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Representation of SEBP Orientation by PLCP Orientation

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1. Introduction

Given a set of $d+1$ affinely independent points, how difficult it is to find the smallest d-dimensional ball enclosing all points? We call the problem the *Smallest enclosing ball of points* (SEB). This problem can be solved in linear time w.r.t. the number of points n with a fixed dimension of space [7, 10, 13]. In, these algorithms, however, the constant factor is exponential of dimension of space, hence a subexponential time algorithm algorithm was developed by Matoušek et al. [6]. Fischer and Gärtner gives $O(d^3 \cdot (1.438)^d)$ algorithm [2] to SEBB using the unique sink orientations model, which was introduced by Szabó and Welzl [12].

This paper targets a special case of an SEB, which computes the smallest $(d-1)$-dimensional ball with a fixed one point on its boundary that encloses the other d points. We call this problem the *smallest enclosing ball of points problem with a fixed point on the boundary* (SEBP). An SEBP is also solved by Fischer and Gärtner's algorithm [2] as well as an SEB.We focus on the behavior of Fischer and Gärtner's algorithm to solve an SEBP. In the case of a d-dimensional SEBP, the behavior is represented by a *unique sink orientation* on the graph $G(C_d)$ of d-dimensional cube C_d [2, 12], called an *SEBP-orientation.* An orientation on $G(C_d)$ is called a unique sink orientation (USO) of C_d if the induced orientation on every face F of C_d has a unique sink (and consequently has a unique source). A USO of a cube was introduced by Szabó and Welzl [12] as a framework to represent the behavior of some algorithms of mathematical programs. An outstanding example is obtained via a linear program (LP) solved by a simplex method. Let us consider an LP whose feasible region is combinatorially equivalent to C_d. If its linear function is in general position, every point of the feasible region has a different function value, hence every edge can be oriented from the endpoint with a smaller function value to a larger function value. The output orientation on $G(C_d)$ is certainly a USO of C_d, called an *LP-cube.* The same is true of a *linear complementarity problem* (LCP), which is a unified framework for the two important optimization problems, linear programs and quadratic programs. If an LCP is solved by a series of pivoting algorithms, called *Bard-type algorithms* [1], we call the bahavior the *LCP-orientation.* Particularly if a given matrix of a LCP is a P-matrix whose all principal minors are positive, Stickney and Watson [11] proved the LCP-orientation is also a USO of C_d, called a *PLCP-cube.*

SEBP-orientations, which is a target of this paper, LP-cubes and PLCP-cubes have much importance as combinatorial properties of mathematical programs. In the case of LP-cubes and PLCP-cubes, other properties other than a USO were already discovered. It is easy to find that every LP-cube is acyclic, but as already observed by Stickney and Watson [11] there exists a PLCP-cube which has a directed cycle. On the other hand, LP-cubes and PLCP-cubes have the common property, called *Holt-Klee condition* [5]. Holt and Klee [5] proved that every LP-cube satisfies the Holt-Klee condition. Moreover, Gärtner, Morris and Rüst [4] recently resolved that PLCP-cubes also satisfies the Holt-Klee condition.

Differently from LP-cubes and PLCP-cubes, no properties of SEBP-orientations other than a USO are directly revealed, but Szabó et al. [12, 3] studied the formulation of SEB as finding sink in a cube graph, which is called an *SEB-orientation.* In the case of a d-SEB, the SEB-orientation is a USO of a C_{d+1} such that the induced orientation on the facet not including the global source is a SEBP-orientation. Gärtner, Miyazawa et al. [3] showed that an SEB-orientation has a directed cycle in sufficiently large dimension. However, SEBP-orientations including SEB-orientations, LP-cubes and PLCP-cubes have been not also characterized yet. All that is certain about three orientations is the inclusion relation between LP-cubes and PLCP-cubes, i.e., every LP-cube is a PLCP-cube, which was proved by Morris [8].

In this paper, we focus on another inclusion relation between SEBP-orientations and PLCP-cubes to show that every SEBP-orientation is a PLCP-cube.

2. SEBP-orientation

Definition 1 (A Smallest Enclosing Ball of Points with fixed point on boundary). Given P of $d+1$ affinely independent points in d-space. Choose one point v as origin, and the position of the other points are given as vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$. Smallest enclosing ball of points with fixed point on boundary (SEBP) is to compute the smallest ball with v on its boundary which encloses all points in P whose dimension is d.

