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$\bar{K}_{p,2q}$ - factorization algorithm of symmetric complete tripartite digraphs

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Let $K^*_{n_1,n_2,n_3}$ denote *the symmetric complete tripartite digraph* with partite sets V_1, V_2, V_3 of n_1, n_2, n_3 vertices each, and let $\bar{K}_{p,2q}$ denote *the evenly partite directed bigraph* from p start-vertices to $2q$ end-vertices such that the start-vertices are in V_i and q end-vertices are in V_j and q end-vertices are in V_k with $\{i, j, k\} = \{1, 2, 3\}$. A spanning subgraph F of $K^*_{n_1,n_2,n_3}$ is called a $\bar{K}_{p,2q}$ - *factor* if each component of F is $\bar{K}_{p,2q}$. If $K^*_{n_1,n_2,n_3}$ is expressed as an arc-disjoint sum of $\bar{K}_{p,2q}$ - factors, then this sum is called a $\bar{K}_{p,2q}$ - *factorization* of $K^*_{n_1,n_2,n_3}$.

Theorem 1. If $K^*_{n_1,n_2,n_3}$ has a $\bar{K}_{p,2q}$ - factorization, then (i) $n_1 = n_2 = n_3 \equiv 0 \pmod p$ for $p = q$, (ii) $n_1 = n_2 = n_3 \equiv 0 \pmod{dp'q'(p' + 2q')}$ for $p \neq q$ and p' odd, (iii) $n_1 = n_2 = n_3 \equiv 0 \pmod{dp'q'(p' + 2q')/2}$ for $p \neq q$ and p' even, where $(p, q) = d$, $p = dp'$, $q = dq'$, $(p', q') = 1$.

Theorem 2. If $K^*_{n,n,n}$ has a $\bar{K}_{p,2q}$ - factorization, then $K^*_{sn,sn,sn}$ has a $\bar{K}_{p,2q}$ - factorization.

Theorem 3. When $n \equiv 0 \pmod p$, $K^*_{n,n,n}$ has a $\bar{K}_{p,2p}$ - factorization.
Proof.(Algorithm 1) Put $n = sp$. When $s = 1$, let $V_1 = \{1, 2, ..., p\}$, $V_2 = \{1', 2', ..., p'\}$, and $V_3 = \{1'', 2'', ..., p''\}$. Construct $\bar{K}_{p,2p}$ - factors $F_1 = (V_1; V_2, V_3)$, $F_2 = (V_2; V_1, V_3)$, $F_3 = (V_3; V_1, V_2)$. Then they comprise a $\bar{K}_{p,2p}$ - factorization of $K^*_{p,p,p}$. Applying Theorem 2, $K^*_{n,n,n}$ has a $\bar{K}_{p,2p}$ - factorization.

Theorem 4. Let $(p, q) = d$, $p = dp'$, $q = dq'$, $(p', q') = 1$ for $p \neq q$. When $n \equiv 0 \pmod{dp'q'(p' + 2q')}$ and p' odd, $K^*_{n,n,n}$ has a $\bar{K}_{p,2q}$ - factorization.
Proof.(Algorithm 2) Put $n = sdp'q'(p' + 2q')$ and $N = dp'q'(p' + 2q')$. When $s = 1$, let $V_1 = \{1, 2, ..., N\}$, $V_2 = \{1', 2', ..., N'\}$, and $V_3 = \{1'', 2'', ..., N''\}$. Construct $(p' + 2q')^2$ $\bar{K}_{p,2q}$ - factors F_{ij} $(i = 1, 2, ..., p' + 2q'; j = 1, 2, ..., p' + 2q')$ as following:
F_{ij}= { $((A+1, ..., A+p); (B+f+1, ..., B+f+q), (C+g+1, ..., C+g+q))$
$((A+p+1, ..., A+2p); (B+f+q+1, ..., B+f+2q), (C+g+q+1, ..., C+g+2q))$
...
$((A+(p'q'-1)p+1, ..., A+p'q'p); (B+f+(p'q'-1)q+1, ..., B+f+p'q'q), (C+g+(p'q'-1)q+1, ..., C+g+p'q'q))$
$((B+1, ..., B+p); (C+f+1, ..., C+f+q), (A+g+1, ..., A+g+q))$
$((B+p+1, ..., B+2p); (C+f+q+1, ..., C+f+2q), (A+g+q+1, ..., A+g+2q))$
...
$((B+(p'q'-1)p+1, ..., B+p'q'p); (C+f+(p'q'-1)q+1, ..., C+f+p'q'q), (A+g+(p'q'-1)q+1, ..., A+g+p'q'q))$
$((C+1, ..., C+p); (A+f+1, ..., A+f+q), (B+g+1, ..., B+g+q))$
$((C+p+1, ..., C+2p); (A+f+q+1, ..., A+f+2q), (B+g+q+1, ..., B+g+2q))$
...
$((C+(p'q'-1)p+1, ..., C+p'q'p); (A+f+(p'q'-1)q+1, ..., A+f+p'q'q), (B+g+(p'q'-1)q+1, ..., B+g+p'q'q))$ },
where $f = p'dp'q'$, $g = (p'+q')dp'q'$, $A = (i-1)dp'q'$, $B = (j-1)dp'q'$, $C = (i+j-2)dp'q'$, and the additions are taken modulo N with residues $1, 2, ..., N$, and $(A+x)$, $(B+x)$, $(C+x)$ means $(A+x)$, $(B+x)'$, $(C+x)''$, respectively. Then they comprise a $\bar{K}_{p,2q}$ - factorization of $K^*_{N,N,N}$. Applying Theorem 2, $K^*_{n,n,n}$ has a $\bar{K}_{p,2q}$ - factorization.

