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The Joinability and Related Decision Problems for Semi-constructor TRSs

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The word and unification problems for term rewriting systems (TRSs) are most important ones and their decision algorithms have various useful applications in computer science. Algorithms of deciding joinability for TRSs are often used to obtain algorithms that decide these problems. In this paper, we first show that the joinability problem is undecidable for linear semi-constructor TRSs. Here, a semi-constructor TRS is such a TRS that all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms. Next, we show that this problem is decidable both for confluent semi-constructor TRSs and for confluent semi-monadic TRSs. This result implies that the word problem is decidable for these classes, and will be used to show that unification is decidable for confluent semi-constructor TRSs in our forthcoming paper.

1. Introduction

The word and unification problems for term rewriting systems (TRSs) are most important ones and their decision algorithms have various useful applications in computer science. The word problem is undecidable in general even if we restrict ourselves to right-ground TRSs [9]. This problem is equivalent to the joinability one if TRSs are confluent (Church-Rosser). Here, the joinability problem for TRSs is the problem of deciding, for a TRS R and two terms s and t, whether s and t can be reduced to some common term by applying the rules of R. The unification problem includes the word problem as its special case and its decision algorithm often needs an algorithm to decide joinability as its component (e.g., for confluent right-ground TRSs [11] and confluent simple TRSs [6]).

In this paper, we consider the joinability problem for some subclasses of TRSs. This problem is also undecidable in general even if we restrict ourselves to flat TRSs [4]. On the other hand, it is decidable for some subclasses of TRSs (e.g., right-ground TRSs [10], right-linear semi-monadic TRSs [8], and right-linear finite path overlapping TRSs [12]). Many of these decidability results have been obtained by reducing these problems to decidable ones for tree automata, so that these decidable subclasses are restricted to those of right-linear TRSs.

In this paper, we show that joinability is undecidable for linear semi-constructor TRSs (Th 3), but decidable for confluent semi-constructor TRSs (Th 37). Here, a semi-constructor TRS is such a TRS that all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms. This subclass is a minimal class of non-right-linear TRSs which properly includes right-ground TRSs and simple TRSs. Our latter result shows decidability of joinability for possibly non-right-linear TRSs and is striking compared with the previous decidability results. To our knowledge, such attempts were very few so far. As a consequence, the word problem is decidable for confluent semi-constructor TRSs. Using the decidability result of joinability, we will show that unification is decidable for confluent semi-constructor TRSs in our forthcoming paper [5]. Our proof technique used to show the decidability of joinability can be applied to subclasses other than confluent semi-constructor TRSs. In fact, we show in this paper that joinability is decidable for confluent semi-monadic TRSs (Th 44). This subclass is possibly non-right-linear too.

We also consider the reachability problem, which is also fundamental. Here, the reachability problem for TRSs is the problem of deciding, for a TRS R and two terms s and t, whether s can be reduced to t by applying the rules of R. We show that reachability is undecidable both for linear semi-constructor TRSs (Th 3) and for

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confluent monadic TRSs (Th 46).

2. Preliminaries

We assume that the reader is familiar with standard definitions of rewrite systems [2),13)] and we just recall here the main notations used in this paper.

We use ε to denote the empty string. Let $|\Delta|$ be the cardinality of a set Δ. Let X be a set of variables, F a finite set of operation symbols graded by an arity function $\mathsf{ar}\colon F \to \mathbf{N}(=\{0,1,2,\cdots\})$, $F_n = \{f \in F \mid \mathsf{ar}(f) = n\}$, and T the set of terms constructed from X and F. We use x, y, z as variables, b, c, d as constants, f, g as operation symbols, r, s, t as terms, and σ, θ as substitutions. A term is *ground* if it has no variable. Let G be the set of ground terms. Let $\mathsf{V}(s)$ be the set of variables occurring in s. We use $|s|$ to denote the *size* of s, i.e., the number of symbols occurring in s. The *height* of a term is defined as follows: $\mathsf{height}(a) = 0$ if a is a variable or a constant and $\mathsf{height}(f(t_1,\ldots,t_n)) = 1 + \max\{\mathsf{height}(t_1),\ldots,\mathsf{height}(t_n)\}$ if $\mathsf{ar}(f) > 0$. The *root symbol* of a term is defined as $\mathsf{root}(a) = a$ if a is a variable and $\mathsf{root}(f(t_1,\ldots,t_n)) = f$.

A position in a term is expressed by a sequence of positive integers, and positions are partially ordered by the prefix ordering $\leq$. We use $u|v$ to denote that positions u and v are parallel. Let $\mathcal{O}(s)$ be the set of positions of s. For a set of positions W, let $\mathsf{Min}(W)$ be the set of its minimal positions (w.r.t. $\leq$).

Let $s_{|u}$ be the subterm of s at position u. Let $\mathsf{Psub}(s)$ be the set of proper subterms of s, and for $\Delta \subseteq T$, let $\mathsf{Psub}(\Delta) = \cup_{s\in\Delta}\mathsf{Psub}(s)$. We use $s[t]_u$ to denote the term obtained from s by replacing the subterm $s_{|u}$ by t. For a sequence $(u_1,\cdots,u_n)$ of pairwise parallel positions and terms $t_1,\cdots,t_n$, we use $s[t_1,\cdots,t_n]_{(u_1,\ldots,u_n)}$ to denote the term obtained from s by replacing each subterm $s_{|u_i}$ by $t_i (1 \leq i \leq n)$.

A *rewrite rule* $\alpha \to \beta$ is a directed equation over terms where $\alpha \notin X$ and $\mathsf{V}(\alpha) \supseteq \mathsf{V}(\beta)$. A *TRS* is a finite set of rewrite rules. A term s reduces to t at position u by TRS R, denoted $s \xrightarrow{u}_R t$, if $s_{|u} = \alpha\theta$ and $t = s[\beta\theta]_u$ for some rewrite rule $\alpha \to \beta$ and substitution θ. This reduction is called a *u-reduction*. For $s \xrightarrow{u}_R t$, u and R may be omitted. We write $t \leftarrow s$ if $s \to t$, $s \leftrightarrow t$ if $s \to t$ or $s \leftarrow t$. $\to^*$ is a reflexive transitive closure of $\to$. Term t is *reachable* from s in R if $s \to^*_R t$. Term s and t are *joinable*, denoted $s \downarrow_R t$ if there exists r such that $s \to^*_R r \leftarrow^*_R t$. Let $\gamma\colon s_1 \overset{u_1}{\leftrightarrow} s_2 \cdots \overset{u_{n-1}}{\leftrightarrow} s_n$ be a *rewrite sequence*. This sequence is abbreviated to $\gamma\colon s_1 \leftrightarrow^* s_n$ and $\mathcal{R}(\gamma) = \{u_1,\cdots,u_{n-1}\}$ is the set of the redex positions of γ. For any sequence γ and position set W, if for every $v \in \mathcal{R}(\gamma)$ there exists $u \in W$ such that $v \geq u$, then we write $\gamma\colon s_1 \overset{\geq W}{\leftrightarrow^*} s_n$.