Definition 2 (A SEBP-orientation). Consider the d dimensional cube whose vertex set is labeled $\{Q \mid Q \subseteq P\}$,

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and edge set is $\{\{Q \setminus \{x\}, Q\} \mid Q \subseteq P, x \in Q\}$. $\beta(X)$ denotes smallest ball that have all elements of point set $X \subseteq \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d\}$ and fixed point v on its boundary. Let the edges between $\{Q \setminus \{x\}\}$ and $\{Q\}$ be directed toward $\{Q \setminus \{x\}\}$ iff $x \in \beta(Q \setminus \{x\})$.

The output orientation is a SEBP-orientation whose dimension is d.

If the sink of this orientation is $S \subseteq P$, the smallest enclosing ball of given points is $\beta(S)$, and it is uniquely determined.

Definition 3 (A Linear complementarity problem). Given $M \in \mathcal{R}^{n \times n}$ and $q \in \mathcal{R}^n$, find vectors $w = (w_1, w_2, \ldots, w_n)$, and $z = (z_1, z_2, \ldots, z_n)$ such that

$$w - Mz = q, w \geq 0, z \geq 0, w^\top z = 0. \qquad (1)$$

If all principle minor of M is positive, it is called LCP on P-matrices (PLCP).

Definition 4 (A PLCP-orientation). Consider the d-dimensional cube whose vertex set is labeled $\{X \mid X \subseteq [d] := \{1, 2, \ldots, d\}\}$. Let X be a set of index basis and M'_X be a submatrix of M whose i-th column is M_i if $i \in X$, otherwise unit vector $\mathbf{e}_i$ Let the edges between $X \setminus \{x\}$ and X be directed toward $X \setminus \{x\}$ to X iff $(M_X q)_x > 0$. The output orientation is a PLCP-orientation in $\mathcal{R}^d$.

If the sink of PLCP-orientation is S, the basis of the feasible solution of given P-linear complementarity problem is $\{z_k, w_l \mid k \in S, l \in [d] \setminus S\}$.

Theorem 1. *A SEBP-orientation is represented by A PLCP orientation.*

Theorem 1. implies Corollary 1.

Corollary 1. *A SEBP-orientation satisfies the Holt-Klee condition.*

We prove Theorem 1. by inductive method of dimension. First, we represent SEBP by PLCP.

Lemma 1. *d-dimensional SEB can be formulated to LCP with following P-matrix M and q,*

$$M = \begin{pmatrix} 2\mathbf{x}_1^\top \mathbf{x}_1 & 2\mathbf{x}_1^\top \mathbf{x}_2 & \ldots & 2\mathbf{x}_1^\top \mathbf{x}_d \\ 2\mathbf{x}_2^\top \mathbf{x}_1 & 2\mathbf{x}_2^\top \mathbf{x}_2 & \ldots & 2\mathbf{x}_2^\top \mathbf{x}_d \\ \vdots & \vdots & \ddots & \vdots \\ 2\mathbf{x}_d^\top \mathbf{x}_1 & 2\mathbf{x}_d^\top \mathbf{x}_2 & \ldots & 2\mathbf{x}_d^\top \mathbf{x}_d \end{pmatrix}, \qquad (2)$$

$$q = \begin{pmatrix} -\mathbf{x}_1^\top \mathbf{x}_1 \\ -\mathbf{x}_2^\top \mathbf{x}_2 \\ \vdots \\ -\mathbf{x}_d^\top \mathbf{x}_d \end{pmatrix} \qquad (3)$$

In the case $d = 2$, it is easy to show every SEBP-orientation of 2-cube is PLCP-orientation.

Then we assume every SEBP-orientation is PLCP-orientation in $d(\leq k-1)$. In k-dimensional cube, proper face is PLCP-orientation because of the assumption.

Lemma 2. *Let p_O be the outer point of $k-1$ points $p_1, p_2, \ldots, p_{k-1}$. In k-dimensional cube derived from $p_1, p_2, \ldots, p_k$, the value of k-th column of $M^{-1}_{\{1,\ldots,k-1\}}$ is $\mathbf{x}_k(2\mathbf{x}_O - \mathbf{x}_k)$, where $\mathbf{x}_O$ is position vector of the outer point of p_O.*

Lemma 3. *Given a point p whose position vector is $\mathbf{x}_p$ and set of points P. Let p_O be the outer point of P and $\mathbf{x}_O$ be its position vector. Then $\mathbf{x}_p(2\mathbf{x}_O - \mathbf{x}_p) \geq 0$ iff $p \in \beta(P)$.*

These two lemma imply that every SEBP-orientation is PLCP-orientation in the case $d = k$.

3. Conclusion

We show that every SEBP-orientation is PLCP-orientation. We also show SEB-orientation satisfies Holt-Klee condition [9] from Corollary 1. But the inclusion relation between SEB-orientations and other orientations has not been characterized yet, which is interesting open problem.

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