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# A NEW COMPLEXITY BOUND FOR THE LEAST–SQUARES PROBLEM

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## 1. INTRODUCTION

Let $(x_i, y_i)$, $i = 1 \sim n$, be measured data of $n$ pairs, where $x_i \neq x_j$, if $i \neq j$. We consider the following least–squares problem with the polynomial regression of degree $m$,

$$||y - f(x)||_2 = min \tag{1}$$

where

$$y = (y_1, y_2, ..., y_n)^T, \qquad f(x) = (f(x_1), f(x_2), ..., f(x_n))^T,$$

$$f(x_i) = a_0 + a_1 x_i + \cdots + a_m x_i^m, \qquad i = 1 \sim n,$$

$m + 1 \leq n$. Solving (1) by using some traditional methods, such as Householder transformation or QR decomposition etc., requires $O(n^2 m)$ arithmetic operations. In [7], we have showed a fast algorithm which needs only $O(nm)$ arithmetic operations. This paper further proves that the arithmetic complexity of the least–squares problem (1) does not exceed $O(nlog_2^2 m)$; this result generalizes the complexity results showed by H.T.Kung for the fast polynomial interpolation [5], D.Bini and V.Pan [2], J.F.Canny, E.Kaltofen and Y.Lakshman [4] for the solution of the Vandermonde linear system.

## 2. ALGORITHM

Obviously, (1) can be expressed in the following Vandermonde form

$$||b - V^T x||_2 = min, \tag{2}$$

where

$$b = y \qquad x = (a_0, a_1, \cdots, a_m)^T$$

$$V^T = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^m \\ 1 & x_2 & x_2^2 & \dots & x_2^m \\ & \dots & \dots & & \\ 1 & x_n & x_n^2 & \dots & x_n^m \end{pmatrix}$$

And the normal equations of (2) become

$$VV^T x = Vb. \tag{3}$$

The main results of this paper are the following:

LEMMA 1. Let $A = (v_j^{i-1})_{n\times n}$ be an $n$-order Vandermonde matrix, then the arithmetic operational complexity of computing $AA^T$ is not more than $O(nlog_2^2 n)$.

LEMMA 2. The arithmetic operational complexity of computing the $VV^T$ and $Vb$ in (3) is not more than $O(nlog_2^2 m)$.

THEOREM 1. The arithmetic operational complexity of solving the least square problem (2) is not more than $O(nlog_2^2 m)$.

Proof: From the Lemma 1 and 2, we know that $VV^T$ and $Vb$ can be computed in $O(nlog_2^2 m)$ arithmetic operations. But $VV^T$ is a Hankel matrix, so we can change (3) to a Toeplitz linear system equations by reversing the order of elements of vector $x$ such as

$$(a_0, a_1, \cdots, a_m)^T \rightarrow (a_m, a_{m-1}, \cdots, a_0)^T,$$

then it can be solved in $O(mlog_2^2 m)$ arithmetic operations [3]. Therefore, the all number of arithmetic operations for solving the least square problem (2), is not more than

$$Max(O(nlog_2^2 m), \quad O(mlog_2^2 m)) = O(nlog_2^2 m).$$

Note: Obviously, when $n = m + 1$, (2) becomes Lagrange's interpolation problem. It is to say that the complexity result showed by H.T.Kung for solving Lagrange's interpolation [5], is a special case of the Theorem 1.

## 3. CONCLUSIONS

In this paper, we proved that the arithmetic operational complexity for solving the least square problem of $m$ degree polynomial regression with $n$ measured values ($n \geq m + 1$), is not more than $O(nlog_2^2 m)$. This result also generalized the complexity bound of Lagrange's interpolation showed by H.T.Kung, and the complexity bound of the Vandermonde linear system showed in [2] and [4].

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