Let $\mathcal{O}_G(s) = \{u \in \mathcal{O}(s) \mid s_{|u} \in G\}$. For any set $\Xi \subseteq X \cup F$, let $\mathcal{O}_\Xi(s) = \{u \in \mathcal{O}(s) \mid \mathsf{root}(s_{|u}) \in \Xi\}$. Let $\mathcal{O}_x(s) = \mathcal{O}_{\{x\}}(s)$. The set D_R of *defined symbols* for a TRS R is defined as $D_R = \{\mathsf{root}(\alpha) \mid \alpha \to \beta \in R\}$. If R is clear from the context, we write D instead of D_R. A term s is *semi-constructor* if for every subterm t of s, t is ground or $\mathsf{root}(t)$ is not a defined symbol.

Definition 1 A rule $\alpha \to \beta$ is *ground* if $\alpha, \beta \in G$, *right-ground* if $\beta \in G$, *semi-constructor* if β is semi-constructor, and *linear* if $|\mathcal{O}_x(\alpha)| \leq 1$ and $|\mathcal{O}_x(\beta)| \leq 1$ for every x. A TRS R is *ground*, *right-ground*, *semi-constructor*, *linear* if every rule in R is ground, right-ground, semi-constructor, linear, respectively. A TRS R is *confluent* if $\leftrightarrow^*_R = \downarrow_R$.

Example 2 Let $R_\mathrm{e} = \{\mathsf{nand}(x,x) \to \mathsf{not}(\mathsf{and}(x,x)),\ \mathsf{nand}(\mathsf{not}(x),x) \to \mathsf{true},\ \mathsf{true} \to \mathsf{nand}(\mathsf{false},\mathsf{false}),\ \mathsf{false} \to \mathsf{nand}(\mathsf{true},\mathsf{true})\}$. R_e is semi-constructor, non-terminating, and confluent [3)]. We will use this R_e in examples given in Section 4.

3. Joinability and Reachability for Linear Semi-constructor TRSs

First, we show that joinability and reachability for (non-confluent) semi-constructor TRSs are undecidable.

Theorem 3 Joinability and reachability for linear semi-constructor TRSs are undecidable.

Proof [sketch] The proof is by a reduction from the Post's correspondence problem (PCP). Let $P = \{\langle u_i, v_i\rangle \in \Sigma^* \times \Sigma^* \mid 1 \leq i \leq k\}$ be an instance of the PCP. The corresponding TRS R_P is constructed as follows: Let $F = F_0 \cup F_1 \cup F_2$ where $F_0 = \{\mathsf{c}, \mathsf{d}, \$\}$, $F_1 = \Sigma \cup \{\mathsf{f}, \mathsf{h}\}$, $F_2 = \{\mathsf{g}\}$, and $R_P = \{\mathsf{c} \to \mathsf{h}(\mathsf{c}),\ \mathsf{c} \to \mathsf{d},\ \mathsf{d} \to \mathsf{f}(\mathsf{d})\} \cup \{\mathsf{d} \to \mathsf{g}(u_i(\$), v_i(\$)),\ \mathsf{f}(\mathsf{g}(x,y)) \to \mathsf{g}(u_i(x), v_i(y)) \mid 1 \leq i \leq k\} \cup \{\mathsf{h}(\mathsf{g}(a(x), a(y))) \to \mathsf{g}(x,y) \mid a \in \Sigma\}$. Here, $u(x)$ is an abbreviation for $a_1(a_2(\cdots a_i(x)))$ where $u = a_1a_2\cdots a_i \in \Sigma^*$. R_P is linear and semi-constructor. For R_P, $\mathsf{c} \to^* \mathsf{g}(\$,\$)$ iff there exists a sequence of indexes $i_1 \cdots i_m \in \{1,\cdots,k\}^+$ such that $\mathsf{c} \to^{n+1}$

$\mathsf{h}^n(\mathsf{d}) \to^m \mathsf{h}^n(\mathsf{f}^{m-1}(\mathsf{g}(u_{i_m}(\$), v_{i_m}(\$)))) \to^{m-1} \mathsf{h}^n(\mathsf{g}(u_{i_1}\cdots u_{i_m}(\$), v_{i_1}\cdots v_{i_m}(\$))) \to^n \mathsf{g}(\$,\$)$ where $n = |u_{i_1}\cdots u_{i_m}|$ and $u_{i_1}\cdots u_{i_m} = v_{i_1}\cdots v_{i_m}$. Thus, $\mathsf{c} \to^* \mathsf{g}(\$,\$)$ iff P has a solution. Since $\mathsf{g}(\$,\$)$ is a normal form, $\mathsf{c} \to^* \mathsf{g}(\$,\$)$ iff $\mathsf{c} \downarrow \mathsf{g}(\$,\$)$. Hence, this theorem holds. □

4. Decidability of Joinability for Confluent Semi-constructor TRSs

In this section, we show that joinability for confluent semi-constructor TRSs is decidable, by reducing it to the joinability for right-ground TRSs, which is decidable [10]. First, a given confluent semi-constructor TRS R_0 is transformed into a standard TRS R (where the definition of standard is given in Section 4.1). Next, we add new ground rules called shortcut rules to R, and obtain TRS R' satisfying that two constants are joinable in R iff they are joinable by only right-ground rules in R' (Section 4.2). Finally, we show the decidability of joinability between arbitrary terms (Section 4.3, 4.4).

4.1 Standard Semi-constructor TRSs

We use R_{rg} and R_{nrg} to denote the sets of right-ground and non-right-ground rewrite rules in TRS R, respectively. That is, $R = R_{\mathrm{rg}} \cup R_{\mathrm{nrg}}$.

Definition 4 A TRS R is *standard* if for every $\alpha \to \beta \in R$, either $\alpha \in F_0$ and $\mathsf{height}(\beta) \le 1$ or $\alpha \notin F_0$ and $\mathcal{O}_G(\beta) \subseteq \mathcal{O}_{F_0}(\beta)$ holds.

Let R_0 be a confluent semi-constructor TRS. The corresponding standard TRS is constructed as follows. The construction has a loop structure. We use k as the loop counter. First, we choose $\alpha \to \beta \in R_k (k \ge 0)$ that does not satisfy the standardness condition. If $\alpha \in F_0$ then let $\{u_1, \cdots, u_m\}$ be $\{1, \cdots, \mathsf{ar}(\mathsf{root}(\beta))\} \setminus \mathcal{O}_{F_0}(\beta)$. Otherwise, let $\{u_1, \cdots, u_m\}$ be $\mathsf{Min}(\mathcal{O}_G(\beta)) \setminus \mathcal{O}_{F_0}(\beta)$. Let $R_{k+1} = (R_k \setminus \{\alpha \to \beta\}) \cup \{\alpha \to \beta[d_1, \cdots, d_m]_{(u_1,\cdots,u_m)}\} \cup \{d_i \to \beta_{|u_i} \mid 1 \le i \le m\}$ where $d_1, \cdots, d_m$ are new pairwise distinct constants which do not appear in R_k or T. This procedure is applied repeatedly until the TRS satisfies the condition of standardness. Let S be this construction procedure and $\mathsf{S}(R_0)$ be the output of S for input R_0. It is obvious that S is terminating.

