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On the Convergence Speed for Some Iterative Methods

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We derived two types of iterative methods, each containing two parameters. Then we established that these methods converge globally and monotonically to the zeros of both a polynomial with only real zeros and an entire function of genus 0 and 1 (and in certain cases of genus 2) with only real zeros under some assumptions. In this paper, we discuss the convergence speed in Ostrowski's method, Halley's method and our methods.

1. Introduction

We will consider some iterative methods for the computation of numerical solutions of two types of nonlinear scalar equations.

One type of the said equations is given by the following form:

$$f(x)\equiv\prod_{k=1}^{r}(x-\zeta_k)=0 \tag{1.1}$$

where $r>1$ and $\zeta_{k+1}\geqq\zeta_k$ $(k=1,\cdots,r-1)$.

The other is given by the following form:

$$f(x)\equiv x^p\exp(a+bx-cx^2)\prod_{k=1}^{\infty}\left(1-\frac{x}{\alpha_k}\right)e^{x/\alpha_k}=0 \tag{1.2}$$

where p is a non-negative integer; a, b and c are real with $c\geqq 0$; and all α_k are real with $\sum_{k=1}^{\infty}\alpha_k^{-2}<\infty$.

Here, especially we will deal with Ostrowski's method,[1),2)] Halley's method[2)] and our methods containing two parameters.[4),5)]

In Ref. 1), pp. 110-115 and Ref. 3)-5), it has been shown that the said methods converge globally and monotonically to the solutions of both Eqs. (1.1) and (1.2) under some assumptions. In section 2, it will be shown that both Ostrowski's iteration function and Halley's iteration function are monotonically increasing on a certain set. In section 3, by using the results derived in section 2, we will discuss the convergence speed in the methods mentioned above.

2. Preliminaries

At first, we will introduce Ostrowski's iteration function, Halley's iteration function and our iteration functions.

Let $h\equiv h(x)=\dfrac{f(x)}{f'(x)}$ and $X\equiv X(x)=h\dfrac{f''(x)}{f'(x)}$. Then, Ostrowski's iteration function can be represented in the form:

$$\Phi_1(x)=x-\frac{h}{\sqrt{1-X}}. \tag{2.1}$$

Halley's iteration function can be represented in the form:

$$\Phi_2(x)=x-\frac{h}{1-\frac{1}{2}X}. \tag{2.2}$$

Our iteration functions can be represented in the forms:

$$\Phi_3(x)=x-\frac{\left(\theta+\frac{1}{2}\right)X+1}{\beta X^2+\theta X+1}h$$

$(\beta,\theta$: parameters), (2.3)

$$\Phi_4(x)=x-\frac{a+\sqrt{b}}{a+\sqrt{b}-\sqrt{b}\,(a+\sqrt{b}\,)X}h$$

$(a, b$: parameters) (2.4)

We will need later the following lemma:

Lemma 1. If $a_k>0$ $(k=1,\cdots,r)$ and $r\geqq 2$, then the following inequality holds: $(\sum_{k=1}^{r}a_k^3)^2<(\sum_{k=1}^{r}a_k^2)^3$.

Proof. Applying the Cauchy-Schwarz inequality, we have the following:

$$(\sum_{k=1}^{r}a_k^3)^2=(\sum_{k=1}^{r}a_k^2a_k)^2\leqq\sum_{k=1}^{r}a_k^2\sum_{k=1}^{r}a_k^4<(\sum_{k=1}^{r}a_k^2)^3. \tag{2.5}$$

From Lemma 1, we have:

Lemma 2. If $a_k>0$ $(k=1,\cdots)$ and $\sum_{k=1}^{\infty}a_k^2<\infty$, then the following inequality holds:

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$$(\sum_{k=1}^{\infty} a_k^3)^2 < (\sum_{k=1}^{\infty} a_k^2)^3$$

