langle f(n), g(n)\rangle) = \langle f_0(n), g_0(n)\rangle$ for each $\langle f(n), g(n)\rangle \in Dom_S$.

We sometimes call a rationalized function $\mathcal{A}_h$ of h an algorithm which computes h. A real-valued function h can have many rationalized functions $\mathcal{A}_h$. If f and g are recursive, then so are f_0 and g_0. Thus, if $x \in S$ is recursive, then $h(x)$ is recursive.

Definition 4 A function $h : S \to R$ $(S \subseteq R)$ is a *recursive real-valued function* if there exists a rationalized function $\mathcal{A}_h : Dom_S \to Q \times Q^+$ of h, where Dom_S is a rationalized domain of S. We demand that $\mathcal{A}_h(\langle p, \alpha\rangle)$ does not halt for all $\langle p, \alpha\rangle \notin Dom_S$. Furthermore, by $\mathcal{RRVF}$ we denote the set of all recursive real-valued functions.

Since $h(x)$ is recursive for each recursive real number $x \in S$, we can design an effective procedure to find $h(x)$ from the given x. Thus, recursive real-valued functions are computable. Furthermore, since a recursive real-valued function h always has a rationalized domain $\mathcal{A}_h$ of h, it holds that $h([p-\alpha, p+\alpha]) \subseteq [q-\beta, q+\beta]$, where $\langle p, \alpha\rangle \in Dom_S$ and $\langle q, \beta\rangle = \mathcal{A}_h(\langle p, \alpha\rangle)$.

A set $\mathcal{T} \subseteq \mathcal{RRVF}$ is said to be *recursively enumerable* if there is a recursive function Ψ such that the set $\mathcal{T}$ is equal to the set of all functions computed by algorithms $\Psi(0), \Psi(1), \ldots$.

In this paper, we say also that a set $\mathcal{T} \subseteq \mathcal{RRVF}$ is recursively enumerable if there exists a recursively enumerable set $\mathcal{H} \subseteq \mathcal{RRVF}$ which is a set of extensions of functions in $\mathcal{T}$.

3. Inductive Inference of Recursive Real-Valued Functions

In this section, first we prepare some notions necessary to the later discussion. Next we formulate inductive inference of recursive real-valued functions. Finally we introduce new criteria RealRefEx corresponding to *learning refutably in the limit* and RealRelEx corresponding to *learning reliably in the limit*.

Definition 5 Let $h : S \to R$ and $h_0 : S_0 \to R$ $(S,\ S_0 \subseteq R)$ be recursive real-valued functions. Then, we say that h_0 is a *restriction* of h or h is an *extension* of h_0, denoted by $h_0 = h|_{S_0}$, if $S_0 \subseteq S$ and $h_0(x) = h(x)$ for each $x \in S_0$. Furthermore, for the set $\mathcal{T}$ of recursive real-valued functions, we call a restriction of a function in $\mathcal{T}$ a *restriction* in $\mathcal{T}$ simply.

Since we do not distinguish a function from its extensions, we claim that our learning is successful even when a sequence of conjectures converges to an algorithm which computes an extension of the target [7)].

By φ_j we denote the partial recursive function from N to N computed by a program j. By $\mathcal{P}$ we denote the set $\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ of all partial recursive functions from N to N and by $\mathcal{R}$ the set of all recursive functions from N to N.

Definition 6 Let $S_0 \subseteq N$ be the domain of $\varphi_j \in \mathcal{P}$. Then, a function $h_j : S \to R$ $(S \subseteq R)$ is called the *stair function* of φ_j if h_j satisfies

the following conditions:
(1) $S = \bigcup_{i \in S_0} (i - \frac{1}{2}, i + \frac{1}{2})$,
(2) $h_j(x) = \varphi_j(i)$ for each $x \in (i - \frac{1}{2}, i + \frac{1}{2})$ and $i \in S_0$.

For $\mathcal{S} \subseteq \mathcal{P}$, we call a stair function of a function in $\mathcal{S}$ a *stair function* in $\mathcal{S}$ simply.

Definition 7 For $\varphi_j \in \mathcal{R}$, the following function $h_j : [0, \infty) \to R$ is called the *line function* of φ_j.

$$\begin{aligned} h_j(x) &= (\varphi_j(i+1) - \varphi_j(i))x \\ &+ \varphi_j(i)(i+1) - \varphi_j(i+1)i \end{aligned}$$
for each $x \in [i, i+1]$ and $i \in N$.

For $\mathcal{T} \subseteq \mathcal{R}$, we call a line function of a function in $\mathcal{T}$ a *line function* in $\mathcal{T}$ simply.

Now we formulate *inductive inference of recursive real-valued functions*. In our scientific activities, it is impossible to observe the exact value of a real number x, but possible to observe only approximations of x. Such approximations can be captured as a pair $\langle p, \alpha \rangle$ of rational numbers such that p is an approximate value of the number x and α is its positive error bound, i.e., $x \in [p - \alpha, p + \alpha]$. We call such a pair $\langle p, \alpha \rangle$ a *datum* of x.

Definition 8 An *example* of a function $h : S \to R$ ($S \subseteq R$) is a pair $\langle \langle p, \alpha \rangle, \langle q, \beta \rangle \rangle$ satisfying that there exists a real number $x \in S$ such that $\langle p, \alpha \rangle$ and $\langle q, \beta \rangle$ are data of x and $h(x)$, respectively.

Definition 9 A *presentation* of a function $h : S \to R$ ($S \subseteq R$) is an infinite sequence $\sigma = w_1, w_2, \ldots$ of examples of h in which, for each real number x in the domain of h and each $\zeta > 0$, there exists an example $w_k = \langle \langle p_k, \alpha_k \rangle, \langle q_k, \beta_k \rangle \rangle$ such that $x \in [p_k - \alpha_k, p_k + \alpha_k]$, $h(x) \in [q_k - \beta_k, q_k + \beta_k]$, $\alpha_k \le \zeta$ and $\beta_k \le \zeta$. By $\sigma[n]$ we denote the initial segment of n examples in σ.

We can imagine an example of h as a rectangular box $[p - \alpha, p + \alpha] \times [q - \beta, q + \beta]$. Then, a sequence $w_1, w_2, \ldots$ of boxes is a presentation of h if each box contains a point $(x, h(x))$ on the graph of h, and for each point on the graph there are arbitrarily small boxes w_k having the point in their interior (See **Fig. 2**).

An *inductive inference machine* (*IIM*, for short) is a procedure that requests inputs from time to time and produces from time to time algorithms that compute recursive real-valued functions. These algorithms produced by an IIM while receiving examples are called *conjectures*.

