# Ls in L and Sphinxes in Sphinx

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**Abstract:** We prove that L-shaped trominoes can tile the L-shaped tromino scaled by any positive integer, and Sphinx-shaped hexiamond can tile the Sphinx-shaped hexiamond scaled by any positive integer. We also give an upper bound and a lower bound for the number of such tilings.

## 1. Introduction

Tiling a floor in such a way that identical shapes are endlessly repeated to cover the plane has been investigated in a lot of contexts including statistical physics, nature-inspired computation, enumerative combinatorics, and recreational mathematics. For example, it is well known that periodic tilings had been classified into seventeen categories, and all of them are used at the Alhambra palace (see, e.g., [FAN2010]). On the other hand, non-periodic tilings are also well studied, and the most famous are known as the Penrose tilings, developed by Penrose in 1974 [Gardner1989]. The Penrose tilings force us to tile in an aperiodic way. Moreover, in 1962, Golumb investigated the "replicating figures," and he named them *rep-tiles*, which give another way of constructing aperiodic tilings [Gardner2014].

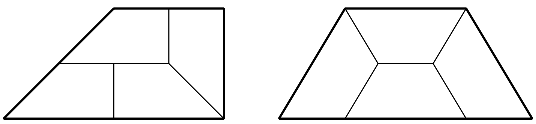

**Fig. 1** Two convex rep-tiles.

Formally, a polygon is called a *rep-tile* if it is filled by a finite set of copies of itself in a certain scale (see Fig. 1). Based on the square lattice and the triangular lattice, there are two major rep-tiles which are well known as puzzles in Fig. 2. We call the left one "L," and the right one "sphinx." Extending them iteratively, we may construct a lot of different tilings including aperiodic ones. However, we wonder if the following basic questions can be answered.

( 1 ) Can Ls tile an L at any scale, and can sphinxes tile a sphinx at any scale?
( 2 ) If any, how many different tilings exist?

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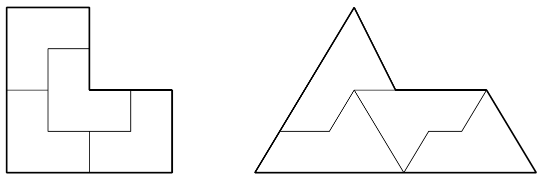

**Fig. 2** Two non-convex rep-tiles, L and Sphinx.

To answer these questions, we introduce the following notation. Let $L_1$ be the L which is composed of three unit squares, and $S_1$ be the sphinx which is composed of six unit equilateral triangles, respectively. We also let $L_n$ and $S_n$ be the scaled copies of $L_1$ and $S_1$ by factor $n$, respectively. That is, $L_n$ consists of $3n^2$ unit squares, and $S_n$ consists of $6n^2$ unit equilateral triangles. We denote the number of ways to tile $Y$ by copies of the figure $X$ by $N(X, Y)$. Note that the figure $X$ can be rotated and reflected.

It is easy to check that $2^2 = 4$ sphinxes can be assembled to form a larger sphinx in a unique way: $N(S_2, S_1) = 1$. However, the larger scale immediately makes the calculation by hand infeasible. With the help of a free puzzle solver "BurrTools," which is well known in puzzle society[*1], we obtain the following values: $N(L_1, L_1) = 1$, $N(L_2, L_1) = 1$, $N(L_3, L_1) = 4$, $N(L_4, L_1) = 409$, $N(L_5, L_1) = 108,388$, and $N(S_1, S_1) = 1$, $N(S_2, S_1) = 1$, $N(S_3, S_1) = 4$, $N(S_4, S_1) = 16$, $N(S_5, S_1) = 153$, $N(S_6, S_1) = 71,838$. It seems to grow "exponentially." In this paper, we will show that it grows more rapidly. We first note that we can estimate $N(S_n, S_1)$ by using the property of the rep-tile itself. For example, we can use $N(S_3, S_1) = 4$, and obtain that $N(S_{3i}, S_1) \geq 4N(S_{3(i-1)}, S_1)$. This inequality implies $N(S_n, S_1) = \Omega(c^n)$ for a constant $c = 4^{1/3}$, which means that we have an exponential lower bound. Replacing 4 by, e.g., 71,838, we will have a larger base of the exponential growth. However, this is too weak as we will show. We develop a stronger analysis, and obtain much better lower bounds. We also give upper bounds in a nontrivial way. The main theorem of this paper is as follows.

**Theorem 1** (1) As the function on $n$, $N(L_n, L_1)$ is bounded by $\Omega(\sqrt{2}^{n^2})$ from below, and by $O((12e)^{n^2})$ from above. (2) As the function on $n$, $N(S_n, S_1)$ is bounded by $\Omega(6.44^{n^2})$ from below, and by $O((36e)^{n^2})$ from above.

