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# Quartic Interpolation on Triangles

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A new method is presented for constructing a surface to interpolate the given boundary curves and cross-boundary slopes on the sides of triangles. On each triangle, the constructed surface patch is a quartic polynomial, which approximates a function with a polynomial precision of degree four or less. Six test functions proposed by Franke are used to test the new method.

## 1. Introduction

In the fields of CAD and free-form surface modeling, the construction of surfaces plays an important role. Usually, the shapes of surfaces are defined by several kinds of surface patches which are mostly four-sided patches and triangular patches. Constructing a surface patch to interpolate the boundary curves and cross-boundary slopes on non-four sided region has been studied[1)~9)]. We also presented a method for constructing a triangular patch, which approximated a function with a polynomial precision of degree three or less[10)].

The purpose of this paper is to present a new method for constructing an interpolant with higher precision to the given boundary curves and cross-boundary slopes in the three sides of a triangle. On the triangle the constructed surface patch is a quartic polynomial, which approximates a function with a polynomial precision of degree four or less. At first we describe the new method. Then we compare it with three other methods[2),3),10)] using six test functions proposed by Franke[11)].

For simplicity, the interpolation method in this paper is described by bivariate surface in a Cartesian $(x, y)$ space. However, the interpolation method is immediately applicable to the parametric representation.

## 2. Construction of Surface Patch on a Triangle

The problem studied in this paper is the construction of triangular patch which interpolates the boundary curves and cross-boundary slopes of a triangle. When three surfaces are met together, a surface patch is required to smoothly connect the three surfaces. In such cases, the construction of the triangular patch, if possible, is very useful. For example, in designing a $C^1$ continuous shape as shown in **Fig. 1**, the six four-sided patches, whose models, namely planes can be easily given, are designed at first, then some pairs of these planes are connected with four-sided patches in $C^1$ continuity. After the construction of these patches, the four triangular regions still remain unpatched, however, the boundary values and cross-boundary slopes of the triangles are determined by each three four-sided patches which have been already constructed. Because, if the triangles are interpolated in $C^1$ continuity, the boundary values and cross-boundary slopes of the triangles must be equal to those of the three patches around them. Therefore, in this example, it is possible to construct a triangular patch having $C^1$ continuity with three four-sided patches that are met together.

Due to such background, we assume that the boundary values and cross-boundary slopes of a triangle are given when the triangle is interpolated.

Let $T$ be an acute triangle with vertices $p_\alpha = (x_\alpha, y_\alpha)$, $\alpha = i, j, k$, as shown in **Fig. 2**, and $e_\alpha$ denote the opposite side of $p_\alpha$. Let $n_\alpha$ denote the unit out normal direction on $e_\alpha$, then the derivative along $n_\alpha$, $\alpha = i, j, k$, is given by Eq. (1).

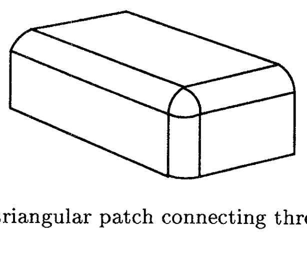

**Fig. 1** A triangular patch connecting three patches.

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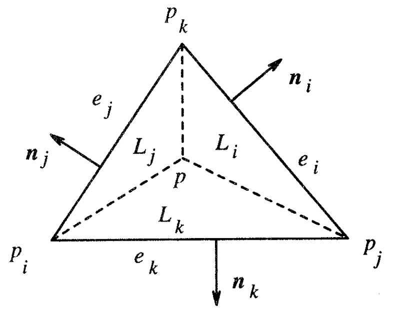


**Fig. 2** A triangle $T$.

$$\frac{\partial}{\partial n_i} = \left[(y_k - y_j)\frac{\partial}{\partial x} + (x_j - x_k)\frac{\partial}{\partial y}\right] / d_{jk},$$
$$\frac{\partial}{\partial n_j} = \left[(y_i - y_k)\frac{\partial}{\partial x} + (x_k - x_i)\frac{\partial}{\partial y}\right] / d_{ki}, \quad (1)$$
$$\frac{\partial}{\partial n_k} = \left[(y_j - y_i)\frac{\partial}{\partial x} + (x_i - x_j)\frac{\partial}{\partial y}\right] / d_{ij},$$

where $d_{rs}$ denotes the distance from $p_r$ to $p_s$, $(r, s = i, j, k)$.