For an IIM $\mathcal{M}$ and a finite sequence $\sigma[n] = \langle w_1, w_2, \ldots, w_n \rangle$, by $\mathcal{M}(\sigma[n])$ we denote the last conjecture of $\mathcal{M}$ after requesting examples $w_1, w_2, \ldots, w_n$ as inputs.

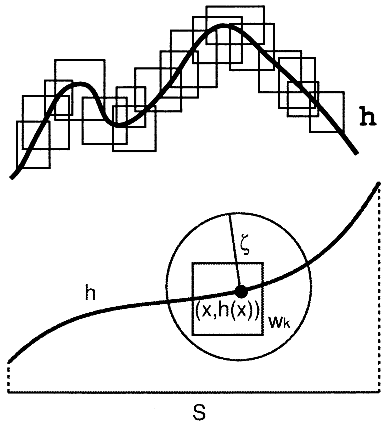

Fig. 2 Data (upper) and a presentation (lower) of a recursive real-valued function h.

Definition 10 Let σ be a presentation of a function and $\{\mathcal{M}(\sigma[n])\}_{n \ge 1}$ the infinite sequence of conjectures produced by an IIM $\mathcal{M}$. The sequence $\{\mathcal{M}(\sigma[n])\}_{n \ge 1}$ *converges* to an algorithm $\mathcal{A}_h$ if there exists a number $n_0 \in N$ such that $\mathcal{M}(\sigma[m])$ equals $\mathcal{A}_h$ for each $m \ge n_0$.

Finally we introduce the new criteria RealRefEx and RealRelEx, together with RealEx, RealFin and RealNum! [11].

Definition 11 Let h be a recursive real-valued function and $\mathcal{T}$ a class of recursive real-valued functions.
(1) An IIM $\mathcal{M}$ *infers h in the limit* (or RealEx-*infers* h), denoted by $h \in$ RealEx($\mathcal{M}$), if, for each presentation σ of h, the sequence $\{\mathcal{M}(\sigma[n])\}_{n \ge 1}$ converges to an algorithm that computes an extension of h.
(2) An IIM $\mathcal{M}$ *infers* $\mathcal{T}$ (or RealEx-*infers* $\mathcal{T}$) if $\mathcal{M}$ infers every $h \in \mathcal{T}$ in the limit.
(3) A class $\mathcal{T}_0$ is *inferable* (or RealEx-*inferable*) if there exists an IIM that infers $\mathcal{T}_0$.

By RealEx we denote the class of all inferable classes of recursive real-valued functions.

Definition 12 Let h be a recursive real-valued function and $\mathcal{T}$ a class of recursive real-valued functions.
(1) An IIM $\mathcal{M}$ *finitely infers h* (or RealFin-*infers* h), denoted by $h \in$ RealFin($\mathcal{M}$), if, for each presentation σ of h, after some finite time the IIM $\mathcal{M}$ presented σ's examples outputs a unique algorithm that computes an extension of h.

(2) An IIM $\mathcal{M}$ *finitely infers* $\mathcal{T}$ (or RealFin-*infers* $\mathcal{T}$) if $\mathcal{M}$ finitely infers every $h \in \mathcal{T}$.
(3) A class $\mathcal{T}_0$ is *finitely inferable* (or RealFin-*inferable*) if there exists an IIM that finitely infers $\mathcal{T}_0$.

By RealFin we denote the class of all finitely inferable classes of recursive real-valued functions.

Definition 13 By RealNum! we denote the class of all recursively enumerable sets of recursive real-valued functions.

Definition 14 We say that $\mathcal{M}$ *refutably infers* $\mathcal{T}$ if $\mathcal{M}$ satisfies the following conditions. Here, $\perp$ is the *refutation symbol*.
(1) $\mathcal{T} \subseteq \text{RealEx}(\mathcal{M})$.
(2) If $h \in \text{RealEx}(\mathcal{M})$, then $\mathcal{M}(\sigma[n]) \neq \perp$ for each σ and $n \in N$.
(3) If $h \in \mathcal{RRVF} \setminus \text{RealEx}(\mathcal{M})$, then there exists a number $n \in N$ such that $\mathcal{M}(\sigma[m]) \neq \perp$ for each σ and $m < n$, and $\mathcal{M}(\sigma[m]) = \perp$ for each σ and $m \geq n$.

By RealRefEx we denote the class of all refutably inferable classes of recursive real-valued functions.

Definition 15 We say that $\mathcal{M}$ *reliably infers* $\mathcal{T}$ if $\mathcal{M}$ satisfies the following conditions:
(1) $\mathcal{T} \subseteq \text{RealEx}(\mathcal{M})$.
(2) If $h \in \mathcal{RRVF} \setminus \text{RealEx}(\mathcal{M})$, then the sequence $\{\mathcal{M}(\sigma[n])\}_{n \geq 1}$ does not converge to an algorithm for each σ.

By RealRelEx we denote the class of all reliably inferable classes of recursive real-valued functions.

4. Examples and Properties of Recursive Real-Valued Functions

In this section, we give several examples and properties of recursive real-valued functions necessary to discuss the interaction of our criteria in Section 5. Furthermore, in the last of this section, we show that RealRefEx and RealRelEx are closed under union.

For each set S, $\#S$ denotes the cardinality of S.

4.1 Self-Describing Functions

Let U be the set of all recursive functions f from N to N such that $\varphi_{f(0)} = f$ and $\mathcal{T}_U$ the set of all stair functions in U.

Lemma 1 $\mathcal{T}_U \in \text{RealFin} \setminus \text{RealNum!}$.

Proof. Since $\mathcal{T}_U$ is the set of all stair functions in U, it holds that $U \in$ Fin (*resp.*, $U \in$ Num!) iff $\mathcal{T}_U \in$ RealFin (*resp.*, $\mathcal{T}_U \in$ RealNum!). Since $U \notin$ Num! [3], it holds that $\mathcal{T}_U \notin$ RealNum!. On the other hand, by the definition of U, for each $f \in U$, it holds that $\varphi_{f(0)} = f$. Consider an IIM $\mathcal{M}$ which receives $\langle 0, f(0) \rangle$ and outputs a unique conjecture $f(0)$ after some finite time. Thus, $\mathcal{M}$ finitely infers U. Hence, it holds that $\mathcal{T}_U \in$ RealFin. □

```
IIM M_U
/* M_{T_U} reliably infers T_U */
begin
k ← 1;
for k = 1 to ∞ do begin
   read the data ⟨n_k, φ(n_k)⟩;
   A_k ← M_{T_U}(σ_h[k(k−1)/2 + 1]);
   j_{A_k} ← the program constructed by A_k;
   output j_{A_k};
end
```

Fig. 3 The IIM $\mathcal{M}_U$ in the proof of Lemma 2.