*1 `http://burrtools.sourceforge.net/`

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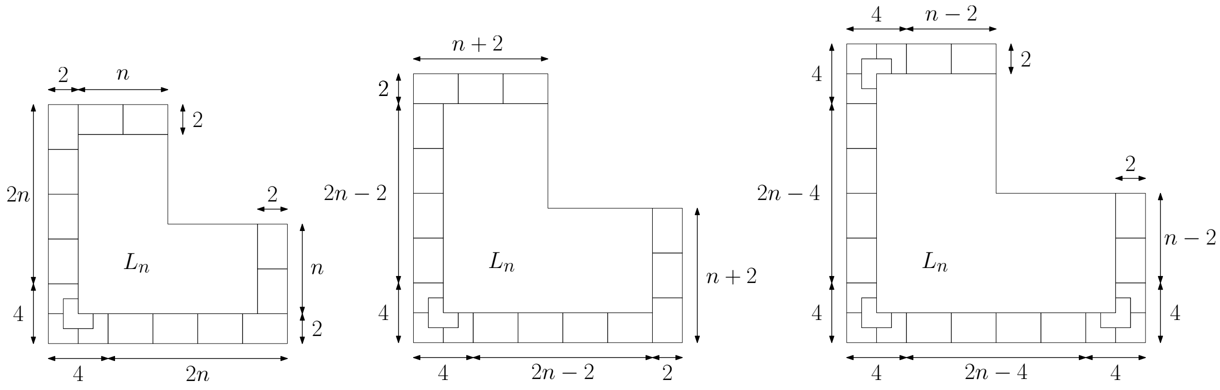



(a) $n \equiv 0 \pmod 3$). (b) $n \equiv 1 \pmod 3$). (c) $n \equiv 2 \pmod 3$).

**Fig. 3** Tilings for $L_n$.

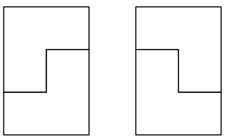


**Fig. 4** Two ways of tiling the rectangle of dimension $2 \times 3$.

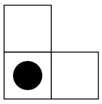


**Fig. 5** A marked unit square in $L_1$.

That is, we can conclude that these functions are of type $c^{n^2}$ with some constant $c$. Note that $\sqrt{2} \geq 1.41$, $12e \leq 32.62$ and $36e \leq 97.86$.

## 2. Ls in L

We first prove that $L_1$ can tile $L_n$ for any $n \geq 1$.

**Theorem 2** $N(L_n, L_1) \geq 1$ holds for any $n \geq 1$.

**Proof.** We have $N(L_1, L_1) = 1$, $N(L_2, L_1) = 1$, and $N(L_3, L_1) = 4$. Therefore, it is sufficient to show that $N(L_n, L_1) \geq N(L_{n-2}, L_1)$ for every $n > 3$. For each of three cases $n \equiv 0, 1, 2 \pmod 3$, we use three patterns as in Fig. 3. The rectangles in Fig. 3 can be tiled in the way in Fig. 4. □

Now, we prove Theorem 1(1).

First, we show the lower bound using the patterns in Fig. 3. Each pattern contains at least $2n - 6$ rectangles of dimension $2\times3$. Moreover, each rectangle has two ways of tiling as shown in Fig. 4, which means that the number of ways to tile these rectangles is $2^{2n-6}$. Therefore, we obtain $N(L_{n+2}, L_1) \geq 2^{2n-6} N(L_n, L_1)$. Thus, we have a lower bound of $N(L_n, L_1) = \Omega(2^{\frac{n^2}{2}}) = \Omega(\sqrt{2}^{n^2})$.

Next, we turn to the upper bound. We put a mark in a unit square of $L_1$ as shown in Fig. 5. We will fill $L_n$ by $n^2$ $L_1$s. Therefore, the number of ways to choose $n^2$ marked unit squares from $3n^2$ unit squares in $L_n$ is bounded from above by $\binom{3n^2}{n^2}$. At each marked unit square, we have at most four ways to put $L_1$. Therefore, $N(L_n, L_1)$ is bounded from above by $4^{n^2}\binom{3n^2}{n^2} = O(4^{n^2}(3e)^{n^2}) = O((12e)^{n^2})$, which concludes the proof of Theorem 1(1).

## 3. Sphinxes in Sphinx

As we have already shown, $N(S_1, S_1) = 1$, $N(S_2, S_1) = 1$, $N(S_3, S_1) = 4$, $N(S_4, S_1) = 16$, $N(S_5, S_1) = 153$, $N(S_6, S_1) = 71,838$. Especially, $N(S_3, S_1) = 4$ is given in Fig. 6.

First, we prove the tilability.