Let $p = (x, y)$ be any point in $T$, and $(L_i, L_j, L_k)$ denote the barycentric coordinates with respect to $T$ [10]. Thus $(L_i, L_j, L_k)$ can be written as Eq. (2).

$$\begin{aligned} L_i &= L_i(p) \\ &= (x_j y_k - x_k y_j + (y_j - y_k)x \\ &\quad + (x_k - x_j)y)/(2S) \\ L_j &= L_j(p) \\ &= (x_k y_i - x_i y_k + (y_k - y_i)x \\ &\quad + (x_i - x_k)y)/(2S) \qquad (2) \\ L_k &= L_k(p) \\ &= (x_i y_j - x_j y_i + (y_i - y_j)x \\ &\quad + (x_j - x_i)y)/(2S). \end{aligned}$$

where, $S$ denotes the area of $T$.

Figure 2 shows that the barycentric coordinates have the following properties:

1. $L_i + L_j + L_k = 1$.
2. $L_\alpha$ is a linear function with value one at $p_\alpha$ and zero along $e_\alpha$, $\alpha = i, j, k$.

The perpendicular projections of $(x, y)$ on the three sides of $T$ are denoted by $p_{jk} = (x_{jk}, y_{jk})$, $p_{ki} = (x_{ki}, y_{ki})$ and $p_{ij} = (x_{ij}, y_{ij})$ respectively, as shown in **Fig. 3**.

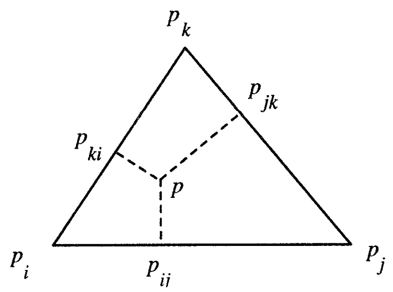


**Fig. 3** Three projection points.

Let the given function values and first derivatives at $p_\alpha$ be $F_\alpha$, $(F_x)_\alpha$ and $(F_y)_\alpha$ $(\alpha = i, j, k)$, and the given function values and cross-boundary slopes at $p_{jk}$, $p_{ki}$ and $p_{ij}$ be $F_{jk}$, $\partial F_{jk}/\partial n_i$, $F_{ki}$, $\partial F_{ki}/\partial n_j$, $F_{ij}$ and $\partial F_{ij}/\partial n_k$ respectively. Thus there are 15 interpolation conditions on $T$. The surface patch satisfying the 15 interpolation conditions is defined by Eq. (3).

$$\begin{aligned} PF(x, y) &= G(p) + A_{jk}L_j^2L_k^2 + A_{ki}L_k^2L_i^2 \\ &\quad + A_{ij}L_i^2L_j^2 + B_{jk}L_i^2L_jL_k \\ &\quad + B_{ki}L_iL_j^2L_k + B_{ij}L_iL_jL_k^2. \qquad (3) \end{aligned}$$