Next, taking the logarithmic derivative of $f(x)$ in Eq. (1. 1), differentiating it again and putting $f \equiv f(x)$, $f' \equiv f'(x)$ and $f'' \equiv f''(x)$, we have

$$\frac{f'}{f} = \sum_{k=1}^{\gamma} \frac{1}{x-\zeta_k},$$

$$-\frac{d}{dx}\left(\frac{f'}{f}\right) = \frac{f'^2 - ff''}{f^2} = \sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^2}. \tag{2. 6}$$

From Eq. (2. 6) we have

$$X = 1 - h^2 \sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^2}. \tag{2. 7}$$

Furthermore,

$$\frac{dh}{dx} = \frac{d}{dx}\left(\frac{f}{f'}\right) = \frac{f'^2 - ff''}{f'^2} = 1 - X. \tag{2. 8}$$

Differentiating Eq. (2. 7), we have

$$\frac{dX}{dx} = -2h \frac{dh}{dx} \sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^2} + 2h^2 \sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^3}.$$

Therefore it follows from Eqs. (2. 7) and (2. 8) that

$$h\frac{dX}{dx} = -2(1-X)^2 + 2h^3 \sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^3}. \tag{2. 9}$$

Next, taking the logarithmic derivative of $f(x)$ in Eq. (1. 2) and differentiating it again, we have

$$\frac{f'}{f} = \frac{p}{x} + b - 2cx + \sum_{k=1}^{\infty}\left(\frac{1}{x-\alpha_k} + \frac{1}{\alpha_k}\right),$$

$$-\left(\frac{f'}{f}\right)' = \frac{f'^2 - ff''}{f^2} = \frac{p}{x^2} + 2c + \sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^2}. \tag{2. 6}'$$

From Eq. (2. 6)′ we have

$$X = 1 - h^2\left\{\frac{p}{x^2} + 2c + \sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^2}\right\} \tag{2. 7}'$$

Differentiating Eq. (2. 7)′, we have

$$\frac{dX}{dx} = -2h\frac{dh}{dx}\left\{\frac{p}{x^2} + 2c + \sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^2}\right\} + 2h^2\left\{\frac{p}{x^3} + \sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^3}\right\}.$$

From Eqs. (2. 7)′ and (2. 8)

$$h\frac{dX}{dx} = -2(1-X)^2 + 2h^3\left\{\frac{p}{x^3} + \sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^3}\right\}. \tag{2. 9}'$$

Let the set $\{x : f(x)f'(x) \neq 0$, and x is real$\}$ be denoted by $\boldsymbol{E}$. Then we have:

Theorem 1. Let $f(x)$ be given by the form in Eq. (1. 1). Then $\Phi_1(x)$ is a monotonically increasing function on $\boldsymbol{E}$.

Proof. Differentiating $\Phi_1(x)$ in Eq. (2. 1), we have

$$\Phi_1'(x) = 1 - \frac{1}{\sqrt{1-X}}\frac{dh}{dx} - \frac{1}{2(1-X)^{3/2}} h \frac{dX}{dx}.$$

Using Eqs. (2. 8) and (2. 9), we have

$$\Phi_1'(x) = 1 - \frac{h^3}{(1-X)^{3/2}} \sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^3}. \tag{2. 10}$$

On the other hand, from Eq. (2. 7) we have

$$(1-X)^{3/2} = |h|^3 \left\{\sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^2}\right\}^{3/2}.$$

Hence it follows that

$$\frac{h^3}{(1-X)^{3/2}} \sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^3} \leqq \frac{\sum_{k=1}^{\gamma} \frac{1}{|x-\zeta_k|^3}}{\left\{\sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^2}\right\}^{3/2}}. \tag{2. 11}$$

Applying Lemma 1 to Eq. (2. 11), we have

$$\frac{h^3}{(1-X)^{3/2}} \sum_{k=1}^{\gamma} \frac{1}{(x-\zeta_k)^3} < 1 \quad \text{for } \forall x \in \boldsymbol{E}. \tag{2. 12}$$

Finally, it follows from Eqs. (2. 10) and (2. 12) that $\Phi_1'(x) > 0$. Thus Theorem 1 is completely proved. Furthermore we have:

Theorem 2. Let $f(x)$ be given by the form in Eq. (1. 2). Then $\Phi_1(x)$ is a monotonically increasing function on $\boldsymbol{E}$.

