A-024

Balanced (C_4, C_{18})-2t-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_{18} be *the* 4*-cycle* and *the* 18*-cycle*, respectively. *The* (C_4, C_{18})*-2t-foil* is a graph of t edge-disjoint C_4's and t edge-disjoint C_{18}'s with a common vertex and the common vertex is called *the center of the* (C_4, C_{18})*-2t-foil.* In particular, the (C_4, C_{18})-2-foil is called *the* (C_4, C_{18})*-bowtie.* When K_n is decomposed into edge-disjoint sum of (C_4, C_{18})-2t-foils, we say that K_n *has a* (C_4, C_{18})*-2t-foil decomposition.* Moreover, when every vertex of K_n appears in the same number of (C_4, C_{18})-2t-foils, we say that K_n *has a balanced* (C_4, C_{18})*-2t-foil decomposition* and this number is called *the replication number.*
Note that (C_4, C_{18})-2t-foil has $20t + 1$ vertices and $22t$ edges.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system.*
In this sense, our balanced (C_4, C_{18})-2t-foil decomposition of K_n is to be known as *a balanced* (C_4, C_{18})*-2t-foil system.*

2. Balanced (C_4, C_{18})-2t-foil decomposition of K_n

Theorem. K_n has a balanced (C_4, C_{18})-2t-foil decomposition if and only if $n \equiv 1$ (mod $44t$).
Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_{18})-2t-foil decomposition. Let b be the number of (C_4, C_{18})-2t-foils and r be the replication number. Then $b = n(n-1)/44t$ and $r = (20t+1)(n-1)/44t$. Among r (C_4, C_{18})-2t-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_{18})-2t-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/44t$ and $r_2 = 20(n-1)/44$. Therefore, $n \equiv 1$ (mod $44t$) is necessary.

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(Sufficiency) Put $n = 44st + 1$ and $T = st$. Then $n = 44T + 1$.
Construct a (C_4, C_{18})-2T-foil as follows:
$\{(44T+1, 36T+1, 27T+1, 37T+1), (44T+1, 1, 2T+2, 13T+2, 18T+3, 32T+3, 6T+3, 24T+3, 38T+4, 17T+3, 40T+4, 25T+3, 8T+3, 33T+3, 20T+3, 16T+2, 4T+2, T+1)\} \cup$
$\{(44T+1, 36T+2, 27T+3, 37T+2), (44T+1, 2, 2T+4, 13T+3, 18T+5, 32T+4, 6T+5, 24T+4, 38T+6, 17T+4, 40T+6, 25T+4, 8T+5, 33T+4, 20T+5, 16T+3, 4T+4, T+2)\} \cup$
$\{(44T+1, 36T+3, 27T+5, 37T+3), (44T+1, 3, 2T+6, 13T+4, 18T+7, 32T+5, 6T+7, 24T+5, 38T+8, 17T+5, 40T+8, 25T+5, 8T+7, 33T+5, 20T+7, 16T+4, 4T+6, T+3)\} \cup ... \cup$
$\{(44T+1, 37T, 29T-1, 38T), (44T+1, T, 4T, 14T+1, 20T+1, 33T+2, 8T+1, 25T+2, 40T+2, 18T+2, 42T+2, 26T+2, 10T+1, 34T+2, 22T+1, 17T+1, 6T, 2T)\}$. ($22T$ edges, $22T$ all lengths)
Decompose the (C_4, C_{18})-2T-foil into s (C_4, C_{18})-2t-foils. Then these s starters comprise a balanced (C_4, C_{18})-2t-foil decomposition of K_n.

Corollary. K_n has a balanced (C_4, C_{18})-bowtie decomposition if and only if $n \equiv 1$ (mod 44).

Example 1. A (C_4, C_{18})-2-foil of K_{45}.
$\{(45, 37, 28, 38), (45, 1, 4, 15, 21, 35, 9, 27, 42, 20, 44, 28, 11, 36, 23, 18, 6, 2)\}$.
(22 edges, 22 all lengths)
This starter comprises a balanced (C_4, C_{18})-2-foil decomposition of K_{45}.