where,

$$G(p) = \sum_{\alpha=i,j,k} \{(3 - 2L_\alpha)F_\alpha + (x - x_\alpha)(F_x)_\alpha + (y - y_\alpha)(F_y)_\alpha\}L_\alpha^2,$$
$$A_{jk} = \{F_{jk} - G(p_{jk})\}/L_j^2(p_{jk})L_k^2(p_{jk}),$$
$$A_{ki} = \{F_{ki} - G(p_{ki})\}/L_k^2(p_{ki})L_i^2(p_{ki}),$$
$$A_{ij} = \{F_{ij} - G(p_{ij})\}/L_i^2(p_{ij})L_j^2(p_{ij}),$$
$$B_{jk} = \frac{C_{11}C_{22}C_{34} - C_{22}C_{31}C_{14} + C_{31}C_{12}C_{24}}{C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31}},$$
$$B_{ki} = \frac{C_{34}C_{12}C_{23} - C_{12}C_{33}C_{24} + C_{33}C_{22}C_{14}}{C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31}},$$
$$B_{ij} = \frac{C_{11}C_{24}C_{33} - C_{11}C_{23}C_{34} + C_{23}C_{31}C_{14}}{C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31}},$$
$$C_{11} = L_j^2(p_{jk})L_k(p_{jk})\frac{\partial L_i}{\partial n_i},$$
$$C_{12} = L_j(p_{jk})L_k^2(p_{jk})\frac{\partial L_i}{\partial n_i},$$
$$C_{14} = \frac{\partial F_{jk}}{\partial n_i} - \frac{\partial G}{\partial n_i}(p_{jk})$$

$$- 2A_{jk}L_j(p_{jk})L_k(p_{jk})\left[L_j(p_{jk})\frac{\partial L_k}{\partial n_i} + L_k(p_{jk})\frac{\partial L_j}{\partial n_i}\right],$$

$$C_{22} = L_i(p_{ki})L_k^2(p_{ki})\frac{\partial L_j}{\partial n_j},$$

$$C_{23} = L_i^2(p_{ki})L_k(p_{ki})\frac{\partial L_j}{\partial n_j},$$

$$C_{24} = \frac{\partial F_{ki}}{\partial n_j} - \frac{\partial G}{\partial n_j}(p_{ki}) - 2A_{ki}L_i(p_{ki})L_k(p_{ki})\left[L_i(p_{ki})\frac{\partial L_k}{\partial n_j} + L_k(p_{ki})\frac{\partial L_i}{\partial n_j}\right],$$

$$C_{31} = L_i(p_{ij})L_j^2(p_{ij})\frac{\partial L_k}{\partial n_k},$$

$$C_{33} = L_i^2(p_{ij})L_j(p_{ij})\frac{\partial L_k}{\partial n_k},$$

$$C_{34} = \frac{\partial F_{ij}}{\partial n_k} - \frac{\partial G}{\partial n_k}(p_{ij}) - 2A_{ij}L_i(p_{ij})L_j(p_{ij})\left[L_i(p_{ij})\frac{\partial L_j}{\partial n_k} + L_j(p_{ij})\frac{\partial L_i}{\partial n_k}\right]. \quad (4)$$

About $PF(x,y)$ defined by Eq. (3), we have the following theorem.

**Theorem 1.** $PF(x,y)$ defined by Eq. (3) satisfies the given boundary curves and cross-boundary slopes on the three sides of $T$.

Proof: By symmetry, it is sufficient to prove that on the side $e_i$, $PF(x,y)$ satisfies the given boundary curves and cross-boundary slopes.

The projection point $p_{jk}$ of $(x,y)$ on $e_i$ is considered. The property 2 of the barycentric coordinates shows that at $p_{jk}$ Eq. (5) holds.

$$L_i(p_{jk}) = 0. \quad (5)$$

Thus, Eqs. (6) and (7) hold.

$$\begin{aligned} PF(x_{jk}, y_{jk}) &= G(p_{jk}) + A_{jk}L_j^2(p_{jk})L_k^2(p_{jk}) \\ &= G(p_{jk}) + [F_{jk} - G(p_{jk})] \\ &= F_{jk}. \end{aligned} \quad (6)$$

$$\begin{aligned} \frac{\partial PF}{\partial n_i}(p_{jk}) &= \frac{\partial G}{\partial n_i}(p_{jk}) + 2A_{jk}L_j(p_{jk})L_k(p_{jk}) \\ &\quad \times \left[L_j(p_{jk})\frac{\partial L_k}{\partial n_i} + L_k(p_{jk})\frac{\partial L_j}{\partial n_i}\right] \\ &\quad + L_j(p_{jk})L_k(p_{jk}) \\ &\quad \times [B_{ki}L_j(p_{jk}) + B_{ij}L_k(p_{jk})]\frac{\partial L_i}{\partial n_i}. \end{aligned} \quad (7)$$

