A_011

Balanced (C_4, C_5)-$2t$-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_k be *the k-cycle. The* (C_4, C_5)*-2t-foil* is a graph of t edge-disjoint C_4's and t edge-disjoint C_5's with a common vertex and the common vertex is called *the center of the* (C_4, C_5)*-2t-foil.* In particular, the (C_4, C_5)-2-foil is called *the* (C_4, C_5)*-bowtie.* When K_n is decomposed into edge-disjoint sum of (C_4, C_5)-$2t$-foils, we say that K_n *has a* (C_4, C_5)*-2t-foil decomposition.* Moreover, when every vertex of K_n appears in the same number of (C_4, C_5)-$2t$-foils, we say that K_n *has a balanced* (C_4, C_5)*-2t-foil decomposition* and this number is called *the replication number.* Note that (C_4, C_5)-$2t$-foil has $7t+1$ vertices and $9t$ edges.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system.*
In this sense, our balanced (C_4, C_5)-$2t$-foil decomposition of K_n is to be known as *a balanced* (C_4, C_5)*-2t-foil system.*

2. Balanced (C_4, C_5)-$2t$-foil decomposition of K_n

Notation. We denote a (C_4, C_5)-$2t$-foil passing through
$v_1 - v_2 - v_3 - v_4 - v_1 - v_5 - v_6 - v_7 - v_8 - v_1$, $v_1 - v_9 - v_{10} - v_{11} - v_1 - v_{12} - v_{13} - v_{14} - v_{15} - v_1$, $v_1 - v_{16} - v_{17} - v_{18} - v_1 - v_{19} - v_{20} - v_{21} - v_{22} - v_1$, ..., $v_1 - v_{7t-5} - v_{7t-4} - v_{7t-3} - v_1 - v_{7t-2} - v_{7t-1} - v_{7t} - v_{7t+1} - v_1$
by
$\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7, v_8)\}$ $\cup$ $\{(v_1, v_9, v_{10}, v_{11}), (v_1, v_{12}, v_{13}, v_{14}, v_{15})\}$ $\cup$ $\{(v_1, v_{16}, v_{17}, v_{18}), (v_1, v_{19}, v_{20}, v_{21}, v_{22})\}$ $\cup \ldots \cup$ $\{(v_1, v_{7t-5}, v_{7t-4}, v_{7t-3}), (v_1, v_{7t-2}, v_{7t-1}, v_{7t}, v_{7t+1})\}$.

Theorem. K_n has a balanced (C_4, C_5)-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{18t}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_5)-$2t$-foil decomposition. Let b be the number of (C_4, C_5)-$2t$-foils and r be the replication number. Then $b = n(n-1)/18t$ and $r = (7t+1)(n-1)/18t$. Among r (C_4, C_5)-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_5)-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/18t$ and $r_2 = 7(n-1)/18$. Therefore, $n \equiv 1 \pmod{18t}$ is necessary.

(Sufficiency) Put $n = 18st + 1$, $T = st$. Then $n = 18T + 1$. Construct n (C_4, C_5)-$2T$-foils as follows:
$B_i = \{(i, i+1, i+3T+2, i+T+1), (i, i+6T+1, i+10T+2, i+15T+3, i+7T+1)\}$ $\cup$ $\{(i, i+2, i+3T+4, i+T+2), (i, i+6T+2, i+10T+4, i+15T+6, i+7T+2)\}$ $\cup$ $\{(i, i+3, i+3T+6, i+T+3), (i, i+6T+3, i+10T+6, i+15T+9, i+7T+3)\}$ $\cup \ldots \cup$ $\{(i, i+T, i+5T, i+2T), (i, i+7T, i+12T, i+18T, i+8T)\}$ $(i = 1, 2, ..., n)$.
Decompose each (C_4, C_5)-$2T$-foil into s (C_4, C_5)-$2t$-foils. Then they comprise a balanced (C_4, C_5)-$2t$-foil decomposition of K_n.

Note. We consider the vertex set V of K_n as

$V = \{1, 2, ..., n\}$. The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Example 1. A balanced (C_4, C_5)-2-foil decomposition of K_{19}.
$B_i = \{(i, i+1, i+5, i+2), (i, i+7, i+12, i+18, i+8)\} \quad (i = 1, 2, ..., 19)$.

Example 2. A balanced (C_4, C_5)-4-foil decomposition of K_{37}.
$B_i = \{(i, i+1, i+8, i+3), (i, i+13, i+22, i+33, i+15)\} \cup \{(i, i+2, i+10, i+4), (i, i+14, i+24, i+36, i+16)\} \quad (i = 1, 2, ..., 37)$.

Example 3. A balanced (C_4, C_5)-6-foil decomposition of K_{55}.
$B_i = \{(i, i+1, i+11, i+4), (i, i+19, i+32, i+48, i+22)\} \cup \{(i, i+2, i+13, i+5), (i, i+20, i+34, i+51, i+23)\} \cup \{(i, i+3, i+15, i+6), (i, i+21, i+36, i+54, i+24)\} \quad (i = 1, 2, ..., 55)$.

Example 4. A balanced (C_4, C_5)-8-foil decomposition of K_{73}.
$B_i = \{(i, i+1, i+14, i+5), (i, i+25, i+42, i+63, i+29)\} \cup \{(i, i+2, i+16, i+6), (i, i+26, i+44, i+66, i+30)\} \cup \{(i, i+3, i+18, i+7), (i, i+27, i+46, i+69, i+31)\} \cup \{(i, i+4, i+20, i+8), (i, i+28, i+48, i+72, i+32)\} \quad (i = 1, 2, ..., 73)$.

Example 5. A balanced (C_4, C_5)-10-foil decomposition of K_{91}.
$B_i = \{(i, i+1, i+17, i+6), (i, i+31, i+52, i+78, i+36)\} \cup \{(i, i+2, i+19, i+7), (i, i+32, i+54, i+81, i+37)\} \cup \{(i, i+3, i+21, i+8), (i, i+33, i+56, i+84, i+38)\} \cup \{(i, i+4, i+23, i+9), (i, i+34, i+58, i+87, i+39)\} \cup \{(i, i+5, i+25, i+10), (i, i+35, i+60, i+90, i+40)\} \quad (i = 1, 2, ..., 91)$.

Example 6. A balanced (C_4, C_5)-12-foil decomposition of K_{109}.
$B_i = \{(i, i+1, i+20, i+7), (i, i+37, i+62, i+93, i+43)\} \cup \{(i, i+2, i+22, i+8), (i, i+38, i+64, i+96, i+44)\} \cup \{(i, i+3, i+24, i+9), (i, i+39, i+66, i+99, i+45)\} \cup \{(i, i+4, i+26, i+10), (i, i+40, i+68, i+102, i+46)\} \cup \{(i, i+5, i+28, i+11), (i, i+41, i+70, i+105, i+47)\} \cup \{(i, i+6, i+30, i+12), (i, i+42, i+72, i+108, i+48)\} \quad (i = 1, 2, ..., 109)$.

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