Lemma 2 $\mathcal{T}_U \notin$ RealRelEx.

Proof. Suppose that there exists an IIM $\mathcal{M}_{\mathcal{T}_U}$ which reliably infers $\mathcal{T}_U$. Let $\varphi \in U$ be a target function and $h \in \mathcal{T}_U$ a stair function of φ. Furthermore, let $\sigma_\varphi = \langle n_1, \varphi(n_1) \rangle, \langle n_2, \varphi(n_2) \rangle, \ldots$ be a presentation of φ. For each $\langle n_k, \varphi(n_k) \rangle$, we can define a constant function h_{n_k} from $(n_k - \frac{1}{2}, n_k + \frac{1}{2})$ to $\{\varphi(n_k)\}$. Let $\sigma^k = w_1^k, w_2^k, \ldots$ be a presentation of h_{n_k} for each $k \in N$, and $\sigma_h = w_1, w_2, \ldots$ an infinite sequence such that $w_k = w_{s-t+1}^t$, where $s, t \in N^+$ satisfying $\frac{1}{2}s(s-1) \leq k - 1 < \frac{1}{2}s(s+1)$ and $t = \frac{1}{2}s(s+1) - k + 1$. Then, σ_h is a presentation of $h \in \mathcal{T}_U$.

For each given algorithm $\mathcal{A}$, we can easily construct a program $j_\mathcal{A}$ which receives $n \in N$ as an input, and works as follows: For the input $n \in N$, if there exist a least number $k \in N$ and a number $m \in N$ such that $\mathcal{A}(\langle n, \frac{1}{2^k} \rangle)$ has an output $\langle q, \beta \rangle$, $|m - q| < \beta$ and $\beta < \frac{1}{2}$, then $j_\mathcal{A}$ outputs $m \in N$ else $j_\mathcal{A}$ never stops. Then, consider the IIM $\mathcal{M}_U$ in **Fig. 3** that requests data $\sigma_\varphi = \langle n_0, \varphi(n_0) \rangle, \langle n_1, \varphi(n_1) \rangle, \ldots$ as inputs from time to time. Thus, the IIM $\mathcal{M}_U$ reliably infers U in the limit (in the sense of RelEx), since the IIM $\mathcal{M}_{\mathcal{T}_U}$ reliably infers $\mathcal{T}_U$. However, it holds that $U \notin$ RelEx [14], which is a contradiction. □

Let $\mathcal{T}_\mathcal{P}$ be the set of all stair functions in $\mathcal{P}$. Then, by the previous work [11], the following statement holds (See Fig. 1).

$$\mathcal{T}_\mathcal{P} \in \text{RealNum!} \setminus \text{RealEx}.$$

4.2 Cardinality Functions of the Inverse Image from 0

For $\varphi \in \mathcal{R}$, $\varphi^{-1}(0)$ denotes the set $\{n \in N \mid \varphi(n) = 0\}$. Then, $\mathcal{R}_{\{0,1\}}$, $\mathcal{R}^m_{\{0,1\}}$ ($m \in N$) and $\mathcal{R}^*_{\{0,1\}}$ are defined as follows.

$$\mathcal{R}_{\{0,1\}} = \{\varphi : N \to \{0,1\} \mid \varphi \in \mathcal{R}\},$$
$$\mathcal{R}^m_{\{0,1\}} = \{\varphi \in \mathcal{R}_{\{0,1\}} \mid \#\varphi^{-1}(0) \leq m\},$$
$$\mathcal{R}^*_{\{0,1\}} = \bigcup_{m \in N} \mathcal{R}^m_{\{0,1\}},$$
$$\mathcal{R}^\infty_{\{0,1\}} = \{\varphi \in \mathcal{R}_{\{0,1\}} \mid \#\varphi^{-1}(0) = \infty\}.$$

Also let $\mathcal{T}^m_{\{0,1\}}$ (*resp.*, $\mathcal{T}^*_{\{0,1\}}$, $\mathcal{T}^\infty_{\{0,1\}}$) be the set of all line functions in $\mathcal{R}^m_{\{0,1\}}$ (*resp.*, $\mathcal{R}^*_{\{0,1\}}$, $\mathcal{R}^\infty_{\{0,1\}}$).

Lemma 3 $\mathcal{T}^m_{\{0,1\}} \in$ RealRefEx *for each* $m \in N$.

Proof. For each finite set $N_0 \subseteq N$, let $\varphi^{N_0} : N \to \{0,1\}$ be a recursive function such that $\varphi^{N_0}(n) = 0$ iff $n \in N_0$. Also let $\mathcal{A}_{N_0}$ be an algorithm which computes the line function of φ^{N_0}. Then, the algorithm $\mathcal{A}_\emptyset$ computes the constant function c_1 defined by $c_1(x) = 1$ for each $x \geq 0$. For each $k \in N$, by $\mathcal{T}_{\{0,1\}}(k)$ we denote the set of all functions h in $\mathcal{T}^*_{\{0,1\}}$ such that $h(x) = 1$ for each $x \geq k+1$. It is obvious that $\mathcal{T}_{\{0,1\}}(k)$ is finite for each $k \in N$.

For $m \in N$, let $Rest(\mathcal{T}^m_{\{0,1\}})$ be the set of all restrictions of every $h \in \mathcal{T}^m_{\{0,1\}}$. Then, we can design an IIM $\mathcal{M}_m$ such that RealEx$(\mathcal{M}_m) = Rest(\mathcal{T}^m_{\{0,1\}})$ for each $m \in N$ (see **Fig. 4**). For $\mathcal{M}_m$, it holds that $\mathcal{T}^m_{\{0,1\}} \subsetneq$ RealEx$(\mathcal{M}_m)$.

Consider a function $h \in \mathcal{RRVF} \setminus$ RealEx$(\mathcal{M}_m)$ and let σ be a presentation of h. Then, there exists a large enough $n \in N$ such that $\mathcal{M}(\sigma[n]) = \bot$. Hence, it holds that $\mathcal{T}^m_{\{0,1\}} \in$ RealRefEx. □

Lemma 4 $\mathcal{T}^m_{\{0,1\}} \notin$ RealFin *for each* $m \in N^+$.