**Theorem 3** $N(S_n, S_1) \geq 1$ holds for any $n \geq 1$.

**Proof.** We have $N(S_1, S_1) = 1$, $N(S_2, S_1) = 1$, and $N(S_3, S_1) = 4$. Therefore, it is sufficient to show that $N(S_{n+3}, S_1) \geq N(S_n, S_1)$ for every $n > 0$. For each of two cases $n \equiv 0, 1 \pmod 2$, we use two patterns as in Fig. 7(a) and Fig. 7(b). Now each parallelogram in Fig. 7(a) and (b) can be tiled in the way in Fig. 7(c), and the area $T$ is filled in one of the three ways in Fig. 8. □

Here we note that each parallelogram in Fig. 7(a) and (b) is tiled in a *unique* way in Fig. 7(c). Therefore, we need a further trick to prove Theorem 1(2).

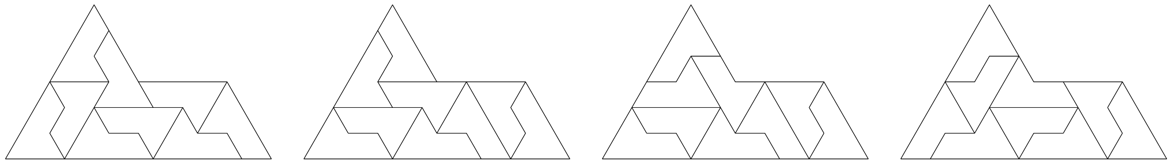
First, we show the lower bound using the patterns in Fig. 7. Since $N(T, L_1) = 3$ as shown in Fig. 8, we have $N(S_{n+3}, S_1) \geq 3N(S_n, S_1)$. Now we split each unit equilateral triangle into 36 small equilateral triangles. Then, we obtain $N(S_{6n+18}, S_6) \geq 3N(S_{6n}, S_6)$. On the other hand, $N(S_6, S_1) = 71838$, and at least $6n + 2$ $S_6$'s are located around $S_n$ in both of Fig. 7(a) and (b). Therefore, we obtain $N(S_{6n+18}, S_1) = 3 \cdot 71838^{6n+2} N(S_{6n}, S_1)$, which implies that $N(S_n, S_1) = \Omega(71838^{\frac{3}{18}n^2}) = \Omega(6.44^{n^2})$. The upper bound can be obtained in a similar way in the case of $L_n$, which concludes the proof of Theorem 1(2).

### References

[FAN2010] Koji Fushimi, Mitsumasa Anno, and Gisaku Nakamura. *Bi no Kikagaku (Geometry in Beauty)*. Hayakawa Shobo, 2010. (in Japanese)

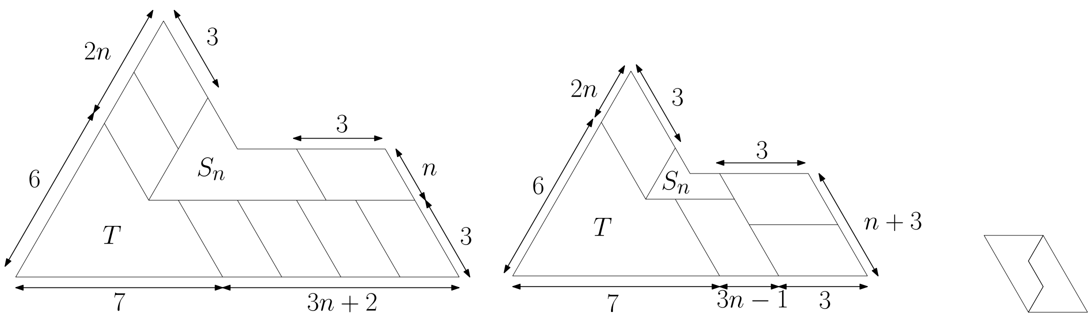
[Gardner1989] Martin Gardner. *Penrose tiles to trapdoor ciphers ... and the return of Dr. Matrix*. W.H. Freeman & Company, 1989.

[Gardner2014] Martin Gardner. *Knots and Borromean Rings, Rep-Tiles, and Eight Queens*. Cambridge University Press, 2014.



(a) (b) (c) (d)

**Fig. 6** Four tilings for $S_3$.




(a) $n = 0 \pmod 2$). (b) $n = 1 \pmod 2$) (c) A parallelogram.

**Fig. 7** Tilings for $S_n$.

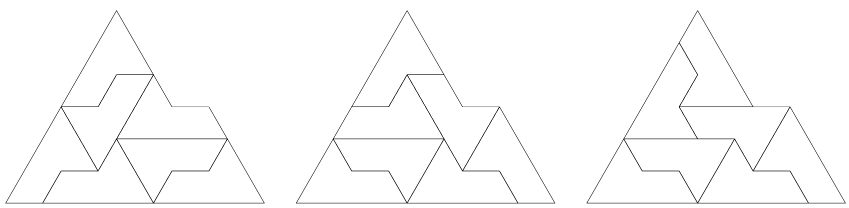


**Fig. 8** Three tilings for $T$.

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