

Balanced  $(C_5, C_{16})$ -Foil Designs and Related Designs

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1. Balanced  $(C_5, C_{16})$ -Foil Designs

Let  $K_n$  denote the complete graph of  $n$  vertices. Let  $C_5$  and  $C_{16}$  be the 5-cycle and the 16-cycle, respectively. The  $(C_5, C_{16})$ - $2t$ -foil is a graph of  $t$  edge-disjoint  $C_5$ 's and  $t$  edge-disjoint  $C_{16}$ 's with a common vertex. When  $K_n$  is decomposed into edge-disjoint sum of  $(C_5, C_{16})$ - $2t$ -foils and every vertex of  $K_n$  appears in the same number of  $(C_5, C_{16})$ - $2t$ -foils, we say that  $K_n$  has a balanced  $(C_5, C_{16})$ - $2t$ -foil decomposition. This decomposition is to be known as a balanced  $(C_5, C_{16})$ -foil design.

**Theorem 1.**  $K_n$  has a balanced  $(C_5, C_{16})$ - $2t$ -foil design if and only if  $n \equiv 1 \pmod{42t}$ .

**Example 1.1.** Balanced  $(C_5, C_{16})$ -2-foil design of  $K_{43}$ . Starter :  $\{(43, 1, 20, 40, 18), (43, 4, 6, 17, 25, 32, 16, 28, 14, 29, 24, 41, 35, 26, 13, 3)\}$ .

**Example 1.2.** Balanced  $(C_5, C_{16})$ -4-foil design of  $K_{85}$ . Starter :  $\{(85, 2, 40, 79, 35), (85, 7, 10, 32, 47, 61, 29, 53, 25, 55, 43, 34, 67, 50, 24, 5), (85, 1, 38, 78, 36), (85, 8, 12, 33, 49, 62, 31, 54, 27, 56, 46, 80, 69, 51, 26, 6)\}$ .

**Example 1.3.** Balanced  $(C_5, C_{16})$ -6-foil design of  $K_{127}$ . Starter :  $\{(127, 3, 60, 118, 52), (127, 10, 14, 47, 69, 90, 42, 78, 36, 81, 63, 50, 99, 74, 35, 7), (127, 2, 58, 117, 53), (127, 11, 16, 48, 71, 91, 44, 79, 38, 82, 65, 51, 101, 75, 37, 8), (127, 1, 56, 116, 54), (127, 12, 18, 49, 73, 92, 46, 80, 40, 83, 68, 119, 103, 76, 39, 9)\}$ .

**Example 1.4.** Balanced  $(C_5, C_{16})$ -8-foil design of  $K_{169}$ . Starter :  $\{(169, 4, 80, 157, 69), (169, 13, 18, 62, 91, 119, 55, 103, 47, 107, 83, 66, 131, 98, 46, 9), (169, 3, 78, 156, 70), (169, 14, 20, 63, 93, 120, 57, 104, 49, 108, 85, 67, 133, 99, 48, 10), (169, 2, 76, 155, 71), (169, 15, 22, 64, 95, 121, 59, 105, 51, 109, 87, 68, 135, 100, 50, 11), (169, 1, 74, 154, 72), (169, 16, 24, 65, 97, 122, 61, 106, 53, 110, 90, 158, 137, 101, 52, 12)\}$ .

## 2. Related Designs

**Theorem 2.**  $K_n$  has a balanced  $C_{21}$ - $t$ -foil design if

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and only if  $n \equiv 1 \pmod{42t}$ .

**Example 2.1.** Balanced  $C_{21}$  design of  $K_{43}$ .

Starter :  $\{(43, 1, 20, 40, 18, 22, 4, 6, 17, 25, 32, 16, 28, 14, 29, 24, 41, 35, 26, 13, 3)\}$ .

**Example 2.2.** Balanced  $C_{21}$ -2-foil design of  $K_{85}$ . Starter :

$\{(85, 2, 40, 79, 35, 42, 7, 10, 32, 47, 61, 29, 53, 25, 55, 43, 34, 67, 50, 24, 5), (85, 1, 38, 78, 36, 44, 8, 12, 33, 49, 62, 31, 54, 27, 56, 46, 80, 69, 51, 26, 6)\}$ .