Proof. Suppose that $\mathcal{T}^m_{\{0,1\}} \in$ RealFin. Then, there exists an IIM $\mathcal{M}$ which finitely infers $\mathcal{T}^m_{\{0,1\}}$. Let h be in $\mathcal{T}^{m-1}_{\{0,1\}}$ and σ a presentation of h. Then, there exists a number $n \in N^+$ such that the IIM $\mathcal{M}$ receives the sequence $\sigma[n]$ and outputs a unique algorithm that computes h. Let h_0 be in $\mathcal{T}^m_{\{0,1\}} \setminus \mathcal{T}^{m-1}_{\{0,1\}}$ such that $\{n \in N \mid h(n) = 0\} \subsetneq \{n \in N \mid h_0(n) = 0\}$ and σ_0 a presentation of h_0 such that $\sigma_0[n] = \sigma[n]$. Thus, $\mathcal{M}$ cannot finitely infer $h_0 \in \mathcal{T}^m_{\{0,1\}}$, which is a contradiction. □

The following lemma is not a direct property for $\mathcal{T}^*_{\{0,1\}}$ but useful for proving other lemmas.

Lemma 5 *For* $\mathcal{T} \in$ RealRefEx, *let* $\mathcal{M}$ *be an IIM which refutably infers* $\mathcal{T}$. *Then, for each* $h \in \mathcal{T}$, *every restriction of* h *is in* RealEx$(\mathcal{M})$.

Proof. Suppose that there exists a restriction h_0 of $h \in \mathcal{T}$ such that $h_0 \notin$ RealEx$(\mathcal{M})$. Then, there exists a number $m \in N$ such that $\mathcal{M}(\sigma_0[m]) = \bot$ for each presentation σ_0 of h_0. Let σ be a presentation of h such that $\sigma[m] = \sigma_0[m]$ for the above $m \in N$. Since σ is a presentation of h, it holds that $\mathcal{M}(\sigma[n]) \neq \bot$ for each $n \in N$, which is a contradiction. □

IIM $\mathcal{M}_m$
begin
$i \leftarrow 1$; $l \leftarrow 0$; $u \leftarrow -1$; $D \leftarrow \emptyset$; $N_0 \leftarrow \emptyset$; $\mathcal{A} \leftarrow \mathcal{A}_\emptyset$;
for $i = 1$ **to** ∞ **do begin**
  read the example $w_i = \langle\langle p_i, \alpha_i\rangle, \langle q_i, \beta_i\rangle\rangle$;
  $D \leftarrow D \cup \{w_i\}$;
  $u \leftarrow \max_{1 \leq j \leq i} \min\{u \in N \mid p_j + \alpha_j < u\}$;
  if D is a set of examples of a function
    in $\mathcal{T}^*_{\{0,1\}}$ **then**
   if $\exists k \in N \setminus N_0$ s.t. $|k - p_i| < \alpha_i < \frac{1}{4}$
     and $|q_i| < \beta_i < \frac{1}{4}$ **then**
    $N_0 \leftarrow N_0 \cup \{k\}$; $l \leftarrow l + 1$; $\mathcal{A} \leftarrow \mathcal{A}_{N_0}$;
   if $m < l$ **then** $\mathcal{A} \leftarrow \bot$;
  else
   $\mathcal{A} \leftarrow \bot$;
  output $\mathcal{A}$;
end

Fig. 4 The IIM $\mathcal{M}_m$ in the proof of Lemma 3.

Lemma 6 $\mathcal{T}^*_{\{0,1\}} \notin$ RealRefEx.

Proof. Suppose that $\mathcal{T}^*_{\{0,1\}} \in$ RealRefEx. Then, there exists an IIM $\mathcal{M}$ which refutably infers $\mathcal{T}^*_{\{0,1\}}$. By Lemma 5, $\mathcal{M}$ also can infer every restriction of a function in $\mathcal{T}^*_{\{0,1\}}$. We note that there exists a function $h \in \mathcal{T}^\infty_{\{0,1\}}$ such that $h \notin$ RealEx$(\mathcal{M})$. For each presentation σ of h, there exists a number $n \in N$ such that $\mathcal{M}(\sigma[n]) = \bot$. Then, there exists a function $h_0 \in \mathcal{T}^*_{\{0,1\}}$ such that $\sigma[n]$ is a sequence of examples of h_0. Let σ_0 be a presentation of h_0 such that $\sigma_0[n] = \sigma[n]$. Then, it holds that $\mathcal{M}(\sigma_0[n]) = \bot$, which is a contradiction. □

Lemma 7 $\mathcal{T}^*_{\{0,1\}} \in$ RealRelEx.

Proof. Let h be a target function and $\sigma = w_1, w_2, \ldots$ a presentation of h such that $w_k = \langle\langle p_k, \alpha_k\rangle, \langle q_k, \beta_k\rangle\rangle$. Without loss of generality, we can assume that $\alpha_k < \frac{1}{4}$, $\beta_k < \frac{1}{4}$ for each $k \in N^+$. Then, consider the IIM $\mathcal{M}_*$ in **Fig. 5** that requests data $w_1, w_2, \ldots$ as inputs from time to time. For each target function h, the IIM $\mathcal{M}_*$ converges to an algorithm iff

```
IIM M_*
begin
D ← ∅; F ← ∅; k ← 1; T ← 0;
for k = 1 to ∞ do begin
   read the data w_k = ⟨⟨p_k, α_k⟩, ⟨q_k, β_k⟩⟩;
   D ← D ∪ {w_k};
   if D is a set of examples of a function
        in T*_{0,1} then
      if ∃s ∈ N s.t. |s − p_k| < α_k and
            |q_k| < β_k then
         F ← F ∪ {s};
      let h_F be the function in T*_{0,1}
         such that h_F(n) = 0 iff n ∈ F;
      A_F ← algo(h_F);
   else
      T ← 1; K ← {m ∈ N | m ≤ k};
      let h_K be the function in T*_{0,1}
         such that h_K(n) = 0 iff n ∈ K;
      A_K ← algo(h_K);
   if T = 0 then output A_F;
   else output A_K;
end
```

Fig. 5 The IIM $\mathcal{M}_*$ in the proof of Lemma 7.

$h \in \mathcal{T}^*_{\{0,1\}}$. □

4.3 Line Functions Based on Non-R.e. Sets

For each subset $F \subseteq N$, let φ_F be the following function:

$$\varphi_F(n) = \begin{cases} 0 & \text{if } n \in F, \\ 1 & \text{otherwise.} \end{cases}$$

Let $S \subsetneq N$ be an infinite subset that is not recursively enumerable and Φ^*_S the set of all line functions of φ_F such that F is a finite subset of S.

Lemma 8 $\Phi^*_S \in \textsc{RealRelEx} \setminus \textsc{RealRefEx}$ *for* $S = \{j \in N \mid \varphi_j \in \mathcal{R}\}$.