From Eq. (4), it follows Eqs. (8)–(11).

$$\frac{\partial G}{\partial n_i}(p_{jk}) + 2A_{jk}L_j(p_{jk})L_k(p_{jk}) \times \left[L_j(p_{jk})\frac{\partial L_k}{\partial n_i} + L_k(p_{jk})\frac{\partial L_j}{\partial n_i}\right] = \frac{\partial F_{jk}}{\partial n_i} - C_{14}. \quad (8)$$

$$B_{ij} = \frac{C_{14} - C_{11}B_{ki}}{C_{12}}. \quad (9)$$

$$C_{12}L_j(p_{jk}) - C_{11}L_k(p_{jk}) = 0. \quad (10)$$

$$\begin{aligned} &L_j(p_{jk})L_k(p_{jk}) \times [B_{ki}L_j(p_{jk}) + B_{ij}L_k(p_{jk})]\frac{\partial L_i}{\partial n_i} \\ &= L_j(p_{jk})L_k(p_{jk})\left[B_{ki}L_j(p_{jk}) + \frac{C_{14} - C_{11}B_{ki}}{C_{12}}L_k(p_{jk})\right]\frac{\partial L_i}{\partial n_i} \\ &= L_j(p_{jk})L_k^2(p_{jk})\frac{C_{14}}{C_{12}}\frac{\partial L_i}{\partial n_i} \\ &= C_{14}. \end{aligned} \quad (11)$$

Substituting Eqs. (8) and (11) into Eq. (7), we obtain Eq. (12).

$$\frac{\partial PF}{\partial n_i}(p_{jk}) = \frac{\partial F_{jk}}{\partial n_i}. \quad (12)$$

Equations (6) and (12) show that at $p_{jk}$, $PF(x,y)$ satisfies the given function value and cross-boundary slope. Since $p_{jk}$ is any point on $e_i$, $PF(x,y)$ satisfies the given boundary curves and cross-boundary slopes on $e_i$. ||

If the given interpolation conditions are taken from a quartic polynomial, then there are unique $A_{jk}$, $A_{ki}$, $A_{ij}$, $B_{jk}$, $B_{ki}$ and $B_{ij}$ to make $PF(x,y)$ satisfy the given interpolation conditions, therefore it follows the following theorem.

**Theorem 2.** The polynomial precision set of $PF(x,y)$ defined by Eq. (3) includes all polynomials of degree four or less.

If the triangle has an obtuse angle, the three projections of $(x,y)$ on the three sides of $T$ are determined by the method in paper[10].

## 3. Experiments

Franke has proposed six bivariate functions for use in comparing various bivariate interpolation methods. The six functions are:

$f_1(x,y)$

$$= 3.9e^{\{-0.25(9x-2)^2-0.25(9y-2)^2\}}$$
$$+3.9e^{\{-(9x+1)^2/49-(9y+1)/10\}}$$
$$+2.6e^{\{-0.25(9x-7)^2-0.25(9y-3)^2\}}$$
$$-1.04e^{\{-(9x-4)^2-(9y-7)^2\}}, \quad (13)$$