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Theorem 5. Let $(p,q)=d$, $p=dp'$, $q=dq'$, $(p',q')=1$ for $p\neq q$. When $n\equiv 0 \pmod{dp'q'(p'+2q')/2}$ and p' even, $K^*_{n,n,n}$ has a $\bar{K}_{p,2q}$ - factorization.

Proof.(Algorithm 3) Put $n=sdp'q'(p'+2q')/2$ and $N=dp'q'(p'+2q')/2$. When $s=1$, let $V_1=\{1,2,...,N\}$, $V_2=\{1',2',...,N'\}$, and $V_3=\{1'',2'',...,N''\}$. Construct $(p'+2q')^2/2$ $\bar{K}_{p,2q}$ - factors $F^{(1)}_{ij}$, $F^{(2)}_{ij}$ $(i=1,2,...,(p'+2q')/2; j=1,2,...,(p'+2q')/2)$ as following:

$F^{(1)}_{ij}=\{$ $((A+1,...,A+p);(B+f+1,...,B+f+q),(C+g+1,...,C+g+q))$
$((A+p+1,...,A+2p);(B+f+q+1,...,B+f+2q),(C+g+q+1,...,C+g+2q))$
...
$((A+(p'q'/2-1)p+1,...,A+(p'q'/2)p);(B+f+(p'q'/2-1)q+1,...,B+f+(p'q'/2)q),(C+g+(p'q'/2-1)q+1,...,C+g+(p'q'/2)q))$
$((B+1,...,B+p);(C+f+1,...,C+f+q),(A+g+1,...,A+g+q))$
$((B+p+1,...,B+2p);(C+f+q+1,...,C+f+2q),(A+g+q+1,...,A+g+2q))$
...
$((B+(p'q'/2-1)p+1,...,B+(p'q'/2)p);(C+f+(p'q'/2-1)q+1,...,C+f+(p'q'/2)q),(A+g+(p'q'/2-1)q+1,...,A+g+(p'q'/2)q))$
$((C+1,...,C+p);(A+f+1,...,A+f+q),(B+g+1,...,B+g+q))$
$((C+p+1,...,C+2p);(A+f+q+1,...,A+f+2q),(B+g+q+1,...,B+g+2q))$
...
$((C+(p'q'/2-1)p+1,...,C+(p'q'/2)p);(A+f+(p'q'/2-1)q+1,...,A+f+(p'q'/2)q),(B+g+(p'q'/2-1)q+1,...,B+g+(p'q'/2)q))$ $\}$,

$F^{(2)}_{ij}=\{$ $((A+1,...,A+p);(C+f+1,...,C+f+q),(B+g+1,...,B+g+q))$
$((A+p+1,...,A+2p);(C+f+q+1,...,C+f+2q),(B+g+q+1,...,B+g+2q))$
...
$((A+(p'q'/2-1)p+1,...,A+(p'q'/2)p);(C+f+(p'q'/2-1)q+1,...,C+f+(p'q'/2)q),(B+g+(p'q'/2-1)q+1,...,B+g+(p'q'/2)q))$
$((B+1,...,B+p);(A+f+1,...,A+f+q),(C+g+1,...,C+g+q))$
$((B+p+1,...,B+2p);(A+f+q+1,...,A+f+2q),(C+g+q+1,...,C+g+2q))$
...
$((B+(p'q'/2-1)p+1,...,B+(p'q'/2)p);(A+f+(p'q'/2-1)q+1,...,A+f+(p'q'/2)q),(C+g+(p'q'/2-1)q+1,...,C+g+(p'q'/2)q))$
$((C+1,...,C+p);(B+f+1,...,B+f+q),(A+g+1,...,A+g+q))$
$((C+p+1,...,C+2p);(B+f+q+1,...,B+f+2q),(A+g+q+1,...,A+g+2q))$
...
$((C+(p'q'/2-1)p+1,...,C+(p'q'/2)p);(B+f+(p'q'/2-1)q+1,...,B+f+(p'q'/2)q),(A+g+(p'q'/2-1)q+1,...,A+g+(p'q'/2)q))$ $\}$,

where $f=(p'/2)dp'q'$, $g=((p'+q')/2)dp'q'$, $A=(i-1)dp'q'$, $B=(j-1)dp'q'$, $C=(i+j-2)dp'q'$, and the additions are taken modulo N with residues $1,2,...,N$, and $(A+x)$, $(B+x)$, $(C+x)$ means $(A+x)$, $(B+x)'$, $(C+x)''$, respectively. Then they comprise a $\bar{K}_{p,2q}$ - factorization of $K^*_{N,N,N}$. Applying Theorem 2, $K^*_{n,n,n}$ has a $\bar{K}_{p,2q}$ - factorization.

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