Proof. Differentiating $\Phi_1(x)$ and using Eqs. (2. 8) and (2. 9)′, we have

$$\Phi_1'(x) = 1 - \frac{h^3}{(1-X)^{3/2}}\left\{\frac{p}{x^3} + \sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^3}\right\}. \tag{2. 10}'$$

It now follows from Eq. (2. 7)′ and $c \geqq 0$ that

$$\frac{h^3}{(1-X)^{3/2}}\left\{\frac{p}{x^3} + \sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^3}\right\} \leqq \frac{p + |x|^3 \sum_{k=1}^{\infty} \frac{1}{|x-\alpha_k|^3}}{\left\{p + x^2 \sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^2}\right\}^{3/2}}. \tag{2. 11}'$$

Since $\sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^2} < \infty$ for $\forall x \in \boldsymbol{E}$, we use Lemma 2 to have

$$\left\{p + |x|^3 \sum_{k=1}^{\infty} \frac{1}{|x-\alpha_k|^3}\right\}^2 \leqq \left\{(p^{\frac{1}{2}})^3 + |x|^3 \sum_{k=1}^{\infty} \frac{1}{|x-\alpha_k|^3}\right\}^2 < \left\{p + x^2 \sum_{k=1}^{\infty} \frac{1}{(x-\alpha_k)^2}\right\}^3 \quad \text{for } \forall x \in \boldsymbol{E}. \tag{2. 12}'$$

Consequently, it follows from Eqs. (2. 10)′, (2. 11)′ and (2. 12)′ that $\Phi_1'(x) > 0$. Thus Theorem 2 is completely proved.

Next we have:

Theorem 3. Let $f(x)$ be given by the form in Eq. (1. 1). $\Phi_2(x)$ is a monotonically increasing function on $\boldsymbol{E}$.

Proof. Differentiating $\Phi_2(x)$ in Eq. (2. 2), we have

$$\Phi_2'(x)=1-\frac{1}{1-\frac{1}{2}X}\frac{dh}{dx}-\frac{1}{2\left(1-\frac{1}{2}X\right)^2}h\frac{dX}{dx}.$$

Using Eqs. (2. 8) and (2. 9), we have

$$\Phi_2'(x)=\frac{(1-X)^{3/2}}{\left(1-\frac{1}{2}X\right)^2}\left[\frac{\frac{3}{4}X^2-\frac{3}{2}X+1}{(1-X)^{3/2}}-\frac{h^3}{(1-X)^{3/2}}\sum_{k=1}^{\gamma}\frac{1}{(x-\zeta_k)^3}\right].$$

Next, we show that the following inequality holds:

$$\frac{\frac{3}{4}X^2-\frac{3}{2}X+1}{(1-X)^{3/2}}\geqq 1 \quad (X<1). \qquad (2.13)$$

Let

$$g(X)=\frac{3}{4}X^2-\frac{3}{2}X+1-(1-X)^{3/2}.$$

From $g'(X)=\frac{3}{2}\sqrt{1-X}(1-\sqrt{1-X})$, we have

$$g'(X)>0 \quad (X>0) \text{ and } g'(X)<0 \quad (X<0).$$

From $g(0)=0$, we have

$$g(X)\geqq 0 \quad (X<1). \qquad (2.14)$$

Hence Eq. (2. 13) holds.

From Eqs. (2. 12) and (2. 13) and

$$\frac{(1-X)^{3/2}}{\left(1-\frac{1}{2}X\right)^2}>0,$$

Theorem 3 is completely proved.

Furthermore, we have:

Theorem 4. Let $f(x)$ be given by the form in Eq. (1. 2). Then $\Phi_2(x)$ is a monotonically increasing function on $\boldsymbol{E}$.

This theorem can be proved in the same way that we proved Theorem 3.