Proof. Consider the set $\mathcal{T}^*_{\{0,1\}}$. By Lemma 7, it holds that $\mathcal{T}^*_{\{0,1\}} \in \textsc{RealRelEx}$. Since $\Phi^*_S \subsetneq \mathcal{T}^*_{\{0,1\}}$, it holds that $\Phi^*_S \in \textsc{RealRelEx}$.

Suppose that $\Phi^*_S \in \textsc{RealRefEx}$. Let $\mathcal{M}$ be an IIM which refutably infers Φ^*_S and Φ^∞_S the sets of all line functions of φ_F for each infinite subset $F \subsetneq S$. Note that $\Phi^*_S \cap \Phi^\infty_S = \emptyset$. Then, there exists a function $h_{S_0} \in \Phi^\infty_S$ such that $h_{S_0} \notin \textsc{RealEx}(\mathcal{M})$, where $S_0 \subseteq S$ is an infinite set.

Let σ be a presentation of h_{S_0}. There exists a number $n \in N$ such that $\mathcal{M}(\sigma[n]) = \perp$. Then, there exists a function $h_0 \in \Phi^*_S$ such that $\sigma[n]$ is a sequence of examples of h_0. Let σ_0 be a presentation of h_0 such that $\sigma_0[n] = \sigma[n]$. Then, it holds that $\mathcal{M}(\sigma_0[n]) = \perp$, which is a contradiction. □

4.4 Compositions

For $m \in N^+$ and $\varphi_j \in \mathcal{R}^m_{\{0,1\}} \setminus \mathcal{R}^{m-1}_{\{0,1\}}$, let $\varphi_{m,j}$ be the following function:

$$\varphi_{m,j}(n) = \begin{cases} m & \text{if } n = 0, \\ \varphi_j(n-1) & \text{otherwise.} \end{cases}$$

For $S \subseteq N$, let $S \circ \mathcal{T}^*_{\{0,1\}}$ be the set of all line functions of $\varphi_{m,j}$ for each $m \in S$ and $\varphi_j \in \mathcal{R}^m_{\{0,1\}} \setminus \mathcal{R}^{m-1}_{\{0,1\}}$. If $S = N$, then we denote the set by $N \circ \mathcal{T}^*_{\{0,1\}}$.

Lemma 9 $S \circ \mathcal{T}^*_{\{0,1\}} \in \textsc{RealFin}$.

Proof. Let h be in $S \circ \mathcal{T}^*_{\{0,1\}}$ such that $h(0) = m$, and σ a presentation of h. Consider an IIM $\mathcal{M}$ which receives the finite initial segment of examples in σ, finds the number m and the m-points $\langle t_1, 0\rangle$, $\langle t_2, 0\rangle, \ldots, \langle t_m, 0\rangle$ such that $h(t_k) = 0$ for each $k \le m$, and outputs a unique algorithm which computes a function h. Since $\mathcal{T}^m_{\{0,1\}} \setminus \mathcal{T}^{m-1}_{\{0,1\}} \in \textsc{RealFin}$ for each $m \in N^+$ and by the definition of h, it holds that $S \circ \mathcal{T}^*_{\{0,1\}} \in \textsc{RealFin}$. □

Lemma 10 $S \circ \mathcal{T}^*_{\{0,1\}} \notin \textsc{RealRefEx}$.

Proof. Suppose that $S \circ \mathcal{T}^*_{\{0,1\}} \in \textsc{RealRefEx}$. There exists an IIM $\mathcal{M}$ which refutably infers $S \circ \mathcal{T}^*_{\{0,1\}}$. By Lemma 5, $\mathcal{M}$ also can infer every restriction in $S \circ \mathcal{T}^*_{\{0,1\}}$ in the limit. We note that there exists a function $h \in \mathcal{T}^\infty_{\{0,1\}}$ such that $h|_{[1,\infty)} \notin \textsc{RealEx}(\mathcal{M})$. For each presentation σ of $h|_{[1,\infty)}$, there exists a number $n \in N$ such that $\mathcal{M}(\sigma[n]) = \perp$. Then, there exists a function $h_0 \in S \circ \mathcal{T}^*_{\{0,1\}}$ such that $\sigma[n]$ is a sequence of examples of h_0. Let σ_0 be a presentation of h_0 such that $\sigma_0[n] = \sigma[n]$. Then, it holds that $\mathcal{M}(\sigma_0[n]) = \perp$, which is a contradiction. □

Lemma 11 $S \circ \mathcal{T}^*_{\{0,1\}} \in \textsc{RealRelEx}$.

Proof. Let h be a target function and $\sigma = w_1, w_2, \ldots$ a presentation of h such that $w_k = \langle\langle p_k, \alpha_k\rangle, \langle q_k, \beta_k\rangle\rangle$. Without loss of generality, we can assume that $\alpha_k < \frac{1}{4}$, $\beta_k < \frac{1}{4}$ for each $k \in N^+$. Then, consider the IIM $\mathcal{M}_{S\circ\mathcal{T}}$ in **Fig. 6** that requests data $w_1, w_2, \ldots, w_n, \ldots$ as inputs from time to time. For each target function h, the IIM $\mathcal{M}_{S\circ\mathcal{T}}$ converges to an algorithm iff $h \in S \circ \mathcal{T}^*_{\{0,1\}}$. Thus, the IIM $\mathcal{M}_{S\circ\mathcal{T}}$ reliably infers $S \circ \mathcal{T}^*_{\{0,1\}}$ in the limit. □

Corollary 1 *The following two statements hold.*

(1) $N \circ \mathcal{T}^*_{\{0,1\}} \in \textsc{RealFin} \setminus \textsc{RealRefEx}$.

(2) $N \circ \mathcal{T}^*_{\{0,1\}} \in \textsc{RealRelEx}$.

Proof. It is straightforward from Lemma 9, 10 and 11. □

```
IIM M_{S∘T}
begin
D ← ∅; F ← ∅; Y ← ∅; k ← 1; T ← 1;
for k = 1 to ∞ do begin
   read the data w_k = ⟨⟨p_k, α_k⟩, ⟨q_k, β_k⟩⟩;
   D ← D ∪ {w_k};
   if D is a set of examples of a function
         in S ∘ T*_{0,1} then
      if ∃s ∈ N+ s.t. |s − p_k| < α_k and |q_k| < β_k
            then F ← F ∪ {s};
      if ∃t ∈ N s.t. |p_k| < α_k and |t − q_k| < β_k
            then Y ← Y ∪ {t};
      y ← max{y | y ∈ Y};
      if #F = y then
         T ← 0;
         let h_{y,F} ∈ S ∘ T*_{0,1} be defined as follows:
            { h_{y,F}(0) = y,
            { h_{y,F}(n) = 0 for every n ∈ F,
            { h_{y,F}(n) = 1 for every n ∈ N+ \ F.
         A_{y,F} ← algo(h_{y,F});
   else
      K ← {m ∈ N | m ≤ k};
      let h_K be the function in T*_{0,1}
         such that h_K(n) = 0 iff n ∈ K;
      A_K ← algo(h_K);
   if T = 0 then output A_{y,F};
   else output A_K;
end
```

Fig. 6 The IIM $\mathcal{M}_{S\circ\mathcal{T}}$ in the proof of Lemma 11.

