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S_6−FACTORIZATION ALGORITHM OF COMPLETE BIPARTITE GRAPHS

Kazuhiko USHIO
Department of Industrial Engineering
Kinki University, 577 Osaka, Japan

1. Introduction

Let S_6 be a ***star*** on 6 vertices and $K_{m,n}$ be a ***complete bipartite graph*** with partite sets V_1 and V_2 of m and n vertices each. A spanning subgraph F of $K_{m,n}$ is called an S_6-***factor*** if each component of F is isomorphic to S_6. If $K_{m,n}$ is expressed as an edge-disjoint sum of S_6-factors, then this sum is called an S_6-***factorization*** of $K_{m,n}$.

2. S_6-factorization of $K_{m,n}$

Notation 1. **r,t,b** : number of S_6-factors, number of S_6-components of each S_6-factor, and total number of S_6-components, respectively, in an S_6-factorization of $K_{m,n}$.

t_1 (t_2) : number of components whose centers are in V_1 (V_2), respectively, among t S_6-components of each S_6-factor.

$r_1(u)$ ($r_2(v)$) : number of components whose centers are all u (v) for any u (v) in V_1 (V_2), respectively, among b S_6-components.

Lemma 1. If $K_{m,n}$ has an S_6-factorization then (i) $b=mn/5$, (ii) $t=(m+n)/6$, (iii) $r=6mn/5(m+n)$, (iv) $t_1=(5n-m)/24$, (v) $t_2=(5m-n)/24$, (vi) $r_1=(5n-m)n/20(m+n)$, (vii) $r_2=(5m-n)m/20(m+n)$, (viii) $m \leqq 5n$ and (ix) $n \leqq 5m$.

Lemma 2. If $K_{m,n}$ has an S_6-factorization, then $K_{sm,sn}$ has an S_6-factorization for every positive integer s.

3. S_6-factorization algorithm of $K_{m,n}$

Case (1) m=5n: From Lemma 2, $K_{5n,n}$ has an S_6-factorization since $K_{5,1}$ is just S_6.

Case (2) n=5m: Obviously, $K_{m,5m}$ has an S_6-factorization.

Case (3) m<5n and n<5m: Let $x=(5n-m)/24$ and $y=(5m-n)/24$. And let $(x,4y)=d$, $x=dp$, $4y=dq$, where $(p,q)=1$.

Lemma 3. Let $(p,q)=1$ and s be an integer. Then

(I) when $q \equiv 1,2,3,4 \pmod 5$,

$m=5(p+q)(5p+q)s, n=(25p+q)(5p+q)s$ ($5p+q \equiv 1,3 \pmod 4$)

or $m=5(p+q)(5p+q)s/2, n=(25p+q)(5p+q)s/2$ ($5p+q \equiv 2 \pmod 4$)

or $m=5(p+q)(5p+q)s/4, n=(25p+q)(5p+q)s/4$ ($5p+q \equiv 0 \pmod 4$)

(II) when $q=5q'$ and $q' \equiv 1,2,3,4 \pmod 5$,

$m=5(p+5q')(p+q')s, n=5(5p+q')(p+q')s$ ($p+q' \equiv 1,3 \pmod 4$)

or $m=5(p+5q')(p+q')s/2, n=5(5p+q')(p+q')s/2$ ($p+q' \equiv 2 \pmod 4$)

or $m=5(p+5q')(p+q')s/4, n=5(5p+q')(p+q')s/4$ ($p+q' \equiv 0 \pmod 4$)

(III) when $q=25q''$,

$m=(p+25q'')(p+5q'')s, n=5(p+q'')(p+5q'')s$ ($p+5q'' \equiv 1,3 \pmod 4$)

or $m=(p+25q'')(p+5q'')s/2, n=5(p+q'')(p+5q'')s/2$ ($p+5q'' \equiv 2 \pmod 4$)

or $m=(p+25q'')(p+5q'')s/4, n=5(p+q'')(p+5q'')s/4$ ($p+5q'' \equiv 0 \pmod 4$).

Notation 2. Let A and B be two sequences of the same size such as

A: $a_1, a_2, \ldots, a_u$

B: $b_1, b_2, \ldots, b_u$.