3. Convergence Speed

We will consider the iterative methods for Eqs. (2. 1), (2. 2), (2. 3) and (2. 4). These methods can be represented in the following forms:

Ostrowski's method:

$$x_{n+1}=\Phi_1(x_n) \quad (n=0, 1, \cdots). \qquad (3.1)$$

Halley's method:

$$y_{n+1}=\Phi_2(y_n) \quad (n=0, 1, \cdots). \qquad (3.2)$$

Our methods:

$$z_{n+1}=\Phi_3(z_n) \quad (n=0, 1, \cdots) \qquad (3.3)$$

$$w_{n+1}=\Phi_4(w_n) \quad (n=0, 1, \cdots). \qquad (3.4)$$

We will need later the following lemmas:

Lemma 3 If $\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2\leqq\beta\leqq-\frac{1}{2}\left(\theta+\frac{1}{2}\right)$, then

$$\frac{1}{1-\frac{1}{2}X}\leqq\frac{\left(\theta+\frac{1}{2}\right)X+1}{\beta X^2+\theta X+1}\leqq\frac{1}{\sqrt{1-X}} \quad (X<1)$$

where the sign of equality holds only for $X=0$.

Lemma 4 If $-\sqrt{b}<a\leqq 0$, then

$$\frac{1}{1-\frac{1}{2}X}\leqq\frac{a+\sqrt{b}}{a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}}\leqq\frac{1}{\sqrt{1-X}} \quad (X<1)$$

where the sign of equality holds only for $X=0$.

Here, in Ref. 4), it has been shown that Lemma 3 holds.

As to Lemma 4, in Ref. 5), it has been shown that if $-\sqrt{b}<a\leqq 0$, then

$$0<\frac{a+\sqrt{b}}{a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}}\leqq\frac{1}{\sqrt{1-X}} \quad (X<1).$$

Therefore, it suffices to show that if $-\sqrt{b}<a\leqq 0$, then

$$a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}\leqq(a+\sqrt{b})\left(1-\frac{1}{2}X\right) \quad (X<1). \quad (3.5)$$

In order to show that Eq. (3. 5) holds, putting

$$P(X)=(a+\sqrt{b})\left(1-\frac{1}{2}X\right)-\{a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}\},$$

we must show that $P(X)$ is non-negative for $X<1$.

Now, since

$$P'(X)=\frac{1}{2}(a+\sqrt{b})\left\{\frac{\sqrt{b}}{\sqrt{b-\sqrt{b}(a+\sqrt{b})X}}-1\right\},$$

$$P'(X)<0 \quad (X<0) \text{ and } P'(X)>0 \quad (X>0).$$

Hence, $P(X)\geqq P(0)=0$.

Thus Lemma 4 is completely proved.

Let $f(x)$ be given by the form in Eq. (1. 1). Then it follows from Rolle's theorem that $f'(x)$ has $r-1$ real zeros ζ'_k, so that

$$\zeta_k \leqq \zeta'_k \leqq \zeta_{k+1} \quad (k=1, \cdots, r-1).$$

Especially, if $\zeta_k < \zeta_{k+1}$, we have $\zeta_k < \zeta'_k < \zeta_{k+1}$.

For any real x distinct from ζ_k and ζ'_k, we can define the associated zero $\zeta(x)$ of $f(x)$ to be $\zeta(x)=\zeta_1$ if $x<\zeta_1$, $\zeta(x)=\zeta_r$ if $x>\zeta_r$, $\zeta(x)=\zeta_k$ if $\zeta_k<x<\zeta'_k$, and $\zeta(x)=\zeta_{k+1}$ if $\zeta'_k<x<\zeta_{k+1}$.

If $\zeta_k=\zeta_1 (k=2, \cdots, r)$, then we can define the associated zero $\zeta(x)$ of $f(x)$ to be $\zeta(x)=\zeta_1$ if $x \neq \zeta_1$.

In addition, we have $\operatorname{sgn}(h(x))=\operatorname{sgn}(x-\zeta(x))$.