While it holds that $S \circ \mathcal{T}^*_{\{0,1\}} \notin$ REALNUM! for each set $S \subsetneq N$ that is not recursively enumerable, it holds that $N \circ \mathcal{T}^*_{\{0,1\}} \in$ REALNUM!.

4.5 Constant Functions

For a set $S \subsetneq N$ that is not recursively enumerable, let $\mathcal{C}_S$ be the set of all constant functions $c_s : [0,1] \to S$ such that $c_s(x) = s$ for each $s \in S$. Furthermore, let $\mathcal{C}_Q$ be the set of all constant functions $c_q : [0,1] \to Q$ such that $c_q(x) = q$ for each $q \in Q$.

Lemma 12 $\mathcal{C}_S \in$ REALREFEX.

Proof. Let $\mathcal{C}_N$ be the set of all constant functions defined by $c_n(x) = n$ for each $n \in N$. Every $c_n \in \mathcal{C}_N$ is defined on R. Furthermore, let $Rest(\mathcal{C}_N)$ be the set of all restrictions of every $c_n \in \mathcal{C}_N$ and $\mathcal{A}_n$ an algorithm which computes a constant function c_n for each $n \in N$. Then, we can design an IIM $\mathcal{M}_{\mathcal{C}_S}$ such that REALEX$(\mathcal{M}_{\mathcal{C}_S}) = Rest(\mathcal{C}_N)$ (see **Fig. 7**). For $\mathcal{M}_{\mathcal{C}_S}$, it holds that $\mathcal{C}_S \subsetneq$ REALEX$(\mathcal{M}_{\mathcal{C}_S})$. Consider a function $h \in \mathcal{RRVF} \setminus$ REALEX$(\mathcal{M}_{\mathcal{C}_S})$ and let σ be a presentation of h. Then, it holds that there exists a large enough $n \in N$ such that $\mathcal{M}_{\mathcal{C}_S}(\sigma[n]) = \perp$, because there exist numbers $x, y \in Q$ such that $h(x) \notin N$ or $h(x) \neq h(y)$ $(x \neq y)$. Hence, it holds that $\mathcal{C}_S \in$ REALREFEX. □

```
IIM M_{C_S}
begin
i ← 1; z ← −1; A ← A_0;
for i = 1 to ∞ do begin
   read the example w_i = ⟨⟨p_i, α_i⟩, ⟨q_i, β_i⟩⟩;
   if β_i < 1/4 then
      if ∃k ∈ N s.t. |k − q_i| < β_i then
         if z = −1 then z ← k; A ← A_k;
         if z ≠ k then A ← ⊥;
      else A ← ⊥;
   output A;
end
```

Fig. 7 The IIM $\mathcal{M}_{\mathcal{C}_S}$ in the proof of Lemma 12.

Lemma 13 $\mathcal{C}_Q \cup \mathcal{T}_U \in$ REALEX.

Proof. By the definition of $\mathcal{C}_Q$, it holds that $\mathcal{C}_Q \in$ REALEX. By Lemma 1, it holds that $\mathcal{T}_U \in$ REALEX. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be IIMs such that $\mathcal{C}_Q \subseteq$ REALEX$(\mathcal{M}_1)$ and $\mathcal{T}_U \subseteq$ REALEX$(\mathcal{M}_2)$. Furthermore, let h be in $\mathcal{C}_Q \cup \mathcal{T}_U$ and $\sigma = w_1, w_2, \ldots$ a presentation of h such that $w_k = \langle\langle p_k, \alpha_k\rangle, \langle q_k, \beta_k\rangle\rangle$ for each $k \in N^+$. Then, we can construct the following IIM $\mathcal{M}$:

$$\mathcal{M}(\sigma[n]) = \begin{cases} \mathcal{M}_1(\sigma[n]) & \text{if } n \in N^+ \text{ and} \\ & lb(n) \leq ub(n), \\ \mathcal{M}_2(\sigma[n]) & \text{otherwise,} \end{cases}$$

where $lb(n) = \max\{q_k - \beta_k \mid 1 \leq k \leq n\}$ and $ub(n) = \min\{q_k + \beta_k \mid 1 \leq k \leq n\}$. If h is a restriction of a function in $\mathcal{C}_Q$, then there exists a number $r \in Q$ such that $h(x) = r$ for each $x \in R$. For the $r \in Q$, we have $|q - q_k| < \beta_k$ for each $k \in N^+$. Thus, it holds that $\mathcal{M}(\sigma[n]) = \mathcal{M}_1(\sigma[n])$ for each $n \in N^+$.

If h is a function in $\mathcal{C}_Q \cup \mathcal{T}_U$ such that h is not a restriction of a function in $\mathcal{C}_Q$, then there exists a number $n \in N^+$ such that $\mathcal{M}(\sigma[m]) = \mathcal{M}_2(\sigma[m])$ for each $m \geq n$.

Hence, it holds that $\mathcal{C}_Q \cup \mathcal{T}_U \in$ REALEX. □

Lemma 14 $\mathcal{C}_Q \cup \mathcal{T}_U \notin$ REALRELEX.

Proof. Assume that $\mathcal{C}_Q \cup \mathcal{T}_U \in$ REALRELEX. Then, there exists an IIM $\mathcal{M}$ which reliably infers $\mathcal{C}_Q \cup \mathcal{T}_U$. It holds that the IIM $\mathcal{M}$ reliably infers $\mathcal{T}_U$. By Lemma 2, we have $\mathcal{T}_U \notin$ REALRELEX, which is a contradiction. □

4.6 Union Property

It is known that REFEX and RELEX are closed under union [7),13),17)]. In this section, we show that REALREFEX and REALRELEX also preserve this property.

Theorem 1 RealRefEx *and* RealRelEx *are closed under union. That is, for each* $I \in \{\text{RealRefEx}, \text{RealRelEx}\}$*, if* $S_1 \in I$ *and* $S_2 \in I$*, then* $S_1 \cup S_2 \in I$.