Example 5 Let $R_0 = \{\mathsf{f}_1(x) \to \mathsf{g}(x, \mathsf{g}(\mathsf{a}, \mathsf{b})), \mathsf{f}_2(x) \to \mathsf{f}_2(\mathsf{g}(\mathsf{c}, \mathsf{d}))\}$, then $\mathsf{S}(R_0) = \{\mathsf{f}_1(x) \to \mathsf{g}(x, \mathsf{d}_1),\ \mathsf{d}_1 \to \mathsf{g}(\mathsf{a}, \mathsf{b}),\ \mathsf{f}_2(x) \to \mathsf{d}_2,\ \mathsf{d}_2 \to \mathsf{f}_2(\mathsf{d}_3),\ \mathsf{d}_3 \to \mathsf{g}(\mathsf{c}, \mathsf{d})\}$.

Lemma 6 Let R_0 be a confluent and semi-constructor TRS.

(1) $\mathsf{S}(R_0)$ is confluent and semi-constructor.

(2) For any terms s, t which do not contain new constants, $s \downarrow_{R_0} t$ iff $s \downarrow_{\mathsf{S}(R_0)} t$.

The proof is given in Appendix A.1. Note that all new defined symbols created in this transformation are constants. By this lemma, we can assume that a given confluent semi-constructor TRS is standardized. In particular, for any right ground rule $\alpha \to \beta \in \mathsf{S}(R_0)_{\mathrm{rg}}$, $\alpha \in F_0$ and $\mathsf{height}(\beta) \le 1$ or $\alpha \notin F_0$ and $\beta \in F_0$ holds.

4.2 Shortcut Rules and Quasi-standard Semi-constructor TRSs

In this section, we add new ground rules called shortcut rules to standard TRS R, and obtain TRS R' satisfying that two constants are joinable in R iff they are joinable by only right-ground rules of R'. Right-hand sides of added shortcut rules may have height greater than 1. These rules are called type C rules and defined as follows.

Definition 7

(1) A rule $\alpha \to \beta$ has *type* C if $\alpha \in F_0, \beta \notin F_0$, and $\mathcal{O}_D(\beta) \subseteq \mathcal{O}_{F_0}(\beta)$. Let R_{C} be the set of type C rules in R.

(2) A TRS R is *quasi-standard* if $R \setminus R_{\mathrm{C}}$ is standard.

Henceforth, we assume that R is confluent, quasi-standard, and semi-constructor. To describe how to produce shortcut rules, we need some definitions and lemmata.

Definition 8 Let $\mathsf{Bud}(R_{\mathrm{C}}) = F_0 \cup \mathsf{Psub}(\{\beta \mid \alpha \to \beta \in R_{\mathrm{C}}\})$.

The following lemma is used in the proofs of Lemmata 11, 25, and 31.

Lemma 9 For any rewrite sequence $\gamma : s \to^*_{R_{\mathrm{rg}}} t$ and $u \in \mathcal{O}(t)$, if there exists $v \in \mathcal{R}(\gamma)$ such that $v < u$, then there exists $s' \in \mathsf{Bud}(R_{\mathrm{C}})$ such that $s \to^*_{R_{\mathrm{rg}}} t[s']_u$ and $s' \to^*_{R_{\mathrm{rg}}} t_{|u}$.

Proof Consider the last v-reduction in γ such that $v < u$. That is, $s \to^*_{R_{\mathrm{rg}}} r \xrightarrow{v}_{R_{\mathrm{rg}}} t'[r']_u$ and $r' \to^*_{R_{\mathrm{rg}}} t_{|u}$ for some r, r'. (Note that $t'[r']_u \to^*_{R_{\mathrm{rg}}} t[r']_u$ holds.) Here, $r = r[\alpha\theta]_v$ and $t'[r']_u = r[\beta]_v$ hold for some right-ground rule $\alpha \to \beta$ and θ. Since $v < u$, $\beta \notin F_0$ holds. This implies that $\alpha \in F_0$ holds and if $\alpha \to \beta \notin R_{\mathrm{C}}$ then $\mathsf{height}(\beta) = 1$ by quasi-standardness of R. Let $u = vw$, then $r' = \beta_{|w}$. If $\alpha \to \beta \notin R_{\mathrm{C}}$ then $r' \in F_0$ otherwise $r' \in \mathsf{Psub}(\{\beta \mid \alpha \to \beta \in R_{\mathrm{C}}\})$. Since $r' \in \mathsf{Bud}(R_{\mathrm{C}})$, we can choose r' as s'. □

Definition 10

(1) The function $\mathsf{linearize}(s)$ linearizes nonlinear term s as follows. For each vari-

able occurring more than once in s, the first occurrence is not renamed, and the other ones are replaced by new pairwise distinct variables. For example, $\mathsf{linearize}(\mathsf{nand}(x,x)) = \mathsf{nand}(x,x_1)$. If function $\mathsf{linearize}$ replaces x by x_1 then we use $x \equiv x_1$ to denote the replacement relation.

(2) A substitution σ is *joinability preserving* under relation $\equiv$ for TRS R_{rg} if $x\sigma \downarrow_{R_{\mathrm{rg}}} x'\sigma$ whenever $x \equiv x'$.

(3) Let $\alpha \to \beta \in R_{\mathrm{nrg}}$ and $\alpha' = \mathsf{linearize}(\alpha)$. Then, $\sigma : \mathsf{V}(\alpha') \to \mathsf{Psub}(s) \cup \mathsf{Bud}(R_{\mathrm{C}})$ is called a bud substitution for s and $\alpha \to \beta$ if $s \to^*_{R_{\mathrm{rg}}} \alpha'\sigma$ and σ is joinability preserving under relation $\equiv$ for R_{rg}. Note that if s is a ground term then $\beta\sigma$ is a ground term. Let $\mathsf{BudMap}_R(s, \alpha \to \beta)$ be the set of such bud substitutions.

Lemma 11 Let $\alpha \to \beta \in R_{\mathrm{nrg}}$.

(1) $\mathsf{BudMap}_R(s, \alpha \to \beta)$ is finite and computable.

(2) Let $\gamma : s \to^*_{R_{\mathrm{rg}}} \alpha\theta$ for some θ. Then, there exists $\sigma \in \mathsf{BudMap}_R(s, \alpha \to \beta)$ such that $s \to^*_{R_{\mathrm{rg}}} \alpha'\sigma \to^*_{R_{\mathrm{rg}}} \alpha\theta$ and $\beta\sigma \to^*_{R_{\mathrm{rg}}} \beta\theta$ where $\alpha' = \mathsf{linearize}(\alpha)$.

(3) For any $\sigma \in \mathsf{BudMap}_R(s, \alpha \to \beta)$, $s \downarrow_R \beta\sigma$ holds.

Proof

(1) Finiteness is obvious. Computability holds since joinability and reachability are decidable for right-ground TRSs [10].

