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Multi-Commodity Source Location Problems and Price of Greed

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1 Introduction

Given an undirected graph $G = (V, E)$ with vertex set V, edge set E and an edge capacity function, the *edge-connectivity* between $S \subseteq V$ and $v \in V$ is the minimum total capacity of a set of edges such that v is disconnected from S by removal of these edges. We say that S *covers* v or that S and v are *k-edge-connected* if the edge-connectivity between S and v is at least k, where $k \geq 1$ is a given integer. The source location problem is to find a minimum-size source set $S \subseteq V$ covering all vertices in V. This problem has been studied widely [2, 3, 7, 10]. These problems are important to design networks resistant to failures of edges.

In recent years, game theory has attracted attention in fields of computer science. In various real problems, e.g., routing, network design and scheduling, selfish actions of agents are obstacles to optimization for social welfare. Such a phenomenon is modeled as games, and analyses of influence of selfish actions of players are studied a lot [1, 5, 9]. In this paper, we propose *multi-commodity source location problem.* In this problem, a network $N = (G = (V, E), w, c)$, and positive integers k, r and p are given, where G is an undirected and connected graph, c is an edge-capacity function, $w : V, E \to \mathbf{R}_+$ is a function of vertex- and edge-weight, and r is the number of players, where $\mathbf{R}_+$ denotes a set of nonnegative real numbers. Players $1, 2, \ldots, r$ locate p sources in this order on the vertices in G. Player i cannot locate sources on the vertices on which one of players $1, \ldots, i-1$ has already located sources. Let player i's profit be the total weight of vertices and edges covered by the sources located by the player i, where a source set S covers an edge $e = vw$ if both v and w are covered by S. The goal of this problem is to maximize the sum of profits of all players. When $r = 1$, the problem is the same with the maximum-cover source-location problem which was proposed in [10]. This problem is NP-hard, but it can be solved in polynomial time when $k \leq 3$ [6, 10, 11].

In this paper, we consider how much worse than the optimal total is the total profit of all players when each player behaves only so as to maximize their own profit. In this case, each selfish player optimally locates p sources on vertices except ones on where other players have already located sources. Hence we can regard their behaviours as a kind of greedy algorithms. As a measure of influence of selfish behaviours, we propose a *price of greed*, which represents the ratio of the maximum total profit of cooperating players to the worst total profit of selfish players. Formally, let the price of greed for the multi-commodity source location problem denote

$$POG_k(N, r, p) = \frac{\text{the optimal total profit}}{\text{the worst selfish total profit}}.$$

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A well-known similar measure, a price of anarchy is a ratio of the worst cost of Nash equilibria of selfish players to the optimal cost [9]. Note that in our problem, the above greedy algorithm can be considered as a Nash equilibrium.

Our Results Our goal is to analyze the worst value of the price of greed $POG_k(r,p) = \max_N POG_k(N, r, p)$. When $k = 1$, $POG_1(r,p) = 1$ for any $r, p \geq 1$, clearly.

First, when $k = 2$, we show the following theorem.

Theorem 1 *$POG_2(r,p) = \min\{r,p\}$ for any $r, p \geq 1$.*

When $k = 2$, G can be reduced to a tree T with every edge of capacity 1, and we can assume that sources are located on only leaves without loss of generality. Because if each 2-edge-connected component X (which is a maximal set of vertices such that each vertex pair in X is 2-edge-connected) is contracted into a vertex x having a total weight of vertices $v \in X$ and edges vw for $v, w \in X$, then a tree with every edge of capacity 1 is obtained. Note that a 2-edge-connected component can consist of one vertex. However, in order to be able to locate $|X|$ sources on x and satisfy the assumption that sources are located on only leaves, we need to add leaves and edges of weight 0 to each vertex x in the tree, appropriately. The consequent tree T is equivalent to G. A vertex or an edge in T is covered if it is contained in a subtree induced by located sources, since $k = 2$.

First, we show that $POG_2(r,p)$ is bounded by at most $\min\{r,p\}$ (Lemma 4) and there exists no better bound by showing existence of an instance with $POG_2(N, r, p) = \min\{r,p\}$. Fig. 1 indicates the instance. Let $|X| = \min\{r,p\}$ and $|Y| = rp - \min\{r,p\}$. Let the weight of v be 1 and the other vertices and edges have weight 0. If selfish player 1 obtains profit 1 by locating $\min\{r,p\}$ sources on X, then the other selfish players obtain no profit. Hence the worst selfish profit is 1. On the other hand, each of optimal players $1, \ldots, \min\{r,p\}$ obtains profit 1 by locating one source on X and $p-1$ sources on Y. Hence the optimal profit is $\min\{r,p\}$. Therefore, this instance has $POG_2(N, r, p) = \min\{r,p\}$.

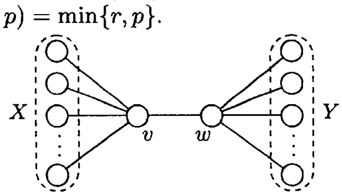

Figure 1: An instance of $POG_2(N, r, p) = \min\{r,p\}$.

Furthermore, we consider that an input graph G is restricted to a vertex-*un*weighted tree with every edge of capacity 1 when $k = 2$. This case is a problem of finding the maximum edge-weight subtree. The maximum edge-weight trees have a lot of application, e.g., communication networks [4, 8]. This case is a quite special case of the original version. However, $POG_2(r,p)$ is at most only one lower than that for the vertex- and edge-weighted case as we show below. If $p = 1$, then any optimal and selfish player obtains no profit and hence we assume $p \geq 2$.