Proof. First we show that RealRefEx is closed under union. For $i = 1$ or 2, let S_i be a set of recursive real-valued functions and $\mathcal{M}_i$ an IIM which refutably infers S_i. Furthermore, let h be in $S_1 \cup S_2$ and σ a presentation of h. Then, we can construct the following IIM $\mathcal{M}$:

$$\mathcal{M}(\sigma[n]) = \begin{cases} \mathcal{M}_1(\sigma[n]) & \text{if } n \in N^+ \text{ and} \\ & \mathcal{M}_1(\sigma[n]) \neq \perp, \\ \mathcal{M}_2(\sigma[n]) & \text{otherwise.} \end{cases}$$

If $h \in \text{RealEx}(\mathcal{M}_1)$, then it holds that $\mathcal{M}(\sigma[n]) = \mathcal{M}_1(\sigma[n])$ for each $n \in N$. If $h \in \text{RealEx}(\mathcal{M}_2) \setminus \text{RealEx}(\mathcal{M}_1)$, then there exists a number $n \in N$ such that $\mathcal{M}(\sigma[m]) = \mathcal{M}_2(\sigma[m])$ for each $m \geq n$. If $h \notin \text{RealEx}(\mathcal{M}_1) \cup \text{RealEx}(\mathcal{M}_2)$, then there exists a number $n \in N$ such that $\mathcal{M}(\sigma[m]) = \perp$ for each $m \geq n$. Thus, it holds that $\mathcal{M}$ refutably infers $S_1 \cup S_2$. Hence, RealRefEx is closed under union.

Next we show that RealRelEx is closed under union. Let S_1 and S_2 be sets of recursive real-valued functions, and $\mathcal{M}_1$ and $\mathcal{M}_2$ IIMs which reliably infer S_1 and S_2, respectively. Furthermore let h be in $S_1 \cup S_2$ and $\sigma = w_1, w_2, \ldots$ be a presentation of h such that $w_k = \langle\langle p_k, \alpha_k\rangle, \langle q_k, \beta_k\rangle\rangle$ for each $k \in N^+$. Then, we can construct the function $i : N^+ \to \{1, 2\}$ defined by $i(1) = 1$ and

$$i(n) = \begin{cases} 1 & \text{if } i(n-1) = 1 \text{ and} \\ & \mathcal{M}_1(\sigma[n-1]) = \mathcal{M}_1(\sigma[n]), \\ 2 & \text{if } i(n-1) = 2 \text{ and} \\ & \mathcal{M}_2(\sigma[n-1]) = \mathcal{M}_2(\sigma[n]), \\ 2 & \text{if } i(n-1) = 1, \\ & \mathcal{M}_1(\sigma[n-1]) \neq \mathcal{M}_1(\sigma[n]), \\ & \mathcal{M}_2(\sigma[n-1]) = \mathcal{M}_2(\sigma[n]) \\ & \text{and} \\ & \mathcal{M}_1(\sigma[n-1]) \neq \mathcal{M}_2(\sigma[n]), \\ 1 & \text{otherwise,} \end{cases}$$

where $n \geq 2$. Furthermore we construct the IIM $\mathcal{M}$ such that $\mathcal{M}(\sigma[n]) = \mathcal{M}_{i(n)}(\sigma[n])$ for each $n \in N$.

If $h \in \text{RealEx}(\mathcal{M}_1) \setminus \text{RealEx}(\mathcal{M}_2)$, then there exists a number $n \in N$ such that $\mathcal{M}(\sigma[m]) = \mathcal{M}_1(\sigma[m])$ for each $m \geq n$. If $h \in \text{RealEx}(\mathcal{M}_2) \setminus \text{RealEx}(\mathcal{M}_1)$, then there exists a number $n \in N$ such that $\mathcal{M}(\sigma[m]) = \mathcal{M}_2(\sigma[m])$ for each $m \geq n$. If $h \in \text{RealEx}(\mathcal{M}_1) \cap \text{RealEx}(\mathcal{M}_2)$, then there exists a number $n \in N$ such that $i(m) = i(m+1)$ and $\mathcal{M}(\sigma[m]) = \mathcal{M}(\sigma[m+1])$ for each $m \geq n$. If $h \notin \text{RealEx}(\mathcal{M}_1) \cup \text{RealEx}(\mathcal{M}_2)$ and there exists a number $n \in N$ such that $i(m_1) = i(m_1 + 1)$ for each $m_1 \geq n$, then a sequence $\{\mathcal{M}(\sigma[m_1])\}_{m_1 \geq 1}$ does not converge to an algorithm. If $h \notin \text{RealEx}(\mathcal{M}_1) \cup \text{RealEx}(\mathcal{M}_2)$ and for each $n \in N$ there exists a number $m_2 \geq n$ such that $i(m_2) \neq i(m_2 + 1)$, then a sequence $\{\mathcal{M}(\sigma[m_2])\}_{m_2 \geq 1}$ does not converge to an algorithm. Thus, it holds that $\mathcal{M}$ reliably infers $S_1 \cup S_2$. Hence, RealRelEx is closed under union. □

Lemma 15 $\mathcal{C}_S \cup \mathcal{T}^m_{\{0,1\}} \in \text{RealRefEx} \setminus (\text{RealFin} \cup \text{RealNum!})$ *for each* $m \in N^+$.

Proof. By Lemma 3, it holds that $\mathcal{T}^m_{\{0,1\}} \in$ RealRefEx. By the definition of RealRelEx and RealRefEx, it holds that RealRefEx $\subseteq$ RealRelEx, so $\mathcal{T}^m_{\{0,1\}} \in$ RealRelEx. By Lemma 12, it holds that $\mathcal{C}_S \in$ RealRefEx. Hence, by Theorem 1, it holds that $\mathcal{C}_S \cup \mathcal{T}^m_{\{0,1\}} \in$ RealRefEx.

On the other hand, since S is not recursively enumerable, it holds that $S \notin$ Num!, which implies that $\mathcal{C}_S \notin$ RealNum!. By Lemma 4, $\mathcal{T}^m_{\{0,1\}} \notin$ RealFin for each $m \in N^+$. Hence, it holds that $\mathcal{C}_S \cup \mathcal{T}^m_{\{0,1\}} \notin$ RealFin $\cup$ RealNum!. □

5. Comparison of Criteria

In this section, we compare the new criteria RealRefEx and RealRelEx with the previous criteria RealEx, RealFin and RealNum!, by using the examples and the lemmas in Section 4. Note that the following statements hold by the previous work [11] and definitions.

