A Generalized Ryuoh-Nim: A Variant of the classical game of Wythoff Nim

Ryohei Miyadera¹,a) Masanori Fukui²,b) Yushi Nakaya¹,c) Yuki Tokuni¹,d)

Abstract: We introduce the impartial game of Ryuoh Nim, a variant of the classical game of Wythoff Nim. In the latter game, two players take turns in moving a single Queen of Chess on a large board, attempting to be the first to put her in the lower left corner, position (0, 0). Instead of the queen used in Wythoff Nim, we use the Ryuoh that is a promoted Hisha (rook) piece of Japanese chess. The Ryuoh combines the power of the rook and king in western chess. We prove that the Grundy number for this variant is expressed by \( G((x, y)) = \text{mod}(x + y, 3) + 3(\lfloor \frac{x}{3} \rfloor \oplus \lfloor \frac{y}{3} \rfloor) \), where \( \text{mod}(x + y, 3) \) is the remainder of \( x + y \) when divided by 3. We study a generalization of the Ryuoh Nim whose Grundy number is expressed by \( \text{mod}(x + y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor) \) for a natural number \( p \). We also study a generalized Ryuoh Nim with a pass.

Keywords: Wythoff Nim Corner the Queen Problem Grundy Number nim-sum

1. Ryuoh (dragon king) game

We introduce the impartial game of Ryuoh Nim, a variant of the classical game of Wythoff Nim in [1]. Let \( \mathbb{Z}_{\ge 0} \) be the set of non-negative integers and \( N \) be the set of natural numbers. Instead of the queen used in Wythoff Nim, we use the Ryuoh (dragon king) of Japanese chess. The Ryuoh combines the power of the rook and king in western chess. The Ryuoh is placed on a chess board of unbounded size, and two players move Ryuoh in turns. The Ryuoh can be moved horizontally, as far as one wants, he can be moved vertically, as far as one wants, and he can be moved one square diagonally.

Let us break with chess traditions here and name fields on the chess board by pairs of numbers. The field in the lower left corner is \((0, 0)\), and the other ones according to a Cartesian scheme - field \((x, y)\) will be \(x\) fields to the right, and then \(y\) fields up (you get the picture, see Figure 2).

In our game we restrict the Ryuoh to be moved to the left or upwards, or along the upper left diagonal, see Figure 1. Of course, the Ryuoh has to be moved at least one field in each move.

The goal of the game is to move Ryuoh to the "winning field" \((0, 0)\); whoever moves the Ryuoh to this field, wins the game.

In this article we only treat impartial games. See [2] or [3] for a background on impartial games.

In this article we study impartial games without draws, so there will only be two outcome classes.

Definition 1.1. (a) \(N\)-positions, from which the next player can force a win, as long as he plays correctly at every stage.

(b) \(P\)-positions, from which the previous player (the player who will play after the next player) can force a win, as long as he plays correctly at every stage.

Definition 1.2. For any position \(p\) of a game \(G\), there is a set of positions that can be reached by making precisely one move in \(G\), which we will denote by \(\text{move}(p)\).

The move of Ryuoh is expressed in (15), (16) and (17).

\[
\text{move}(x, y) = \{ (u, y) : u < x \} \quad (1)
\]

\[
\cup \{ (x, v) : v < y \} \quad (2)
\]

\[
\cup \{ (x - 1, y - 1) \} \quad (3)
\]

The set (15) stands for the horizontal move, the set (16) stands for the vertical move, and the set (17) stands for the diagonal move in Figure 2.

![Fig. 1 definition of coordinates](image)

![Fig. 2 move of Ryuoh](image)

Definition 1.3. (i) The minimum excluded value (mex) of a set, \(S\), of non-negative integers is the least non-negative integer which is not in \(S\).

(ii) Each position \(p\) of an impartial game \(G\) has an associated Grundy number, and we denoted it by \(G(p)\).

