1B-1 Balanced (C_7, C_8)-$2t$-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_7 and C_8 be *the 7-cycle* and *the 8-cycle*, respectively. *The (C_7, C_8)-$2t$-foil* is a graph of t edge-disjoint C_7's and t edge-disjoint C_8's with a common vertex and the common vertex is called *the center of the (C_7, C_8)-$2t$-foil.* When K_n is decomposed into edge-disjoint sum of (C_7, C_8)-$2t$-foils, we say that *K_n has a (C_7, C_8)-$2t$-foil decomposition.* Moreover, when every vertex of K_n appears in the same number of (C_7, C_8)-$2t$-foils, we say that *K_n has a balanced (C_7, C_8)-$2t$-foil decomposition* and this number is called *the replication number.*

2. Balanced (C_7, C_8)-$2t$-foil decomposition of K_n

Theorem. K_n has a balanced (C_7, C_8)-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{30t}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_7, C_8)-$2t$-foil decomposition. Let b be the number of (C_7, C_8)-$2t$-foils and r be the replication number. Then $b = n(n-1)/30t$ and $r = (13t+1)(n-1)/30t$. Among r (C_7, C_8)-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_7, C_8)-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/30t$ and $r_2 = 13(n-1)/30$. Therefore, $n \equiv 1 \pmod{30t}$ is necessary.
(Sufficiency) Put $n = 30st + 1$ and $T = st$. Then $n = 30T + 1$.
Case 1. n = 31. (Example 1. Balanced (C_7, C_8)-2-foil decomposition of K_{31}.)

Case 2. $\mathbf{n = 30T + 1,\ T \geq 2}$. Construct a (C_7, C_8)-$2T$-foil as follows:
$\{(30T+1, 1, 21T+2, 29T+2, 8T+2, 28T+2, 14T), (30T+1, T+1, 19T+2, 24T+2, 17T+2, 23T+2, 5T+2, 2T+1)\} \cup$
$\{(30T+1, 2, 21T+4, 29T+3, 8T+4, 28T+3, 14T-1), (30T+1, T+2, 19T+4, 24T+3, 17T+4, 23T+3, 5T+4, 2T+2)\} \cup$
$\{(30T+1, 3, 21T+6, 29T+4, 8T+6, 28T+4, 14T-2), (30T+1, T+3, 19T+6, 24T+4, 17T+6, 23T+4, 5T+6, 2T+3)\} \cup$
$\ldots \cup$
$\{(30T+1, T-2, 23T-4, 30T-1, 10T-4, 29T-1, 13T+3), (30T+1, 2T-2, 21T-4, 25T-1, 19T-4, 24T-1, 7T-4, 3T-2)\} \cup$
$\{(30T+1, T-1, 23T-2, 30T, 10T-2, 29T, 13T+2), (30T+1, 2T-1, 21T-2, 25T, 19T-2, 24T, 7T-2, 4T-1)\} \cup$
$\{(30T+1, T, 23T, 3T-1, 10T, 29T+1, 13T+1), (30T+1, 2T, 21T, 25T+1, 19T, 24T+1, 7T, 3T)\}$.
Decompose the (C_7, C_8)-$2T$-foil into s (C_7, C_8)-$2t$-foils. Then these starters comprise a balanced (C_7, C_8)-$2t$-foil decomposition of K_n.

Example 1. Balanced (C_7, C_8)-2-foil decomposition of K_{31}.
$\{(31, 1, 23, 2, 10, 30, 14), (31, 5, 24, 26, 19, 25, 7, 3)\}$.
This starter comprises a balanced (C_7, C_8)-2-foil decomposition of K_{31}.

Example 2. Balanced (C_7, C_8)-4-foil decomposition of K_{61}.
$\{(61, 1, 44, 60, 18, 58, 28)$,
$(61, 2, 46, 5, 20, 59, 27)$,
$(61, 3, 40, 50, 36, 48, 12, 7)$,
$(61, 4, 42, 51, 38, 49, 14, 6)\}$.
This starter comprises a balanced (C_7, C_8)-4-foil decomposition of K_{61}.