(2) Let $\{u_1, \cdots, u_n\}$ be $\mathcal{O}_X(\alpha)$. For u_1, if there exists $v_1 \in \mathcal{R}(\gamma)$ such that $v_1 < u_1$, then there exists $s'_1 \in \mathsf{Bud}(R_{\mathrm{C}})$ such that $s \to^*_{R_{\mathrm{rg}}} \alpha\theta[s'_1]_{u_1}$ and $s'_1 \to^*_{R_{\mathrm{rg}}} \alpha\theta_{|u_1}$ by Lemma 9. Otherwise, $s_{|u_1} \to^*_{R_{\mathrm{rg}}} \alpha\theta_{|u_1}$, so let s'_1 be $s_{|u_1}$. Thus, $s \to^*_{R_{\mathrm{rg}}} \alpha\theta[s'_1]_{u_1} \overset{\geq\{u_1\}}{\to^*}{}_{R_{\mathrm{rg}}} \alpha\theta$. Let $\gamma' : s \to^*_{R_{\mathrm{rg}}} \alpha\theta[s'_1]_{u_1}$. By similar arguments, if there exists $v_2 \in \mathcal{R}(\gamma')$ such that $v_2 < u_2$, then there exists $s'_2 \in \mathsf{Bud}(R_{\mathrm{C}})$ such that $s \to^*_{R_{\mathrm{rg}}} \alpha\theta[s'_1]_{u_1}[s'_2]_{u_2}$ and $s'_2 \to^*_{R_{\mathrm{rg}}} (\alpha\theta[s'_1]_{u_1})_{|u_2}$ by Lemma 9. Otherwise, let s'_2 be $s_{|u_2}$. By $u_1|u_2$, $\alpha\theta[s'_1]_{u_1}[s'_2]_{u_2} = \alpha\theta[s'_1, s'_2]_{(u_1,u_2)}$ and $(\alpha\theta[s'_1]_{u_1})_{|u_2} = \alpha\theta_{|u_2}$. Thus, $s \to^*_{R_{\mathrm{rg}}} \alpha\theta[s'_1, s'_2]_{(u_1,u_2)} \overset{\geq\{u_1,u_2\}}{\to^*_{R_{\mathrm{rg}}}} \alpha\theta$. By repeating similar arguments to the above, there exists $\{s'_1, \cdots, s'_n\} \subseteq \mathsf{Psub}(s) \cup \mathsf{Bud}(R_{\mathrm{C}})$ such that $s \to^*_{R_{\mathrm{rg}}} \alpha\theta[s'_1, \cdots, s'_n]_{(u_1,\cdots,u_n)} \overset{\geq\mathcal{O}_X(\alpha)}{\to^*_{R_{\mathrm{rg}}}} \alpha\theta$ since $u_1, \cdots, u_n$ are pairwise parallel. Hence, $s \to^*_{R_{\mathrm{rg}}} \alpha'\sigma \to^*_{R_{\mathrm{rg}}} \alpha\theta$ and $\beta\sigma \to^*_{R_{\mathrm{rg}}} \beta\theta$ where $\sigma = \{\alpha'_{|u_i} \to s'_i \mid 1 \leq i \leq n\}$. $\sigma : \mathsf{V}(\alpha') \to \mathsf{Psub}(s) \cup \mathsf{Bud}(R_{\mathrm{C}})$ and for any $x \in \mathsf{V}(\alpha)$ and $x' \in \mathsf{V}(\alpha')$, if $x' \equiv x$ then $x\sigma \to^*_{R_{\mathrm{rg}}} x\theta$ and $x'\sigma \to^*_{R_{\mathrm{rg}}} x\theta$, so that σ is a bud substitution.

(3) By the definition of BudMap, $s \to^*_{R_{\mathrm{rg}}} \alpha'\sigma$ and σ is joinability preserving under relation $\equiv$ for R_{rg}, where $\alpha' = \mathsf{linearize}(\alpha)$. So, there exists a substitution θ such that $\alpha'\sigma \to^*_{R_{\mathrm{rg}}} \alpha\theta$, and $\beta\sigma \to^*_{R_{\mathrm{rg}}} \beta\theta$. Since $\alpha\theta \to_R \beta\theta$, $s \downarrow_R \beta\sigma$ holds. $\square$

By Lemma 11(2), for any constant d and rewrite sequence $d \to^*_{R_{\mathrm{rg}}} \alpha\theta \to_{R_{\mathrm{nrg}}} \beta\theta$, there exists $\alpha'\sigma$ such that $d \to^*_{R_{\mathrm{rg}}} \alpha'\sigma \to^*_{R_{\mathrm{rg}}} \alpha\theta$ and $\beta\sigma \to^*_{R_{\mathrm{rg}}} \beta\theta$ where $\alpha' = \mathsf{linearize}(\alpha)$. So, we have $d \to^*_{R'} \beta\theta$ for $R' = R_{\mathrm{rg}} \cup \{d \to \beta\sigma\}$. Thus, by adding shortcut rules such as $d \to \beta\sigma$, we can remove applications of the non-right-ground rule $\alpha \to \beta$. Note that confluence and joinability properties are preserved even if we add $d \to \beta\sigma$ since $d \downarrow_R \beta\sigma$. However, shortcut rules may be added infinitely in this procedure. To avoid this, we will apply a procedure which bounds the number of shortcut rules. To describe this procedure, we need some preliminaries.

Definition 12 For a ground term s, let $\#(s) = (\mathsf{height}(s), \tau(s))$ where $\tau : G \to \mathbf{N}$ is an injective mapping, and we assume that the ordering derived by this function is closed under context, i.e., for any r, s, t and any position $u \in \mathcal{O}(r)$, if $\tau(s) < \tau(t)$ then $\tau(r[s]_u) < \tau(r[t]_u)$. There exists such a function τ which is effectively computable (see Appendix A.2). In order to compare $\#(s)$ and $\#(t)$, we use lexicographic order $<_{\mathrm{lex}}$. Note that $<_{\mathrm{lex}}$ is a total order. A term s_0 is *minimum* in a set Δ iff $\#(s_0)$ is minimum in $\{\#(s) \mid s \in \Delta\}$.

Definition 13

(1) For a term α, let $\mathsf{Rhs}(\alpha, R) = \{\beta \mid \alpha \to \beta \in R\}$.

(2) For $\Delta \subseteq G$, let $\mathsf{Cut}(\Delta) = \{(u, d) \mid u \in \mathsf{Min}(\cup_{s\in\Delta}\mathcal{O}_{F_0}(s))$, and d is the minimum constant in $\{s_{|u} \in F_0 \mid s \in \Delta\}\}$. For example, $\mathsf{Cut}(\{\mathsf{not}(\mathsf{not}(\mathsf{true})), \mathsf{not}(\mathsf{false})\}) = \{(1, \mathsf{false})\}$.

The following lemma is used in the proof of Lemma 16.

Lemma 14 Let $\mathsf{Cut}(\mathsf{Rhs}(d, R_{\mathrm{C}})) = \{(u_1, d_1), \cdots, (u_n, d_n)\}$.

(1) For every $j \in \{1, \cdots, n\}$, $u_j \neq \varepsilon$ holds.

(2) For every $s \in \mathsf{Rhs}(d, R_{\mathrm{C}})$, $\mathcal{O}(s) \supseteq \{u_1, \cdots, u_n\}$ holds.

(3) For every $s, t \in \mathsf{Rhs}(d, R_{\mathrm{C}})$, $s_{|u_j} \downarrow t_{|u_j}$ for every $j \in \{1, \cdots, n\}$, and $s[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)} = t[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)}$.