Example 2. A (C_4, C_{18})-4-foil of K_{89}.
$\{(89, 73, 55, 75), (89, 1, 6, 28, 39, 67, 15, 51, 80, 37, 84, 53, 19, 69, 43, 34, 10, 3)\} \cup$
$\{(89, 74, 57, 76), (89, 2, 8, 29, 41, 68, 17, 52, 82, 38, 86, 54, 21, 70, 45, 35, 12, 4)\}$.
(44 edges, 44 all lengths)
This starter comprises a balanced (C_4, C_{18})-4-foil decomposition of K_{89}.

Example 3. A (C_4, C_{18})-6-foil of K_{133}.
$\{(133, 109, 82, 112), (133, 1, 8, 41, 57, 99, 21, 75, 118, 54, 124, 78, 27, 102, 63, 50, 14, 4)\} \cup$
$\{(133, 110, 84, 113), (133, 2, 10, 42, 59, 100, 23, 76, 120, 55, 126, 79, 29, 103, 65, 51, 16, 5)\} \cup$
$\{(133, 111, 86, 114), (133, 3, 12, 43, 61, 101, 25, 77, 122, 56, 128, 80, 31, 104, 67, 52, 18, 6)\}$.
(66 edges, 66 all lengths)

This starter comprises a balanced (C_4, C_{18})-6-foil decomposition of K_{133}.

Example 4. A (C_4, C_{18})-8-foil of K_{177}.
$\{(177,145,109,149),(177,1,10,54,75,131,27,99,156,71,164,103,35,135,83,66,18,5)\} \cup$
$\{(177,146,111,150),(177,2,12,55,77,132,29,100,158,72,166,104,37,136,85,67,20,6)\} \cup$
$\{(177,147,113,151),(177,3,14,56,79,133,31,101,160,73,168,105,39,137,87,68,22,7)\} \cup$
$\{(177,148,115,152),(177,4,16,57,81,134,33,102,162,74,170,106,41,138,89,69,24,8)\}$.
(88 edges, 88 all lengths)
This starter comprises a balanced (C_4, C_{18})-8-foil decomposition of K_{177}.

Example 5. A (C_4, C_{18})-10-foil of K_{221}.
$\{(221,181,136,186),(221,1,12,67,93,163,33,123,194,88,204,128,43,168,103,82,22,6)\} \cup$
$\{(221,182,138,187),(221,2,14,68,95,164,35,124,196,89,206,129,45,169,105,83,24,7)\} \cup$
$\{(221,183,140,188),(221,3,16,69,97,165,37,125,198,90,208,130,47,170,107,84,26,8)\} \cup$
$\{(221,184,142,189),(221,4,18,70,99,166,39,126,200,91,210,131,49,171,109,85,28,9)\} \cup$
$\{(221,185,144,190),(221,5,20,71,101,167,41,127,202,92,212,132,51,172,111,86,30,10)\}$.
(110 edges, 110 all lengths)
This starter comprises a balanced (C_4, C_{18})-10-foil decomposition of K_{221}.

Example 6. A (C_4, C_{18})-12-foil of K_{265}.
$\{(265,217,163,223),(265,1,14,80,111,195,39,147,232,105,244,153,51,201,123,98,26,7)\} \cup$
$\{(265,218,165,224),(265,2,16,81,113,196,41,148,234,106,246,154,53,202,125,99,28,8)\} \cup$
$\{(265,219,167,225),(265,3,18,82,115,197,43,149,236,107,248,155,55,203,127,100,30,9)\} \cup$
$\{(265,220,169,226),(265,4,20,83,117,198,45,150,238,108,250,156,57,204,129,101,32,10)\} \cup$
$\{(265,221,171,227),(265,5,22,84,119,199,47,151,240,109,252,157,59,205,131,102,34,11)\} \cup$
$\{(265,222,173,228),(265,6,24,85,121,200,49,152,242,110,254,158,61,206,133,103,36,12)\}$.
(132 edges, 132 all lengths)
This starter comprises a balanced (C_4, C_{18})-12-foil decomposition of K_{265}.

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