$$f_2(x,y) = 5.2e^{\{18y-18x\}}/(9e^{\{18y-18x\}}+9), \quad (14)$$

$$f_3(x,y) = 5.2\{1.25+\cos(5.4y)\} /\{6+6(3x-1)^2\}, \quad (15)$$

$$f_4(x,y) = 5.2e^{\{-81((x-0.5)^2+(y-0.5)^2)/16\}}/3, \quad (16)$$

$$f_5(x,y) = 5.2e^{\{-81((x-0.5)^2+(y-0.5)^2)/4\}}/3, \quad (17)$$

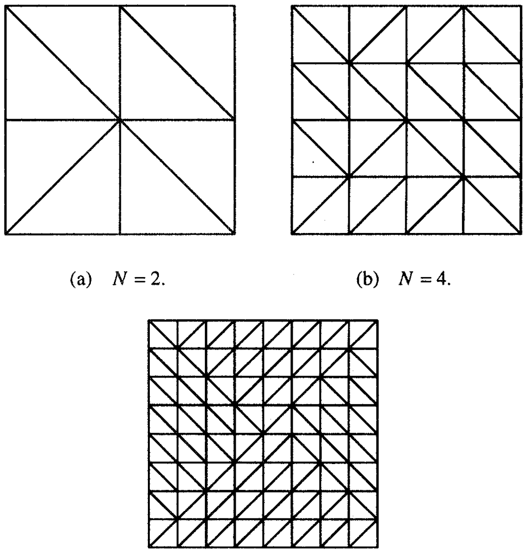
$$f_6(x,y) = 5.2sqrt\{64-81((x-0.5)^2 +(y-0.5)^2)\}/9-2.6. \quad (18)$$

In this section the six functions are used to compare the new method with the Gregory's method [2], the Nielson's method [3] and the method [10]. For brevity, the surface produced by the method [10] is called surface-1, the one by the new method is called surface-2.

Three data sets are used to produce triangles for comparing the four methods, which are defined by Eq. (19), for $N = 2, 4, 8$.

$$\left(\frac{i}{N}, \frac{j}{N}\right) \quad i,j = 0,1,\cdots,N \quad (19)$$

Three triangulations of the three data sets are produced automatically by using max-min criterion proposed by Lawson [12], which are given in **Fig. 4**.



(a) $N=2$. (b) $N=4$. (c) $N=8$.

**Fig. 4** Three triangulations.

The interpolation conditions for comparisons are boundary curves and cross-boundary slopes on the sides of the triangles in Fig. 4, which are taken from $f_1(x,y)$ to $f_6(x,y)$ above. For these functions, the maximum absolute errors of the four methods are given in **Tables 1–3**.

In each table, G-error, N-error, error-1 and error-2 are defined as follows for each $i$ ($i = 1,\cdots,6$):

G-error
$= \max\{|\text{Gregory-surface} - f_i(x,y)|\}$,
N-error
$= \max\{|\text{Nielson-surface} - f_i(x,y)|\}$,
error-1 $= \max\{|\text{surface-1} - f_i(x,y)|\}$,
error-2 $= \max\{|\text{surface-2} - f_i(x,y)|\}$.

As an example, four error surfaces are given in **Fig. 5**, which are produced by interpolating $f_1(x,y)$ based on the triangulation in (c) of Fig. 4. Tables 1–3 and Fig. 5 show that in the sense of approximation, the new method is better than the other three ones.

## 4. Conclusions

This paper presents a method for constructing a curved triangular surface patch. The constructed surface patch is a quartic polynomial, and satisfies the given boundary curves and cross-boundary slopes on the three sides of a triangle. The polynomial precision set of the new method includes all polynomials of degree four or less. The experiments also show that the surface patches produced by the new method have better precision.

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**Table 1** Maximum absolute errors in the case of $N = 2$.