Proof

(1) By $\mathsf{height}(s) > 0$ for any $s \in \mathsf{Rhs}(d, R_{\mathrm{C}})$.

(2) We assume to the contrary that there exist $s \in \mathsf{Rhs}(d, R_{\mathrm{C}})$ and $i \in \{1, \cdots, n\}$ such that $u_i \notin \mathcal{O}(s)$. Since $(u_i, d_i) \in \mathsf{Cut}(\mathsf{Rhs}(d, R_{\mathrm{C}}))$, there exists $t \in \mathsf{Rhs}(d, R_{\mathrm{C}})$ such that $u_i \in \mathcal{O}_{F_0}(t)$. By confluence of R, $s \downarrow_R t$ holds. Thus, there exists $v \in \mathcal{O}_{F \setminus F_0}(s) \cap \mathcal{O}_{F \setminus F_0}(t)$ such that $v < u_i$ and $s_{|v} \downarrow_R t_{|v}$. But, for such a maximal occurrence v, $\mathsf{root}(s_{|v})$ and $\mathsf{root}(t_{|v})$ must be different constructors(non-defined symbols), a contradiction.

(3) Since R is confluent, $s \downarrow_R t$. By the definition of the type C rule, $\mathcal{O}_D(s) \subseteq \mathcal{O}_{F_0}(s)$ and $\mathcal{O}_D(t) \subseteq \mathcal{O}_{F_0}(t)$. By (2), $\{u_1, \cdots, u_n\} \subseteq \mathcal{O}(s) \cap \mathcal{O}(t)$. Thus, (3) holds. $\Box$

Definition 15 Let $\mathsf{Rhs}(d, R_{\mathrm{C}}) = \{s_1, \cdots, s_m\}$ and $\mathsf{Cut}(\mathsf{Rhs}(d, R_{\mathrm{C}})) = \{(u_1, d_1), \cdots, (u_n, d_n)\}$. Then we define $\mathsf{Normalize}(d, R_{\mathrm{C}}) = \{d \to s_1[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)}\} \cup \{d_j \to s_{i|u_j} \mid 1 \leq i \leq m, 1 \leq j \leq n, d_j \neq s_{i|u_j}\}$. For example, $\mathsf{Normalize}(\mathsf{true}, \{\mathsf{true} \to \mathsf{not}(\mathsf{not}(\mathsf{true})), \mathsf{true} \to \mathsf{not}(\mathsf{false})\}) = \{\mathsf{true} \to \mathsf{not}(\mathsf{false}), \mathsf{false} \to \mathsf{not}(\mathsf{true})\}$.

We use $\{\cdots\}_{\mathrm{m}}$ to denote a multiset. Let $\ll$ be the multiset extension of relation $<_{lex}$. We use $\sqcup$ to denote multiset union.

Lemma 16 Let $|\mathsf{Rhs}(d, R_{\mathrm{C}})| > 1$ and $Q = \mathsf{Normalize}(d, R_{\mathrm{C}})$.

(1) $\{\#(s) \mid s \in \mathsf{Rhs}(d, R_{\mathrm{C}})\}_{\mathrm{m}} \gg \{\#(\beta) \mid \alpha \to \beta \in Q\}_{\mathrm{m}}$.

(2) For any $s \in \mathsf{Rhs}(d, R_{\mathrm{C}})$, $d \to_Q^+ s$ holds.

(3) $\to_Q \subseteq \downarrow_R$.

(4) $Q' = (R \setminus \{d \to s \mid s \in \mathsf{Rhs}(d, R_{\mathrm{C}})\}) \cup Q$ is confluent.

Proof Let $\mathsf{Rhs}(d, R_{\mathrm{C}}) = \{s_1, \cdots, s_m\}$ where $m > 1$, and $\mathsf{Cut}(\mathsf{Rhs}(d, R_{\mathrm{C}})) = \{(u_1, d_1), \cdots, (u_n, d_n)\}$.

(1) The proposition is expressed as $\{\#(s_i) \mid 1 \leq i \leq m\}_{\mathrm{m}} \gg \{\#(s_1[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)})\}_{\mathrm{m}} \sqcup \{\#(s_{i|u_j}) \mid 1 \leq i \leq m, 1 \leq j \leq n\}_{\mathrm{m}}$. By Lemma 14(1) and (2), for any $i \in \{1, \cdots, m\}$ and $j \in \{1, \cdots, n\}$, $u_j \in \mathcal{O}(s_i)$ and $\#(s_i) > \#(s_{i|u_j})$ hold, and $\#(s_1) \geq \#(s_1[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)})$, since d_i is minimum and the ordering derived by $\#$ is closed under context. If $\#(s_1) > \#(s_1[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)})$ then the proposition obviously holds. If $\#(s_1) = \#(s_1[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)})$ then $s_1 = s_1[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)}$ must hold by the injectivity of $\#$. Since $\#(s_i) > \#(s_{i|u_j})$ for any $i \in \{2, \cdots, m\}$ and $j \in \{1, \cdots, n\}$, the proposition holds.

(2) For any $i \in \{1, \cdots, m\}$, we have $d \to_Q^+ s_1[s_{i|u_1}, \cdots, s_{i|u_n}]_{(u_1, \cdots, u_n)}$ by the definition of $\mathsf{Normalize}$, and $s_1[s_{i|u_1}, \cdots, s_{i|u_n}]_{(u_1, \cdots, u_n)} = s_i$ by Lemma 14(3).

(3) It is sufficient to show that $d \downarrow_R s_1[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)}$ and $d_j \downarrow_R s_{i|u_j}$ for any $i \in \{1, \cdots, m\}$ and $j \in \{1, \cdots, n\}$ where $d_j \to s_{i|u_j} \in Q$. By Lemma 14(3), $d_j \downarrow_R s_{i|u_j}$ holds. Thus, $s_1[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)} \downarrow_R s_i$ holds, so that $d \downarrow_R s_1[d_1, \cdots, d_n]_{(u_1, \cdots, u_n)}$ holds by $d \to s_i \in R$.

(4) Let $s \leftarrow_{Q'}^* r \to_{Q'}^* t$. Since R is confluent, $R \cup Q$ is confluent by (3). Hence, $s \downarrow_{R \cup Q} t$ holds. By (2), $s \downarrow_{Q'} t$ holds. $\Box$

Each of the following functions takes as input a quasi-standard confluent and semi-constructor TRS R. Note that if $R' = \mathsf{Determinize}(R)$ then $|\mathsf{Rhs}(d, R'_C)| \leq 1$ for any d by the termination condition of $\mathsf{Determinize}$. Henceforth, we use $(A \circ B)(x)$ to denote $A(B(x))$ for functions A, B.

function $\mathsf{M}(R)$
  $R' := (\mathsf{Determinize} \circ \mathsf{AddShortcut})(R)$;
  if $R = R'$
    then return R
    else return $\mathsf{M}(R')$

function $\mathsf{AddShortcut}(R)$
  $R' := R$;
  for each $d \in F_0, \alpha \to \beta \in R_{\mathrm{nrg}}$ **do**
    $R' := R' \cup$
  $\{d \to \beta\sigma \mid \sigma \in \mathsf{BudMap}_R(d, \alpha \to \beta)\}$;
  return R'

function $\mathsf{Determinize}(R)$
  if $^{\exists}d \in F_0.\ |\mathsf{Rhs}(d, R_{\mathrm{C}})| > 1$

then return
$\mathsf{Determinize}($
$(R\backslash\{d \to s \mid s \in \mathsf{Rhs}(d, R_{\mathrm{C}})\})$
$\cup\, \mathsf{Normalize}(d, R_{\mathrm{C}}))$
else return R