By applying $\operatorname{sgn}(h(x))=\operatorname{sgn}(x-\zeta(x))$ to Theorem 1 in Ref. 4), we have:

Theorem 5. Let $f(x)$ be given by the form in Eq. (1. 1). Then, if $\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2 \leqq \beta \leqq -\frac{1}{2}\left(\theta+\frac{1}{2}\right)$, and if we take the real starting value in Eq. (3. 3) z_0 such that $f(z_0)f'(z_0) \neq 0$, then we have

$$z_n \downarrow \zeta(z_0) \quad (h(z_0)>0) \text{ and } z_n \uparrow \zeta(z_0) \quad (h(z_0)<0) \quad (n=0, 1, \cdots).$$

Also, by applying sgn $(h(x))=\operatorname{sgn}(x-\zeta(x))$ to Theorem 2 in Ref. 5), we have:

Theorem 6. Let $f(x)$ be given by the form in Eq. (1. 1). Then, if $-\sqrt{b}<a\leqq 0$ and if we take the real starting value in Eq. (3. 4) w_0 such that $f(w_0)f'(w_0) \neq 0$, then we have

$$w_n \downarrow \zeta(w_0) \quad (h(w_0)>0) \text{ and } w_n \uparrow \zeta(w_0) \quad (h(w_0)<0) \quad (n=0, 1, \cdots).$$

Furthermore, for $\beta=-\frac{1}{2}\left(\theta+\frac{1}{2}\right)\left(-\frac{3}{2}\leqq\theta\leqq-\frac{1}{2}\right)$, Eq. (3. 3) coincides with Halley's method. For $b=1$, Eq. (3. 4) coincides with Hansen and Patrick's methods.[6] For $b=1$ and $a=0$, Eq. (3. 4) coincides with Ostrowski's method.

Next, as to the convergence speed, we have:

Theorem 7. Let $f(x)$ be given by the form in Eq. (1. 1). Then, if $\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2 \leqq \beta < -\frac{1}{2}\left(\theta+\frac{1}{2}\right)$, and if we take the real starting values x_0, y_0 and z_0 respectively in Eqs. (3. 1), (3. 2) and (3. 3) such that $x_0=y_0=z_0$ and $f(x_0)f'(x_0)f''(x_0)\neq 0$, then we have

$$\zeta(x_0)<x_n<z_n<y_n \quad (h(x_0)>0) \text{ and}$$
$$\zeta(x_0)>x_n>z_n>y_n \quad (h(x_0)<0) \quad (n=1, 2, \cdots).$$

Proof. Assume $h(x_0)>0$. Then, from Lemma 3 $x_1<z_1<y_1$. On the other hand, from Theorem 6, Theorem 5, the definition of $\zeta(x)$, and $\operatorname{sgn}(h(x)) =\operatorname{sgn}(x-\zeta(x))$, we have

$$\zeta(x_0)\equiv\zeta(z_1)<x_1<z_1<y_1 \text{ and } h(z_1)>0.$$

It now follows from Theorem 1, Lemma 3 and Theorem 3 that

$$\Phi_1(x_1)<\Phi_1(z_1)\leqq\Phi_3(z_1)\leqq\Phi_2(z_1)<\Phi_2(y_1)$$

that is

$$x_2<\Phi_1(z_1)\leqq z_2\leqq\Phi_2(z_1)<y_2.$$

Therefore, similarly, we see that

$$\zeta(x_0)\equiv\zeta(z_2)<x_2<z_2<y_2 \text{ and } h(z_2)>0.$$

Consequently, by repetition of the same discussion, we have

$$\zeta(x_0)<x_n<z_n<y_n. \quad (n=1, 2, \cdots).$$

Next, if $h(x_0)<0$, the discussion is completely symmetric. Thus Theorem 7 is completely proved.