If $b_i=a_i+c$ $(i=1,2,\ldots,u)$, then we write **B=A+c**. If $b_i=((a_i+c) \bmod w)$ $(i=1,2,\ldots,u)$, then we write **B=A+c mod w**, where the residuals $a_i+c \bmod w$ are integers in the set $\{1,2,\ldots,w\}$.

Lemma 4. $(p,q)=1$ and $q \equiv 1,2,3,4 \pmod 5$
$m=5(p+q)(5p+q)s$, $n=(25p+q)(5p+q)s$
$\Longrightarrow$ $K_{m,n}$ has an S_6-factorization.

Proof.(Algorithm I) Put $s=1$. Let $x=(5n-m)/24$, $y=(5m-n)/24$, $t=(m+n)/6$, $r=6mn/5(m+n)$. Then $x=5p(5p+q)$, $y=q(5p+q)$, $t=(5p+q)^2$, $r=(p+q)(25p+q)$. Let $r_m=p+q$, $r_n=25p+q$, $m_0=m/r_m=5(5p+q)$, $n_0=n/r_n=5p+q$.
Consider two sequences R and C of the same size $25(5p+q)$.
R: $1,1,1,1,1,2,2,2,2,2,\ldots,5(5p+q),5(5p+q),5(5p+q),5(5p+q),5(5p+q)$
C: $1,2,\ldots,25(5p+q)-1,25(5p+q)$.
Construct $R_i=R+5(i-1)(5p+q)$ $(i=1,2,\ldots,p)$.
Construct $C_i=(C+5(i-1) \bmod 25(5p+q))+25(i-1)(5p+q)$ $(i=1,2,\ldots,p)$.
Consider two sequences R' and C' of the same size $5(5p+q)$.
R': $r_1,r_2,\ldots,r_{5(5p+q)}$, where $r_i=(i-1)p+1 \bmod 5(5p+q)$ $(i=1,2,\ldots,5(5p+q))$
C': $c_1,c_2,\ldots,c_{5(5p+q)}$, where $c_i=n-(i-1)q \bmod q(5p+q)$ $(i=1,2,\ldots,5(5p+q))$.
Construct $R_i'=R'+5(i-1)(5p+q)+5p(5p+q)$ $(i=1,2,\ldots,q)$.
Construct $C_i'=(C'-(i-1) \bmod q(5p+q))+25p(5p+q)$ $(i=1,2,\ldots,q)$.
Consider two sequences I and J of the same size $5t$.
I: $R_1,R_2,\ldots,R_p,R_1',R_2',\ldots,R_q'$
J: $C_1,C_2,\ldots,C_p,C_1',C_2',\ldots,C_q'$.
Let i_k and j_k be the k-th element of I and J, respectively $(k=1,2,\ldots,5t)$. Join two vertices i_k in V_1 and j_k in V_2 with an edge (i_k,j_k) $(k=1,2,\ldots,5t)$. Construct a graph F with two vertex sets $\{i_k\}$ and $\{j_k\}$ and an edge set $\{(i_k,j_k)\}$. Then F is an S_6-factor of $K_{m,n}$. This graph is called an ***S_6-factor constructed with two sequences I and J.***
Construct I_i such that $I_i=I+(i-1)m_0 \bmod m$ $(i=1,2,\ldots,r_m)$.
Construct J_j such that $J_j=J+(j-1)n_0 \bmod n$ $(j=1,2,\ldots,r_n)$.
Construct S_6-factors F_{ij} with I_i and J_j $(i=1,2,\ldots,r_m; j=1,2,\ldots,r_n)$. Then F_{ij} are edge-disjoint and their sum is an S_6-factorization of $K_{m,n}$. By Lemma 2, $K_{m,n}$ has an S_6-factorization for every positive integer s. □

Lemma 5. $(p,q)=1$ and $q=5q'$ and $q' \equiv 1,2,3,4 \pmod 5$
$m=5(p+5q')(p+q')s$, $n=5(5p+q')(p+q')s$
$\Longrightarrow$ $K_{m,n}$ has an S_6-factorization.

Lemma 6. $(p,q)=1$ and $q=25q''$
$m=(p+25q'')(p+5q'')s$, $n=5(p+q'')(p+5q'')s$
$\Longrightarrow$ $K_{m,n}$ has an S_6-factorization.

References

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