Example 3. Balanced (C_7, C_8)-6-foil decomposition of K_{91}.
$\{(91, 1, 65, 89, 26, 86, 42),$
$(91, 2, 67, 90, 28, 87, 41),$
$(91, 3, 69, 8, 30, 88, 40),$
$(91, 4, 59, 74, 53, 71, 17, 7),$
$(91, 5, 61, 75, 55, 72, 19, 11),$
$(91, 6, 63, 76, 57, 73, 21, 9)\}.$
This starter comprises a balanced (C_7, C_8)-6-foil decomposition of K_{91}.

Example 4. Balanced (C_7, C_8)-8-foil decomposition of K_{121}.
$\{(121, 1, 86, 118, 34, 114, 56),$
$(121, 2, 88, 119, 36, 115, 55),$
$(121, 3, 90, 120, 38, 116, 54),$
$(121, 4, 92, 11, 40, 117, 53),$
$(121, 5, 78, 98, 70, 94, 22, 9),$
$(121, 6, 80, 99, 72, 95, 24, 10),$
$(121, 7, 82, 100, 74, 96, 26, 15),$
$(121, 8, 84, 101, 76, 97, 28, 12)\}.$
This starter comprises a balanced (C_7, C_8)-8-foil decomposition of K_{121}.

Example 5. Balanced (C_7, C_8)-10-foil decomposition of K_{151}.
$\{(151, 1, 107, 147, 42, 142, 70),$
$(151, 2, 109, 148, 44, 143, 69),$
$(151, 3, 111, 149, 46, 144, 68),$
$(151, 4, 113, 150, 48, 145, 67),$
$(151, 5, 115, 14, 50, 146, 66),$
$(151, 6, 97, 122, 87, 117, 27, 11),$
$(151, 7, 99, 123, 89, 118, 29, 12),$
$(151, 8, 101, 124, 91, 119, 31, 13),$
$(151, 9, 103, 125, 93, 120, 33, 19),$
$(151, 10, 105, 126, 95, 121, 35, 15)\}.$
This starter comprises a balanced (C_7, C_8)-10-foil decomposition of K_{151}.

Example 6. Balanced (C_7, C_8)-12-foil decomposition of K_{181}.
$\{(181, 1, 128, 176, 50, 170, 84),$
$(181, 2, 130, 177, 52, 171, 83),$
$(181, 3, 132, 178, 54, 172, 82),$
$(181, 4, 134, 179, 56, 173, 81),$
$(181, 5, 136, 180, 58, 174, 80),$
$(181, 6, 138, 17, 60, 175, 79),$
$(181, 7, 116, 146, 104, 140, 32, 13),$
$(181, 8, 118, 147, 106, 141, 34, 14),$
$(181, 9, 120, 148, 108, 142, 36, 15),$
$(181, 10, 122, 149, 110, 143, 38, 16),$
$(181, 11, 124, 150, 112, 144, 40, 23),$
$(181, 12, 126, 151, 114, 145, 42, 18)\}.$
This starter comprises a balanced (C_7, C_8)-12-foil decomposition of K_{181}.

Example 7. Balanced (C_7, C_8)-14-foil decomposition of K_{211}.
$\{(211, 1, 149, 205, 58, 198, 98),$
$(211, 2, 151, 206, 60, 199, 97),$
$(211, 3, 153, 207, 62, 200, 96),$
$(211, 4, 155, 208, 64, 201, 95),$
$(211, 5, 157, 209, 66, 202, 94),$
$(211, 6, 159, 210, 68, 203, 93),$
$(211, 7, 161, 20, 70, 204, 92),$
$(211, 8, 135, 170, 121, 163, 37, 15),$
$(211, 9, 137, 171, 123, 164, 39, 16),$
$(211, 10, 139, 172, 125, 165, 41, 17),$
$(211, 11, 141, 173, 127, 166, 43, 18),$
$(211, 12, 143, 174, 129, 167, 45, 19),$
$(211, 13, 145, 175, 131, 168, 47, 27),$
$(211, 14, 147, 176, 133, 169, 49, 21)\}.$
This starter comprises a balanced (C_7, C_8)-14-foil decomposition of K_{211}.

References

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