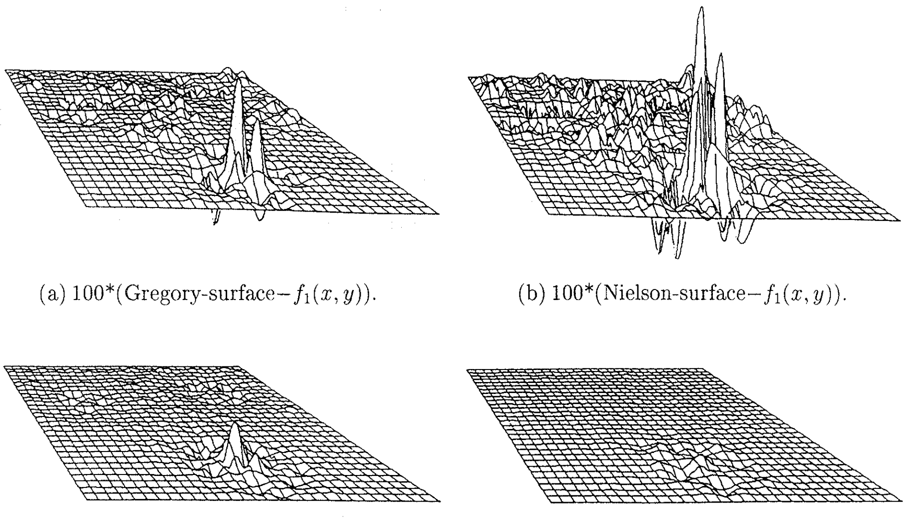
| | $f_1(x,y)$ | $f_2(x,y)$ | $f_3(x,y)$ | $f_4(x,y)$ | $f_5(x,y)$ | $f_6(x,y)$ |
|---|---|---|---|---|---|---|
| G-error | 7.788e-2 | 2.599e-2 | 1.204e-2 | 3.465e-2 | 1.205e-2 | 9.766e-4 |
| N-error | 1.189e-1 | 1.742e-2 | 1.699e-2 | 5.143e-2 | 2.809e-2 | 2.658e-3 |
| error-1 | 3.429e-2 | 8.892e-3 | 4.961e-3 | 1.031e-3 | 6.296e-3 | 6.254e-4 |
| error-2 | 2.760e-2 | 1.550e-2 | 2.588e-3 | 5.652e-4 | 4.814e-3 | 2.949e-4 |

**Table 2** Maximum absolute errors in the case of $N = 4$.

| | $f_1(x,y)$ | $f_2(x,y)$ | $f_3(x,y)$ | $f_4(x,y)$ | $f_5(x,y)$ | $f_6(x,y)$ |
|---|---|---|---|---|---|---|
| G-error | 2.046e-2 | 4.409e-3 | 1.006e-3 | 3.536e-4 | 3.480e-3 | 1.832e-4 |
| N-error | 1.184e-2 | 3.987e-3 | 1.788e-3 | 7.145e-4 | 5.145e-3 | 4.720e-4 |
| error-1 | 7.187e-3 | 3.276e-3 | 3.996e-4 | 1.403e-4 | 1.022e-3 | 9.307e-5 |
| error-2 | 5.775e-3 | 2.438e-3 | 1.527e-4 | 1.689e-5 | 5.700e-4 | 2.637e-5 |

**Table 3** Maximum absolute errors in the case of $N = 8$.

| | $f_1(x,y)$ | $f_2(x,y)$ | $f_3(x,y)$ | $f_4(x,y)$ | $f_5(x,y)$ | $f_6(x,y)$ |
|---|---|---|---|---|---|---|
| G-error | 2.964e-3 | 3.648e-4 | 8.681e-5 | 2.336e-5 | 3.293e-4 | 2.674e-5 |
| N-error | 5.479e-3 | 9.025e-4 | 1.864e-4 | 5.275e-5 | 6.951e-4 | 6.221e-5 |
| error-1 | 1.298e-3 | 4.137e-4 | 3.587e-5 | 9.532e-6 | 1.231e-4 | 1.063e-5 |
| error-2 | 3.260e-4 | 7.816e-5 | 5.136e-6 | 6.097e-7 | 1.576e-5 | 1.664e-6 |



(a) 100*(Gregory-surface$-f_1(x,y)$). (b) 100*(Nielson-surface$-f_1(x,y)$).

(c) 100*(surface-1$-f_1(x,y)$). (d) 100*(surface-2$-f_1(x,y)$).

**Fig. 5** (a), (b), (c) and (d) are error surfaces produced by interpolating $f_1(x,y)$ on the boundaries of the triangles in (c) of Fig. 4 by the four methods, respectively.

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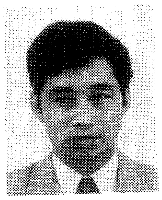

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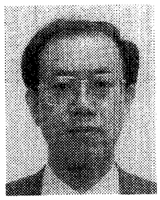

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