**Example 2.3.** Balanced  $C_{21}$ -3-foil design of  $K_{127}$ . Starter :

$\{(127, 3, 60, 118, 52, 62, 10, 14, 47, 69, 90, 42, 78, 36, 81, 63, 50, 99, 74, 35, 7), (127, 2, 58, 117, 53, 64, 11, 16, 48, 71, 91, 44, 79, 38, 82, 65, 51, 101, 75, 37, 8), (127, 1, 56, 116, 54, 66, 12, 18, 49, 73, 92, 46, 80, 40, 83, 68, 119, 103, 76, 39, 9)\}$ .

**Example 2.4.** Balanced  $C_{21}$ -4-foil design of  $K_{169}$ . Starter :

$\{(169, 4, 80, 157, 69, 82, 13, 18, 62, 91, 119, 55, 103, 47, 107, 83, 66, 131, 98, 46, 9), (169, 3, 78, 156, 70, 84, 14, 20, 63, 93, 120, 57, 104, 49, 108, 85, 67, 133, 99, 48, 10), (169, 2, 76, 155, 71, 86, 15, 22, 64, 95, 121, 59, 105, 51, 109, 87, 68, 135, 100, 50, 11), (169, 1, 74, 154, 72, 88, 16, 24, 65, 97, 122, 61, 106, 53, 110, 90, 158, 137, 101, 52, 12)\}$ .

**Theorem 3.**  $K_n$  has a balanced  $C_{42}$ - $t$ -foil design if and only if  $n \equiv 1 \pmod{84t}$ .

**Example 3.1.** Balanced  $C_{42}$  design of  $K_{85}$ .

Starter :  $\{(85, 2, 40, 79, 35, 42, 7, 10, 32, 47, 61, 29, 53, 25, 55, 43, 34, 67, 50, 24, 5, 11, 6, 26, 51, 69, 80, 46, 56, 27, 54, 31, 62, 49, 33, 12, 8, 44, 36, 78, 38, 1)\}$ .

**Example 3.2.** Balanced  $C_{42}$ -2-foil design of  $K_{169}$ . Starter :

$\{(169, 4, 80, 157, 69, 82, 13, 18, 62, 91, 119, 55, 103, 47, 107, 83, 66, 131, 98, 46, 9, 19, 10, 48, 99, 133, 67, 85, 108, 49, 104, 57, 120, 93, 63, 20, 14, 84, 70, 156, 78, 3), (169, 2, 76, 155, 71, 86, 15, 22, 64, 95, 121, 59, 105, 51, 109, 87, 68, 135, 100, 50, 11, 23, 12, 52, 101, 137, 158, 90, 110, 53, 106, 61, 122, 97, 65, 24, 16, 88, 72, 154, 74, 1)\}$ .

**Example 3.3.** Balanced  $C_{42}$ -3-foil design of  $K_{253}$ . Starter :

$\{(253, 6, 120, 235, 103, 122, 19, 26, 92, 135, 177, 81, 153, 69, 159, 123, 98, 195, 146, 68, 13, 27, 14, 70, 147, 197, 99, 125, 160, 71, 154, 83, 178, 137, 93, 28, 20, 124, 104, 234, 118, 5), (253, 4, 116, 233, 105, 126, 21, 30, 94, 139, 179, 85, 155, 73, 161, 127, 100, 199, 148, 72, 15, 31, 16, 74, 149, 201, 101, 129, 162, 75, 156, 87, 180, 141, 95, 32, 22, 128, 106, 232, 114, 3), (253, 2, 112, 231, 107, 130, 23, 34, 96, 143, 181, 89, 157, 77,$

163, 131, 102, 203, 150, 76, 17, 35, 18, 78, 151, 205, 236, 134, 164, 79, 158, 91, 182, 145, 97, 36, 24, 132, 108, 230, 110, 1)}.  
**Example 3.4. Balanced  $C_{42}$ -4-foil design of  $K_{337}$ .** Starter :