Example 17 For TRS R_{e} of Example 2, $\mathsf{M}(R_{\mathrm{e}})$ is computed as follows. $\mathsf{AddShortcut}(R_{\mathrm{e}})$ is first called and a new shortcut rule $\mathsf{true} \to \mathsf{not}(\mathsf{and}(\mathsf{false}, \mathsf{false}))$ is added to R_{e} since $\mathsf{true} \to \mathsf{nand}(\mathsf{false}, \mathsf{false})$, $\mathsf{nand}(x, x) \to \mathsf{not}(\mathsf{and}(x, x)) \in R_{\mathrm{e}}$. By $\mathsf{false} \to \mathsf{nand}(\mathsf{true}, \mathsf{true}) \in R_{\mathrm{e}}$, $\mathsf{false} \to \mathsf{not}(\mathsf{and}(\mathsf{true}, \mathsf{true}))$ is also added. Thus, $\mathsf{AddShortcut}(R_{\mathrm{e}}) = R'$ where $R' = R_{\mathrm{e}} \cup \{\mathsf{true} \to \mathsf{not}(\mathsf{and}(\mathsf{false}, \mathsf{false})),\ \mathsf{false} \to \mathsf{not}(\mathsf{and}(\mathsf{true}, \mathsf{true}))\}$. Next, $\mathsf{Determinize}(R')$ is called and returns the same R' as output. Since $R' \neq R_{\mathrm{e}}$, $(\mathsf{Determinize} \circ \mathsf{AddShortcut})(R')$ is computed. Note that $R'_{\mathrm{C}} = \{\mathsf{true} \to \mathsf{not}(\mathsf{and}(\mathsf{false}, \mathsf{false})),\ \mathsf{false} \to \mathsf{not}(\mathsf{and}(\mathsf{true}, \mathsf{true}))\}$. $\mathsf{AddShortcut}(R')$ returns the same R' and so $\mathsf{Determinize}(R')$. Thus, this algorithm halts. $\mathsf{M}(R_{\mathrm{e}})$ returns R' as output. That is, $\mathsf{M}(R_{\mathrm{e}}) = R_{\mathrm{e}} \cup \{\mathsf{true} \to \mathsf{not}(\mathsf{and}(\mathsf{false}, \mathsf{false})),\ \mathsf{false} \to \mathsf{not}(\mathsf{and}(\mathsf{true}, \mathsf{true}))\}$.

Note that $\mathsf{M}(R) = (\mathsf{Determinize} \circ \mathsf{AddShortcut})^l(R)$ for some $l \geq 1$, $R_{nrg} = \mathsf{M}(R)_{nrg}$, and $\mathsf{M}(\mathsf{M}(R)) = \mathsf{M}(R)$. In the produced TRS $\mathsf{M}(R)$, the heights of some right-hand side terms of type C rules may become greater than 1.

First, we show that $\mathsf{M}(R)$ is confluent, quasi-standard, and semi-constructor. Next, we show that M is terminating. Finally, we show that two constants are joinable in R iff they are joinable in $\mathsf{M}(R)_{\mathrm{rg}}$. For these purpose, we need some lemmata.

Definition 18 A rule $\alpha \to \beta$ has *type* F_0^2 if $\alpha, \beta \in F_0$. Let $R_{\mathrm{F}_0^2}$ be the set of type F_0^2 rules in R.

Lemma 19 Let $Q = \mathsf{AddShortcut}(R)$ and $R' = \mathsf{Determinize}(Q)$.

(1) Both Q and R' are quasi-standard and semi-constructor.
(2) For any constant d, if $\mathsf{Rhs}(d, R_{\mathrm{C}}) \neq \emptyset$ then $\mathsf{Rhs}(d, R'_{\mathrm{C}}) \neq \emptyset$.

Proof

(1) For any $d, \alpha \to \beta \in R_{\mathrm{nrg}}$, and $\sigma \in \mathsf{BudMap}_R(d, \alpha \to \beta)$, $d \to \beta\sigma$ has type F_0^2 or C, since $x\sigma \in \mathsf{Bud}(R_{\mathrm{C}})$ for any $x \in \mathsf{V}(\beta)$. Thus, Q is quasi-standard and semi-constructor. By the definition of $\mathsf{Normalize}$, all rules produced in functions $\mathsf{Determinize}$ have type F_0^2 or C. Thus, R' is quasi-standard and semi-constructor.
(2) Since no rule is deleted in $\mathsf{AddShortcut}$, $\mathsf{Rhs}(d, Q_{\mathrm{C}}) \neq \emptyset$. By the definition of $\mathsf{Normalize}$, $\mathsf{Rhs}(d, R'_{\mathrm{C}}) \neq \emptyset$. □

Lemma 20 Let $Q = \mathsf{AddShortcut}(R)$ and $R' = \mathsf{Determinize}(Q)$.

(1) $\to_{Q_{\mathrm{rg}}} \subseteq \to^{+}_{R'_{\mathrm{rg}}}$.
(2) $\leftrightarrow_Q \subseteq \downarrow_R$.
(3) $\to_{R'} \subseteq \downarrow_Q$.
(4) Q and R' are confluent.

Proof

(1) If $|\mathsf{Rhs}(d, Q_{\mathrm{C}})| \leq 1$ for every d then (1) obviously holds since $R' = Q$. If there exists d such that $|\mathsf{Rhs}(d, Q_{\mathrm{C}})| > 1$ then (1) holds by Lemma 16 (2).
(2) Assume that $d \to \beta\sigma$ is added as a new rule by $\mathsf{AddShortcut}$ where $\sigma \in \mathsf{BudMap}_R(d, \alpha \to \beta)$. By Lemma 11 (3), $d \downarrow_R \beta\sigma$ holds.
(3) If $|\mathsf{Rhs}(d, Q_{\mathrm{C}})| \leq 1$ for every d then this lemma holds since $R' = Q$. If there exists d such that $|\mathsf{Rhs}(d, Q_{\mathrm{C}})| > 1$ then (3) holds by Lemma 16 (3).
(4) Q is confluent by (2) and $Q \supseteq R$. R' is also confluent by Lemma 16 (4). □

By Lemmata 19 (1) and 20 (4), every TRS produced by M is confluent, quasi-standard, and semi-constructor if so is an input TRS.

Corollary 21 $\mathsf{M}(R)$ is confluent.

Now, we show that M is terminating. For this purpose, we need the following definition and lemma.

Definition 22 We define $@(R)$ as $(@_1(R), @_2(R))$, where

$$@_1(R) = (|F_0|^2 - |R_{\mathrm{F}_0^2}|) + (|F_0| - |R_{\mathrm{C}}|),$$
$$@_2(R) = \{\#(\beta) \mid \alpha \to \beta \in R_{\mathrm{C}}\}_{\mathrm{m}}$$

Note that if $|\mathsf{Rhs}(d, R_{\mathrm{C}})| \leq 1$ for each d then $@_1(R) \geq 0$. In order to compare $@(R)$ and $@(R')$, we use lexicographic order $<_{lex}$.