Furthermore, we have:

Theorem 8. Let $f(x)$ be given by the form in Eq. (1. 1). Then, if $-\sqrt{b}<a<0$, and if we take the real starting values x_0, y_0 and w_0 respectively in Eqs. (3. 1), (3. 2) and (3. 4) such that $x_0=y_0=w_0$ and $f(x_0)f'(x_0)f''(x_0)\neq 0$, then we have

$$\zeta(x_0)<x_n<w_n<y_n. \quad (h(x_0)>0) \text{ and}$$
$$\zeta(x_0)>x_n>w_n>y_n. \quad (h(x_0)<0) \quad (n=1, 2, \cdots).$$

By using Lemma 4, Theorems 1, 3 and 6, this theorem can be proved in a way similar to that of Theorem 7.

Next, let $f(x)$ be given by the form in Eq. (1. 2), and the distinct zeros of $f(x)$ be ordered consecutively so that $\eta_0<\eta_1$. Then, since the right hand side of Eq. (2. 6)′ is positive, we see that $\frac{f'(x)}{f(x)}$ is monotonically decreasing in the open interval (η_0, η_1). Furthermore, since $\lim_{x\to\eta_0+0}\frac{f'(x)}{f(x)}=+\infty$, $\lim_{x\to\eta_1-0}\frac{f'(x)}{f(x)}=-\infty$, we see that $f'(x)$ has exactly one zero η'_0 in the open interval (η_0, η_1). Then, for $\forall x\in(\eta_0, \eta'_0)\cup(\eta'_0, \eta_1)$ we can define the associated zero $\alpha(x)$ of $f(x)$ to be $\alpha(x)=\eta_0$ if $\eta_0<x<\eta'_0$, and $\alpha(x)=\eta_1$ if $\eta'_0<x<\eta_1$. In addition, we have $\operatorname{sgn}(h(x))=\operatorname{sgn}(x-\alpha(x))$. By applying $\operatorname{sgn}(h(x))=\operatorname{sgn}(x-\alpha(x))$ to Theorem 2 in Ref. 4), we have:

Theorem 9. Let $f(x)$ be given by the form in Eq. (1. 2).

Then, if $\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2 \leqq \beta \leqq -\frac{1}{2}\left(\theta+\frac{1}{2}\right)$, and if we take the real starting value in Eq. (3. 3) z_0 such that $f(z_0)f'(z_0)\neq 0$ and z_0 is neither less nor greater than all α_k, then we have

$z_n \downarrow \alpha(z_0)$ $(h(z_0)>0)$ and $z_n \uparrow \alpha(z_0)$ $(h(z_0)<0)$ $(n=0, 1, \cdots)$.

Also, by applying $\mathrm{sgn}(h(x))=\mathrm{sgn}(x-\zeta(x))$ to Theorem 1 in Ref. 5), we have:

Theorem 10. Let $f(x)$ be given by the form in Eq. (1. 2). Then, if $-\sqrt{b}<a\leqq 0$, and if we take the real starting value in Eq. (3. 4) w_0 such that $f(w_0)f'(w_0)\neq 0$ and w_0 is neither less nor greater than all α_k, then we have

$w_n \downarrow \alpha(w_0)$ $(h(w_0)>0)$ and $w_n \uparrow \alpha(w_0)$ $(h(w_0)<0)$ $(n=0, 1, \cdots)$.

Furthermore, we have two theorems analogous to Theorems 7 and 8. More precisely:

Theorem 11. Let $f(x)$ be given by the form in Eq. (1. 2). Then, if $\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2 \leqq \beta < -\frac{1}{2}\left(\theta+\frac{1}{2}\right)$, and if we take the real starting values x_0, y_0 and z_0 respectively in Eqs. (3. 1), (3. 2) and (3. 3) such that $x_0=y_0=z_0$, $f(x_0)f'(x_0)f''(x_0)\neq 0$ and x_0 is neither less nor greater than all α_k, then we have

$\alpha(x_0)<x_n<z_n<y_n$ $(h(x_0)>0)$ and
$\alpha(x_0)>x_n>z_n>y_n$ $(h(x_0)<0)$ $(n=1, 2, \cdots)$.

Theorem 12. Let $f(x)$ be given by the form in Eq. (1. 2).