{(337, 8, 160, 313, 137, 162, 25, 34, 122, 179, 235, 107, 203, 91, 211, 163, 130, 259, 194, 90, 17, 35, 18, 92, 195, 261, 131, 165, 212, 93, 204, 109, 236, 181, 123, 36, 26, 164, 138, 312, 158, 7), (337, 6, 156, 311, 139, 166, 27, 38, 124, 183, 237, 111, 205, 95, 213, 167, 132, 263, 196, 94, 19, 39, 20, 96, 197, 265, 133, 169, 214, 97, 206, 113, 238, 185, 125, 40, 28, 168, 140, 310, 154, 5), (337, 4, 152, 309, 141, 170, 29, 42, 126, 187, 239, 115, 207, 99, 215, 171, 134, 267, 198, 98, 21, 43, 22, 100, 199, 269, 135, 173, 216, 101, 208, 117, 240, 189, 127, 44, 30, 172, 142, 308, 150, 3), (337, 2, 148, 307, 143, 174, 31, 46, 128, 191, 241, 119, 209, 103, 217, 175, 136, 271, 200, 102, 23, 47, 24, 104, 201, 273, 314, 178, 218, 105, 210, 121, 242, 193, 129, 48, 32, 176, 144, 306, 146, 1)}.

**Theorem 4.**  $K_n$  has a balanced  $C_{63}$ - $t$ -foil design if and only if  $n \equiv 1 \pmod{126t}$ .

**Example 4.1. Balanced  $C_{63}$  design of  $K_{127}$ .**

Starter : {(127, 3, 60, 118, 52, 62, 10, 14, 47, 69, 90, 42, 78, 36, 81, 63, 50, 99, 74, 35, 7, 15, 8, 37, 75, 101, 51, 65, 82, 38, 79, 44, 91, 71, 48, 16, 11, 64, 53, 117, 58, 2, 57, 55, 56, 116, 54, 66, 12, 18, 49, 73, 92, 46, 80, 40, 83, 68, 119, 103, 76, 39, 9)}.

**Example 4.2. Balanced  $C_{63}$ -2-foil design of  $K_{253}$ .** Starter :

{(253, 6, 120, 235, 103, 122, 19, 26, 92, 135, 177, 81, 153, 69, 159, 123, 98, 195, 146, 68, 13, 27, 14, 70, 147, 197, 99, 125, 160, 71, 154, 83, 178, 137, 93, 28, 20, 124, 104, 234, 118, 113, 117, 4, 116, 233, 105, 126, 21, 30, 94, 139, 179, 85, 155, 73, 161, 127, 100, 199, 148, 72, 15), (253, 3, 114, 232, 106, 128, 22, 32, 95, 141, 180, 87, 156, 75, 162, 129, 101, 201, 149, 74, 16, 33, 17, 76, 150, 203, 102, 131, 163, 77, 157, 89, 181, 143, 96, 34, 23, 130, 107, 231, 112, 2, 111, 109, 110, 230, 108, 132, 24, 36, 97, 145, 182, 91, 158, 79, 164, 134, 236, 205, 151, 78, 18)}.

**Theorem 5.**  $K_n$  has a balanced  $C_{84}$ - $t$ -foil design if and only if  $n \equiv 1 \pmod{168t}$ .

**Example 5.1. Balanced  $C_{84}$  design of  $K_{169}$ .**

Starter : {(169, 4, 80, 157, 69, 82, 13, 18, 62, 91, 119, 55, 103, 47, 107, 83, 66, 131, 98, 46, 9, 19, 10, 48, 99, 133, 67, 85, 108, 49, 104, 57, 120, 93, 63, 20, 14, 84, 70, 156, 78, 75, 77, 2, 76, 155, 71, 86, 15, 22, 64, 95, 121, 59, 105, 51, 109, 87, 68, 135, 100, 50, 11, 23, 12, 52, 101, 137, 158, 90, 110, 53, 106, 61, 122, 97, 65, 24, 16, 88, 72, 154, 74, 1)}.