Theorem 2 *For vertex-unweighted trees and any* $r \geq 1$, $POG_2(r, 2) = \min\{r, 2\}$ *and* $POG_2(r, p) = \min\{r, p-1\}$ *for* $p \geq 3$.

At first, we tried to construct an instance having $POG_2(N, r, p) = p$ for $r \geq p \geq 3$ in a similar way to the vertex-weighted case. However, in Fig. 1, if we let the weight of the edge vw be 1 instead of v, then X must satisfy $|X| \leq p-1$ so that selfish player 1's sources can occupy X. However, in this case, the total profit of optimal players is at most $p-1$. Hence even if $|X| = p-1$, $POG_2(N, r, p) = p-1$ for this instance. In fact, it can be shown that this instance has the worst price of greed, i.e., $POG_2(r, p) \leq \min\{r, p-1\}$ can be shown in Lemma 7.

2 Analyses of $POG_2(r, p)$ for General Cases

Given a network N, let W_i $(1 \leq i \leq r)$ be the profit of an optimal player i. We assume $W_1 \geq W_2 \geq \cdots \geq W_r$ without loss of generality. Let W'_i $(1 \leq i \leq r)$ be the profit of a selfish player i for the worst case, i.e., where the total profit is least. From the definition, $W'_1 \geq W'_2 \geq \cdots \geq W'_r$ holds, and $POG_k(N, r, p) = \sum_{i=1}^{r} W_i / \sum_{i=1}^{r} W'_i$. In addition, let S_i $(1 \leq i \leq r)$ be the set of sources located by the optimal player i and S'_i $(1 \leq i \leq r)$ be the set located by the selfish player i. For a source set $S \subseteq V$, let $w(S)$ denote the total weight of vertices and edges covered by S. For a vertex set $\{v, \ldots, w\}$, we may write $w(\{v, \ldots, w\})$ as $w(v, \ldots, w)$ for notational simplicity.

We show the upper bounds of $POG_2(r, p)$ in Lemma 4 and show that the bounds are tight in Lemma 5. Lemma 4 depends on the following lemma.

Lemma 3 *For any* i $(1 \leq i \leq \lfloor \frac{r-1}{p} \rfloor + 1)$, $W'_i \geq W_{(i-1)p+1}$.

We omit the proof due to limitations of space.

Lemma 4 $POG_2(r, p) \leq \min\{r, p\}$ *for any* $r, p \geq 1$.

Proof. When $p < r$, Lemma 3 implies $POG_2(N, r, p) \leq p$ for any N, since $pW'_i \geq W_{(i-1)p+1} + \cdots + W_{ip}$ and hence $p \sum_{i=1}^{\ell} W'_i \geq \sum_{i=1}^{r} W_i$, where ℓ is the minimum i such that $ip \geq r$. Hence $POG_2(r, p) \leq p$ for $p < r$.

On the other hand, since $W'_1 \geq W_i$ for any i, $POG_2(N, r, p) \leq \sum_{i=1}^{r} W_i / W'_1 \leq r$ for any N, r and p. Therefore, $POG_2(r, p) \leq \min\{r, p\}$. ▮

The next lemma shows that the upper bound is tight.

Lemma 5 *There exists an instance with* $POG_2(N, r, p) = \min\{r, p\}$ *for each* $r \geq 1$ *and* $p \geq 1$.

Proof. When $r = 1$ or $p = 1$, $POG_2(N, r, p) = 1$ for any network N, clearly. For $r, p \geq 2$, we have shown the result already in Sect. 1. ▮

From Lemmas 4 and 5, Theorem 1 holds.

3 Analyses of $POG_2(r, p)$ for Vertex-Unweighted Trees

The upper bounds are obtained from Lemmas 6 and 7.

Lemma 6 *Assume that* $r \geq 2$ *and* $p \geq 3$. *Let* i *be an integer with* $1 \leq i \leq r-1$. *Let* $S'_i = \{s'_1, \ldots, s'_p\}$ *and* $v \notin S'_1 \cup \cdots \cup S'_i$ *be a vertex. Then* $\sum_{j=1}^{p} w(s'_j, v) \leq (p-1)W'_i$.

Lemma 7 *For vertex-unweighted trees and any* $r \geq 1$, $POG_2(r, 2) \leq \min\{r, 2\}$ *and* $POG_2(r, p) \leq \min\{r, p-1\}$ *for* $p \geq 3$.

We omit the proofs of Lemmas 6 and 7 because of space limitations. The following lemma shows that the upper bounds are tight.

Lemma 8 *For any* $r \geq 1$ *and* $p \geq 2$, *there exists an instance with* $POG_2(N, r, 2) = \min\{r, 2\}$ *and one with* $POG_2(N, r, p) = \min\{r, p-1\}$ *for* $p \geq 3$.

Proof. For $r \geq p \geq 3$, in Fig. 1, let weight of vw be 1, and $|X| = p-1$, then selfish player 1 may locate $p-1$ sources on X and one source on Y. The other selfish players cannot obtain profits from the assumption that sources are located only on leaves. Each optimal player's profit is 1 and hence $POG_2(N, r, p) = p-1$.

For $p = 2$, let $|X| = 2$ and $|Y| = 2r-2$. If the weight of the three edges incident to v is 1, then optimal and selfish total profits are 4 and 2, respectively. The rest cases are similar to the vertex- and edge-weighted case. ▮

Theorem 2 holds from Lemmas 7 and 8.

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