Then, if $-\sqrt{b}<a<0$, and if we take the real starting values x_0, y_0 and w_0 respectively in Eqs. (3. 1), (3. 2) and (3. 4) such that $x_0=y_0=w_0$, $f(x_0)f'(x_0)f''(x_0)\neq 0$ and x_0 is neither less nor greater than all α_k, then we have

$\alpha(x_0)<x_n<w_n<y_n$ $(h(x_0)>0)$ and
$\alpha(x_0)>x_n>w_n>y_n$ $(h(x_0)<0)$ $(n=1, 2, \cdots)$.

By using Lemma 3, Theorems 2, 4, 9 and 10, we can prove Theorem 11 in a way similar to that of Theorem 7. Similarly, by using Lemma 4, Theorems 2, 4 and 10, we can prove Theorem 12.

4. Concluding Remarks

(a) In this paper, when we find the numerical solutions of the zeros of a polynomial with only real zeros or an entire function of genus 0 and 1 (and in certain cases of genus 2) with only real zeros by using the methods mentioned above, it is shown that with regard to the convergence speed, Ostrowski's method is the fastest, Halley's method is the slowest and our methods are intermediate.

(b) We consider the case where we find the numerical solutions of Eq. (1. 1) or Eq. (1. 2) by using the method (3. 3). Let us denote the iteration function of Eq. (2. 3) by $\Phi_3(x, \theta, \beta)$. Then if for any fixed $\theta\in\left(-\frac{3}{2}, -\frac{1}{2}\right)$, β_0 and β_1 are given by $\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2 \leqq \beta_0 < \beta_1 \leqq -\frac{1}{2}\left(\theta+\frac{1}{2}\right)$, then the following inequality holds.

$$\frac{\left(\theta+\frac{1}{2}\right)X+1}{\beta_1 X^2+\theta X+1} \leqq \frac{\left(\theta+\frac{1}{2}\right)X+1}{\beta_0 X^2+\theta X+1}.$$

Furthermore, if $\Phi_3(x, \theta, \beta_0)$ or $\Phi_3(x, \theta, \beta_1)$ is monotonically increasing function, then it can be seen that the method (3. 3) for $\beta=\beta_0$ is faster than that for $\beta=\beta_1$ under the same assumptions as those on Theorem 7 or Theorem 11. We will continue to consider the convergence speed in our methods.

(c) We consider the case where Eqs. (1. 1) and (1. 2) have complex zeros. Let ζ be any real zero of Eq. (1. 1), and $\bar{\zeta}_j$ also be a zero of Eq. (1. 1) if ζ_j is any complex zero of Eq. (1. 1).

Then, if $h^2\sum_{k=1}^{\gamma}\frac{1}{(x-\zeta_k)^2}-\frac{h^2}{(x-\zeta)^2}>0$, it can be seen that both Theorems 5 and 6 hold[see Remarks (a) in Ref. 4)].

Furthermore, if $\sum_{k=1}^{\gamma}\frac{1}{(x-\zeta_k)^3}\Big/\left\{\sum_{k=1}^{\gamma}\frac{1}{(x-\zeta_k)^2}\right\}^{3/2}<1$, then we have both Theorems 7 and 8.

Similarly, let α be any real zero of Eq. (1. 2) and $\bar{\alpha}_j$ also be a zero of Eq. (1. 2) if α_j is any complex zero of Eq. (1. 2).

Then, if $h^2\left\{\frac{p}{x^2}+2c+\sum_{k=1}^{\infty}\frac{1}{(x-\alpha_k)^2}\right\}-\frac{h^2}{(x-\alpha)^2}>0$ and $\left\{p+x^3\sum_{k=1}^{\infty}\frac{1}{(x-\alpha_k)^3}\right\}\Big/\left\{p+2cx^2+x^2\sum_{k=1}^{\infty}\frac{1}{(x-\alpha_k)^2}\right\}^{3/2}<1$, then we have both Theorems 11 and 12.

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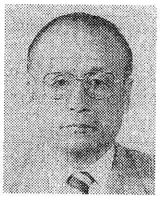

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