Grundy number is calculated recursively: \(G(p) = \text{mex}(G(h) : h \in \text{move}(p))\).
Example 1.1. Examples of calculation of mex.
\[
mex\{0, 1, 2, 3\} = 4, \mex\{1, 2, 3\} = 0,
\]
\[
mex\{0, 2, 3, 5\} = 1 \text{ and } \mex\{0, 0, 1\} = 2.
\]

Theorem 1.1. Let \( G \) be the Grundy number. Then, \( h \) is a \( P \)-position if and only if \( G(h) = 0 \).

This is a well known theorem in combinatorial game theory.

Example 1.2. Figure 3 is a table of Grundy numbers of the Ryūō-nim.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
x & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
y & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
\end{array}
\]

Fig. 3 the Grundy number \( G(x, y) \) of Ryūō

Example 1.3. Figure 5 is the move of the Ryūō. Figure 6 and Figure 7 are the move of the generalized Ryūō for \( p = 4 \) and \( p = 8 \) respectively. Figure 8 represent the move of generalized Ryūō for a natural number \( p \).

It is easy to see that Figure 3 is the same as Figure 4. This implies that \( G(x, y) = \text{mod}(x + y, 3) + 3(\lfloor \frac{x}{4} \rfloor \oplus \lfloor \frac{y}{4} \rfloor) \). This fact is presented in Theorem 1.2.

Next, we generalize Ryūō, and we define a generalized Ryūō.

Definition 1.4. Let \( p \) a natural number. We define a generalized Ryūō for \( p \). For the generalized Ryūō, move is expressed in (4), (5) and (6).

\[
\text{move}(x, y) = \{(a, u) : u < x\} \cup \{(x, v) : v < y\} \cup \{(x - s, y - t) : 1 \leq s, t \leq p - 1\}.
\]

The set (4) stands for the horizontal move, the set (5) stands for the vertical move, and the set (6) stands for the upper left move.

Example 1.3. Figure 5 is the move of the Ryūō. Figure 6 and Figure 7 are the move of the generalized Ryūō for \( p = 4 \) and \( p = 8 \) respectively. Figure 8 represent the move of generalized Ryūō for a natural number \( p \).

It is easy to see that Figure 9 is the same as Figure 10. This implies that \( G(x, y) = \text{mod}(x + y, 4) + 4(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor) \). In general the Grundy number of the generalized \( G(x, y) \) of Ryūō for \( p \) is \( G(x, y) = \text{mod}(x + y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor) \). We present this fact in Theorem 1.2.
We present some lemmas that are needed for Theorem 1.2 without proof. We present Theorem 1.2 without proof, since the proof is too lengthy.

**Lemma 1.1.**

\[ k \oplus h = \text{mex}( (k \ominus t) \oplus h : t = 1, 2, \ldots, k) \]
\[ \cup \{ k \oplus (h \ominus t) : t = 1, 2, \ldots, h \} \]

(7)

**Lemma 1.2.** Let \( A_{k,h} = \bigcup_{p=0}^{h-1} \{ p(k \ominus t) \oplus u : t = 1, 2, \ldots, k \} \cup \bigcup_{p=0}^{h-1} \{ p(k \ominus (h-t)) \oplus u : t = 1, 2, \ldots, h \} \). (a) For any \( v \in 1, \ldots, p-1 \)

\[ p(k \ominus h) + v = \text{mex}(A_{k,h} \cup \{ p(k \ominus h) + w : w = 0, \ldots, v-1 \}) \]

(8)

(b) For \( v = 0 \), \( p(k \ominus h) = \text{mex}(A_{k,h}) \).

For any arbitrary non-negative integer \( x \), we denote by \( \text{mod}(x, p) \) the remainder of \( x \) when divided by \( p \).

**Lemma 1.3.** Let \( x, y, k \in \mathbb{Z}_{>0} \). If \( 0 \leq k < \text{mod}(x + y, p) \), then

\[ k \in \{ \text{mod}(s+y-t, p) : 0 \leq s \leq y, 0 \leq t \leq y \text{ and } s+t \leq p-1 \} \]  

(9)

**Lemma 1.4.** Let \( V \) be a subset of \( \mathbb{Z}_{>0} \), and let \( v \in \mathbb{Z}_{>0} \) such that

\[ v = \text{mex}(V) \]

(10)

If \( W \) is a subset of \( \mathbb{Z}_{>0} \) such that \( V \subseteq W \) and \( v \notin W \), then \( v = \text{mex}(W) \).