Lemma 23

(1) $\mathsf{AddShortcut}$ is terminating.
(2) $\mathsf{Determinize}$ is terminating.

Proof

(1) By Lemma 11 (1).
(2) If there exists d such that $|\mathsf{Rhs}(d, R_{\mathrm{C}})| > 1$, then by Lemma 16 (1), the size $@_2$ strictly decreases in each call of $\mathsf{Determinize}$: $@_2(R) \gg @_2(Q)$ where $Q = (R \setminus \{d \to s \mid s \in \mathsf{Rhs}(d, R_{\mathrm{C}})\}) \cup \mathsf{Normalize}(d, R_{\mathrm{C}})$. □

Lemma 24 M is terminating.

Proof By Lemma 23, $\mathsf{AddShortcut}$ and $\mathsf{Determinize}$ are terminating. Let $Q = (\mathsf{Determinize} \circ \mathsf{AddShortcut})(R)$. Then, for each d, $|\mathsf{Rhs}(d, Q_{\mathrm{C}})| \leq 1$ holds. If $R = Q$ then M is obviously terminating. So, consider the case of $R \neq Q$. Let $R' = (\mathsf{Determinize} \circ \mathsf{AddShortcut})(Q)$. Then, for each d, $|\mathsf{Rhs}(d, R'_{\mathrm{C}})| \leq 1$ also holds. If $Q = R'$ then M is obviously terminating. In the case of $Q \neq R'$, it is sufficient to show that $@(Q) > @(R')$. Since every rule in $Q_{\mathrm{F}_0^2}$ is never deleted by functions $\mathsf{AddShortcut}$ and $\mathsf{Determinize}$, $|F_0|^2 - |Q_{\mathrm{F}_0^2}| \geq |F_0|^2 - |R'_{\mathrm{F}_0^2}|$ holds. Moreover, $|F_0| - |Q_{\mathrm{C}}| \geq |F_0| - |R'_{\mathrm{C}}|$ by Lemma 19 (2). Thus, $@_1(Q) \geq @_1(R') \geq 0$ holds. If $@_1(Q) = @_1(R')$ then $Q_{\mathrm{F}_0^2} = R'_{\mathrm{F}_0^2}$ and $|Q_{\mathrm{C}}| = |R'_{\mathrm{C}}|$, so that $|\mathsf{Rhs}(d, Q_{\mathrm{C}})| = |\mathsf{Rhs}(d, R'_{\mathrm{C}})| \leq 1$ for every d. By $Q \neq R'$, there exists d such that $\mathsf{Rhs}(d, Q_{\mathrm{C}}) \neq \mathsf{Rhs}(d, R'_{\mathrm{C}})$. Let $\mathsf{Rhs}(d, Q_{\mathrm{C}}) = \{d \to t\}$ and $\mathsf{Rhs}(d, R'_{\mathrm{C}}) = \{d \to t'\}$ for some t, t' where $t \neq t'$. This implies that $\mathsf{Determinize}$ deletes $d \to t$ and produces $d \to t'$, so that $\#(t) > \#(t')$ holds as we described in the proof of Lemma 16 (1). Thus, $@_2(Q) \gg @_2(R')$, so that $@(Q) > @(R')$, as claimed. □

Now, we show that two constants are joinable in R iff they are joinable in $\mathsf{M}(R)_{\mathrm{rg}}$. For this purpose, we need the following lemma.

Lemma 25

(1) $\to_{R_{\mathrm{rg}}} \subseteq \to^+_{\mathsf{M}(R)_{\mathrm{rg}}}$.

(2) For any $d, \alpha \to \beta \in \mathsf{M}(R)_{\mathrm{nrg}}$, and $\sigma \in \mathsf{BudMap}_{\mathsf{M}(R)}(d, \alpha \to \beta)$, $d \to^+_{\mathsf{M}(R)_{\mathrm{rg}}} \beta\sigma$ holds.

(3) For any d and s, if $d \to^*_R s$ then $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} s$.

(4) $\to_{\mathsf{M}(R)} \subseteq \downarrow_R$.

Proof

(1) For any confluent, quasi-standard, and semi-constructor TRS R', let $Q = \mathsf{AddShortcut}(R')$ and $R'' = \mathsf{Determinize}(Q)$. Since R'' is confluent, quasi-standard, and semi-constructor by Lemmata 19 (1) and 20 (4), it is sufficient to show that $\to_{R'_{\mathrm{rg}}} \subseteq \to^+_{R''_{\mathrm{rg}}}$. Obviously, $\to_{R'_{\mathrm{rg}}} \subseteq \to_{Q_{\mathrm{rg}}}$. By Lemma 20 (1), $\to_{R'_{\mathrm{rg}}} \subseteq \to^+_{R''_{\mathrm{rg}}}$ holds, as claimed.

(2) Since M is terminating, there exists Q such that $Q = \mathsf{AddShortcut}(\mathsf{M}(R))$ and $\mathsf{M}(R) = \mathsf{Determinize}(Q)$. By the definition of $\mathsf{AddShortcut}$, $d \to_{Q_{\mathrm{rg}}} \beta\sigma$ holds. By Lemma 20 (1), $d \to^+_{\mathsf{M}(R)_{\mathrm{rg}}} \beta\sigma$ holds.

(3) Let $\gamma : d \to^*_R s$. We show by induction on the number of applications of non-right-ground rules in γ. Basis: If $d \to^*_{R_{\mathrm{rg}}} s$ then $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} s$ by (1). Induction step: Let $\gamma : d \to^*_R t[\alpha\theta]_p \to t[\beta\theta]_p \to^*_{R_{\mathrm{rg}}} s$ where $p \in \mathcal{O}(t), \alpha \to \beta \in R_{\mathrm{nrg}}$. By the induction hypothesis and (1), $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} t[\alpha\theta]_p \to t[\beta\theta]_p \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} s$.

Case(a) If $p = \varepsilon$ then $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \alpha\theta \to \beta\theta$. By Lemma 11 (2), there exists $\sigma \in \mathsf{BudMap}_{\mathsf{M}(R)}(d, \alpha \to \beta)$ such that $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \alpha'\sigma \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \alpha\theta$ and $\beta\sigma \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \beta\theta$. By (2), $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \beta\sigma$. Thus $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \beta\theta$.

Case (b) If $p \neq \varepsilon$ then there exists $s' \in \mathsf{Bud}(\mathsf{M}(R)_{\mathrm{C}})$ such that $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} t[s']_p$ and $s' \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \alpha\theta$ by Lemma 9. Since $\mathsf{root}(s') \in D$, s' must be a constant, so that we can use the same proof as that of case (a) to show $s' \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \beta\theta$. Thus, $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} t[s']_p \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} t[\beta\theta]_p \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} s$.