(1) RealFin $\subsetneq$ RealEx.
(2) RealFin $\cap$ RealNum! $\neq \emptyset$.
(3) RealNum! $\setminus$ RealEx $\neq \emptyset$.
(4) RealRefEx $\subseteq$ RealRelEx $\subseteq$ RealEx.

Theorem 2 *The following statement holds.*

RealRefEx $\cap$ RealFin $\cap$ RealNum! $\neq \emptyset$.

Proof. It is obvious that $\mathcal{T}^0_{\{0,1\}} \in$ RealRefEx $\cap$ RealFin $\cap$ RealNum!. □

Theorem 3 *The following statement holds.*

RealRefEx $\subsetneq$ RealRelEx $\subsetneq$ RealEx.

Proof. By the above statement (4), it is sufficient to show the properness. By Lemma 1, 2 and the above statement (1), it holds that RealRelEx $\subsetneq$ RealEx. By Lemma 6 and 7, it also holds that RealRefEx $\subsetneq$ RealRelEx. □

Theorem 4 *The following statement holds.*
$\textsc{RealFin} \setminus (\textsc{RealRelEx} \cup \textsc{RealNum!}) \neq \emptyset$.

Proof. By Lemma 1 and 2, it holds that $\mathcal{T}_U \in \textsc{RealFin} \setminus (\textsc{RealRelEx} \cup \textsc{RealNum!})$. □

Theorem 5 *The following statement holds.*
$\textsc{RealRelEx} \setminus (\textsc{RealRefEx} \cup \textsc{RealFin} \cup \textsc{RealNum!}) \neq \emptyset$.

Proof. Let S be $\{j \in N \mid \varphi_j \in \mathcal{R}\}$. By Lemma 8, it holds that $\Phi_S^* \in \textsc{RealRelEx} \setminus \textsc{RealRefEx}$. By the definition of Φ_S^*, it holds that $\Phi_S^* \notin \textsc{RealFin}$. Since S is not recursively enumerable, it holds that $\Phi_S^* \notin \textsc{RealNum!}$. □

Theorem 6 *The following statement holds for* $I_i \in \{\textsc{RealRefEx}, \textsc{RealFin}, \textsc{RealNum!}\}$ $(i = 1, 2, 3)$ *such that* $I_i \neq I_j$ $(i \neq j)$.
$(I_1 \cap I_2) \setminus I_3 \neq \emptyset$.

Proof. It is sufficient to show the following three cases.

($I_3 = \textsc{RealFin}$) By Lemma 3 and 4, it holds that $\mathcal{T}_{\{0,1\}}^m \in \textsc{RealRefEx} \setminus \textsc{RealFin}$ for each $m \in N^+$. Furthermore, it is obvious that $\mathcal{T}_{\{0,1\}}^m \in \textsc{RealNum!}$.

($I_3 = \textsc{RealRefEx}$) By Corollary 1, it holds that $N \circ \mathcal{T}_{\{0,1\}}^* \in (\textsc{RealFin} \cap \textsc{RealNum!}) \setminus \textsc{RealRefEx}$.

($I_3 = \textsc{RealNum!}$) By Lemma 12, it holds that $\mathcal{C}_S \in \textsc{RealRefEx}$. Furthermore, it is obvious that $\mathcal{C}_S \in \textsc{RealFin}$. By the definition of $\mathcal{C}_S$, it also holds that $\mathcal{C}_S \notin \textsc{RealNum!}$. □

Theorem 7 *The following statement holds for* $I_i \in \{\textsc{RealRefEx}, \textsc{RealFin}, \textsc{RealNum!}\}$ $(i = 1, 2, 3)$ *such that* $I_i \neq I_j$ $(i \neq j)$.
$(\textsc{RealRelEx} \cap I_1) \setminus (I_2 \cup I_3) \neq \emptyset$.

Proof. It is sufficient to show the following three cases.

($I_1 = \textsc{RealFin}$) For a set $S \subsetneq N$ that is not recursively enumerable, consider a function $S \circ \mathcal{T}_{\{0,1\}}^*$. By Lemma 9, 10 and 11, it holds that $S \circ \mathcal{T}_{\{0,1\}}^* \in (\textsc{RealRelEx} \cap \textsc{RealFin}) \setminus \textsc{RealRefEx}$. Since S is not recursively enumerable, it holds that $S \notin \textsc{Num!}$, which implies that $S \circ \mathcal{T}_{\{0,1\}}^* \notin \textsc{RealNum!}$.

($I_1 = \textsc{RealRefEx}$) In this case, it is sufficient to show that $\textsc{RealRefEx} \setminus (\textsc{RealFin} \cup \textsc{RealNum!}) \neq \emptyset$, which directly follows from Lemma 15.

($I_1 = \textsc{RealNum!}$) By Lemma 6 and 7, it holds that $\mathcal{T}_{\{0,1\}}^* \in \textsc{RealRelEx} \setminus \textsc{RealRefEx}$. By the definition of $\mathcal{T}_{\{0,1\}}^*$, it holds that $\mathcal{T}_{\{0,1\}}^* \in \textsc{RealNum!}$ but $\mathcal{T}_{\{0,1\}}^* \notin \textsc{RealFin}$. □

Theorem 8 *The following statement holds.*
$\textsc{RealEx} \setminus (\textsc{RealRelEx} \cup \textsc{RealFin} \cup \textsc{RealNum!}) \neq \emptyset$.

Proof. By Lemma 13 and 14, it holds that $\mathcal{C}_Q \cup \mathcal{T}_U \in \textsc{RealEx}$ and $\mathcal{C}_Q \cup \mathcal{T}_U \notin \textsc{RealRelEx}$. By Lemma 1, it holds that $\mathcal{T}_U \notin \textsc{RealNum!}$, which implies that $\mathcal{C}_Q \cup \mathcal{T}_U \notin \textsc{RealNum!}$. Since $\mathcal{C}_Q \notin \textsc{RealFin}$, it holds that $\mathcal{C}_Q \cup \mathcal{T}_U \notin \textsc{RealFin}$. □

6. Conclusion

In this paper, we have introduced the criteria RealRefEx and RealRelEx for *refutably* and *reliably* inductive inference of recursive real-valued functions, and compared them with RealEx, RealFin and RealNum!, as described in Fig. 1 in Section 1. In particular, we have shown that RealRefEx and RealRelEx are closed under union.

The shapes marked by '?' in Fig. 1 remain open, so it is a future work to clarify them. In this paper, we have adopted the refutability introduced by Jain, et al.[13] It is also a future work to realize the definition of refutability by Mukouchi and Arikawa[19] for inductive inference of recursive real-valued functions and investigate its properties.

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