**Example 5.2. Balanced  $C_{84}$ -2-foil design of  $K_{337}$ .** Starter :

{(337, 8, 160, 313, 137, 162, 25, 34, 122, 179, 235, 107, 203, 91, 211, 163, 130, 259, 194, 90, 17, 35, 18, 92, 195, 261, 131, 165, 212, 93, 204, 109, 236, 181, 123, 36, 26, 164, 138, 312, 158, 151, 157, 6, 156, 311, 139, 166, 27, 38, 124, 183, 237, 111, 205, 95, 213, 167, 132, 263, 196, 94, 19, 39, 20, 96, 197, 265, 133, 169, 214, 97, 206, 113, 238, 185, 125, 40, 28, 168, 140, 310, 154, 5), (337, 4, 152, 309, 141, 170, 29, 42, 126, 187, 239, 115, 207, 99, 215, 171, 134, 267, 198, 98, 21, 43, 22, 100, 199, 269, 135, 173, 216, 101, 208, 117, 240, 189, 127, 44, 30, 172, 142, 308, 150, 147,

149, 2, 148, 307, 143, 174, 31, 46, 128, 191, 241, 119, 209, 103, 217, 175, 136, 271, 200, 102, 23, 47, 24, 104, 201, 273, 314, 178, 218, 105, 210, 121, 242, 193, 129, 48, 32, 176, 144, 306, 146, 1)}.

**Theorem 6.**  $K_n$  has a balanced  $C_{105}$ - $t$ -foil design if and only if  $n \equiv 1 \pmod{210t}$ .

**Example 6.1. Balanced  $C_{105}$  design of  $K_{211}$ .**

Starter : {(211, 5, 100, 196, 86, 102, 16, 22, 77, 113, 148, 68, 128, 58, 133, 103, 82, 163, 122, 57, 11, 23, 12, 59, 123, 165, 83, 105, 134, 60, 129, 70, 149, 115, 78, 24, 17, 104, 87, 195, 98, 4, 97, 93, 96, 194, 88, 106, 18, 26, 79, 117, 150, 72, 130, 62, 135, 107, 84, 167, 124, 61, 13, 27, 14, 63, 125, 169, 85, 109, 136, 64, 131, 74, 151, 119, 80, 28, 19, 108, 89, 193, 94, 2, 3, 1, 92, 192, 90, 110, 20, 30, 81, 121, 152, 76, 132, 66, 137, 112, 197, 171, 126, 65, 15)}.

**Theorem 7.**  $K_n$  has a balanced  $C_{126}$ - $t$ -foil design if and only if  $n \equiv 1 \pmod{252t}$ .

**Example 7.1. Balanced  $C_{126}$  design of  $K_{253}$ .**

Starter : {(253, 6, 120, 235, 103, 122, 19, 26, 92, 135, 177, 81, 153, 69, 159, 123, 98, 195, 146, 68, 13, 27, 14, 70, 147, 197, 99, 125, 160, 71, 154, 83, 178, 137, 93, 28, 20, 124, 104, 234, 118, 113, 117, 4, 116, 233, 105, 126, 21, 30, 94, 139, 179, 85, 155, 73, 161, 127, 100, 199, 148, 72, 15, 31, 16, 74, 149, 201, 101, 129, 162, 75, 156, 87, 180, 141, 95, 32, 22, 128, 106, 232, 114, 3, 5, 2, 112, 231, 107, 130, 23, 34, 96, 143, 181, 89, 157, 77, 163, 131, 102, 203, 150, 76, 17, 35, 18, 78, 151, 205, 236, 134, 164, 79, 158, 91, 182, 145, 97, 36, 24, 132, 108, 230, 110, 1)}.

**Theorem 8.**  $K_n$  has a balanced  $C_{147}$ - $t$ -foil design if and only if  $n \equiv 1 \pmod{294t}$ .

**Theorem 9.**  $K_n$  has a balanced  $C_{168}$ - $t$ -foil design if and only if  $n \equiv 1 \pmod{336t}$ .

**Theorem 10.**  $K_n$  has a balanced  $C_{189}$ - $t$ -foil design if and only if  $n \equiv 1 \pmod{378t}$ .

**Theorem 11.**  $K_n$  has a balanced  $C_{210}$ - $t$ -foil design if and only if  $n \equiv 1 \pmod{420t}$ .

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