**Lemma 1.5.** Let \( k, h, v, w \in \mathbb{Z}_{>0} \) such that \( 0 \leq v, w \leq p-1 \), and let

\[ C_{k,h,v,w} = \{ p(k \ominus h) + \text{mod}(v-t+w, p) : t = 1, 2, \ldots, v \}, \]
\[ \cup \{ p(k \ominus h) + \text{mod}(v+w-t, p) : t = 1, 2, \ldots, w \}, \]
\[ \cup \{ p(\lfloor \frac{pk+v-s}{p} \rfloor \oplus \lfloor \frac{ph+w-t}{p} \rfloor) + \text{mod}(v+w-s-t, p) \}
\[ : 1 \leq s, t \text{ and } s+t \leq p-1 \}. \]

(11)

Then we have the following (a) and (b).

(a) \( p(k \ominus h) + \text{mod}(v+w, p) \notin C_{k,h,v,w} \)

(b) \( p(k \ominus h) + u \in C_{k,h,v,w} \) for any non-negative integer \( u \) such that

\[ 0 < u < \text{mod}(v+w, p) \].

(12)

**Theorem 1.2.**

\[ G((x, y)) = \text{mod}(x+y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor) \]

(13)

Here, \( \text{mod}(x+y, p) \) is the remainder of \( x + y \) when divided by \( p \).

Theorem 1.2 and Theorem 1.3 present a sufficient condition and a necessary condition for a chess piece to have Grundy numbers expressed by (13) respectively.

**Theorem 1.3.** Suppose that we make a variant of "Corner the Queen" with a new chess piece. If the Grundy number of this game satisfies (14), then the move of this piece is defined by (4), (5) and (6) of Definition 1.4.

\[ G'(x, y) = \text{mod}(x+y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor) \]

(14)

\[ \text{move}(x, y, 0) \]
\[ = \{(u, y, 0) : u < x \} \]  

(15)

\[ \cup \{(x, v, 0) : v < y \} \]  

(16)

\[ \cup \{(x-1, y-1, 0)\} \]

(17)

\[ \text{move}(x, y, 1) \]
\[ = \{(u, y, 1) : u < x \} \]  

(18)

\[ \cup \{(x, v, 1) : v < y \} \]  

(19)

\[ \cup \{(x-1, y-1, 1)\} \]

(20)

\[ \cup \{(x, y, 0)\} \]

(21)

The sets (15) and (16) stand for the horizontal move, the sets (17) and (19) stand for the vertical move, the sets (17) and (20) stand for the diagonal move in Figure 2, and the set (21) stands for a pass move.

Let \( P_0 = \{(x, y, 0) : x+y = \text{mod}(p) \} \) and \( \lfloor \frac{n}{p} \rfloor = \lfloor \frac{m}{p} \rfloor \).

Let \( P_{1,1} = \{(1 + m, p-m) : 0 \leq m \leq p-1 \} \) and \( m \in \mathbb{Z}_{>0} \).

\[ \text{P}_{1,2} = \{(1+pn, 1+pn) : n \in N \} \]

and \( P_{1,3} = \{(k+pn, p+2k+pn) : n \in N, 2 \leq k \leq p \} \) and \( k \in \mathbb{Z}_{>0} \). Let \( P_1 = P_{1,1} \cup P_{1,2} \cup P_{1,3} \). Let \( P = P_0 \cup P_1 \).

**Lemma 2.1.** Let \( (x, y, 0) \) is a \( P \)-position if and only if \( (x, y, 0) \in P_0 \).

**Theorem 2.1.** \( G((x, y, 1)) = 0 \) if and only if \( (x, y, 1) \in P_1 \).

**References**