(4) For any confluent, quasi-standard, and semi-constructor TRS R', let $Q = \mathsf{AddShortcut}(R')$ and $R'' = \mathsf{Determinize}(Q)$. Since R'' is confluent, quasi-standard, and semi-constructor by Lemmata 19 (1) and 20 (4), it is sufficient to show that $\to_{R''} \subseteq \downarrow_{R'}$. By Lemma 20 (2), (3), $\to_{R''} \subseteq \leftrightarrow^*_{R'}$ holds. By confluence of R', $\to_{R''} \subseteq \downarrow_{R'}$ holds, as claimed. □

By Lemma 25 (3), (4), we have the following corollary.

Corollary 26 $c \downarrow_R d$ iff $c \downarrow_{\mathsf{M}(R)_{\mathrm{rg}}} d$.

4.3 Auxiliary terms

We have shown that all rewrite sequences from every constant in R (i.e., $d \to^*_R s$) can be simulated using only right-ground rules (i.e., $d \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} s$). Now, we want to show that this property still holds for rewrite sequences from an arbitrary term. For this purpose, we need the notion of auxiliary terms. The following algorithm Aux produces the set of auxiliary terms of s. We use $\mathsf{Aux}(s)$ to denote the set.

function $\mathsf{Aux}(s)$
  $\Delta := \{s\}$;
  for each $p \in \mathcal{O}_{D \setminus F_0}(s)$,
      $\alpha \to \beta \in \mathsf{M}(R)_{\mathrm{nrg}}$,
    $\sigma \in \mathsf{BudMap}_{\mathsf{M}(R)}(s_{|p}, \alpha \to \beta)$ **do**
    $\Delta := \Delta \cup \mathsf{Aux}(s[\beta\sigma]_p)$;
  return Δ

Example 27 For TRS $\mathsf{M}(R_{\mathrm{e}})$ of Example 17, $\mathsf{Aux}(\mathsf{not}(\mathsf{nand}(\mathsf{true}, \mathsf{true}))) =$

$\{\mathsf{not}(\mathsf{nand}(\mathsf{true},\mathsf{true})), \mathsf{not}(\mathsf{not}(\mathsf{and}(\mathsf{true},\mathsf{true})))\}$.

Definition 28

$$\mathsf{height}_{\mathrm{D}}(s) = \begin{cases} w_f + \mathsf{max}\{\mathsf{height}_{\mathrm{D}}(s_i) \mid 1 \le i \le n\} \\ \text{(if } s = f(s_1, \cdots, s_n), n > 0) \\ 0 \text{ (if } s \in X \cup F_0) \end{cases}$$

Here, $w_f = 1 + 2\mathsf{max}\{\mathsf{height}(\beta) \mid \alpha \to \beta \in \mathsf{M}(R)\}$ if f is a defined symbol, otherwise $w_f = 1$. We define $\mathsf{H}_{\mathrm{D}}(s) = \{\mathsf{height}_{\mathrm{D}}(s_{|u}) \mid u \in \mathcal{O}(s)\}_{\mathrm{m}}$, which is the multiset of the $\mathsf{height}_{\mathrm{D}}$-values of all the subterms of s. For TRS $\mathsf{M}(R_{\mathrm{e}})$ of Example 17, $w_{\mathsf{nand}} = 5$ and $\mathsf{H}_{\mathrm{D}}(\mathsf{nand}(\mathsf{not}(x), x)) = \{0, 0, 1, 6\}_{\mathrm{m}}$.

Let $\ll$ be the multiset extension of the usual relation $<$ on $\mathbf{N}$ and $\underline{\ll}$ be $\ll \cup =$. The relation $\ll$ is closed under context.

Lemma 29 For any s, t, the following propositions hold.

(1) If $\mathsf{height}_{\mathrm{D}}(s) < \mathsf{height}_{\mathrm{D}}(t)$ then $\mathsf{H}_{\mathrm{D}}(s) \ll \mathsf{H}_{\mathrm{D}}(t)$.

(2) If $\mathsf{H}_{\mathrm{D}}(s) \ll \mathsf{H}_{\mathrm{D}}(t)$ then $\mathsf{height}_{\mathrm{D}}(s) \le \mathsf{height}_{\mathrm{D}}(t)$.

(3) For any r and position $u \in \mathcal{O}(r)$, if $\mathsf{H}_{\mathrm{D}}(s) \ll \mathsf{H}_{\mathrm{D}}(t)$ then $\mathsf{H}_{\mathrm{D}}(r[s]_u) \ll \mathsf{H}_{\mathrm{D}}(r[t]_u)$.

Proof

(1) For any subterm s' of s, $\mathsf{height}_{\mathrm{D}}(s') \le \mathsf{height}_{\mathrm{D}}(s)$. By $\mathsf{height}_{\mathrm{D}}(s) < \mathsf{height}_{\mathrm{D}}(t)$, $\mathsf{H}_{\mathrm{D}}(s) \ll \mathsf{H}_{\mathrm{D}}(t)$ holds.

(2) To the contrary, we assume that $\mathsf{height}_{\mathrm{D}}(s) > \mathsf{height}_{\mathrm{D}}(t)$. By (1), $\mathsf{H}_{\mathrm{D}}(s) \gg \mathsf{H}_{\mathrm{D}}(t)$, a contradiction.

(3) Let $\hat{s} = f(r_1, \cdots, r_{i-1}, s, r_{i+1}, \cdots, r_n)$ and $\hat{t} = f(r_1, \cdots, r_{i-1}, t, r_{i+1}, \cdots, r_n)$ where $f \in F_n$ and $i \in \{1, \cdots, n\}$. It suffices to show that $\{\mathsf{height}_{\mathrm{D}}(\hat{s})\}_{\mathrm{m}} \sqcup \mathsf{H}_{\mathrm{D}}(s) \ll \{\mathsf{height}_{\mathrm{D}}(\hat{t})\}_{\mathrm{m}} \sqcup \mathsf{H}_{\mathrm{D}}(t)$. By (2), $\mathsf{height}_{\mathrm{D}}(s) \le \mathsf{height}_{\mathrm{D}}(t)$, so $\mathsf{height}_{\mathrm{D}}(\hat{s}) \le \mathsf{height}_{\mathrm{D}}(\hat{t})$ holds. If $\mathsf{height}_{\mathrm{D}}(\hat{s}) < \mathsf{height}_{\mathrm{D}}(\hat{t})$ then $\mathsf{H}_{\mathrm{D}}(\hat{s}) \ll \mathsf{H}_{\mathrm{D}}(\hat{t})$ holds by (1). If $\mathsf{height}_{\mathrm{D}}(\hat{s}) = \mathsf{height}_{\mathrm{D}}(\hat{t})$ then $\mathsf{H}_{\mathrm{D}}(\hat{s}) \ll \mathsf{H}_{\mathrm{D}}(\hat{t})$ holds by $\mathsf{H}_{\mathrm{D}}(s) \ll \mathsf{H}_{\mathrm{D}}(t)$. □

Lemma 30 For any $s \in G$, $p \in \mathcal{O}_{D \setminus F_0}(s)$, $\alpha \to \beta \in \mathsf{M}(R)_{\mathrm{nrg}}$, and $\sigma \in \mathsf{BudMap}_{\mathsf{M}(R)}(s_{|p}, \alpha \to \beta)$, the following conditions hold.

(1) $s[\beta\sigma]_p$ is a ground term.

(2) $\mathsf{H}_{\mathrm{D}}(s[\beta\sigma]_p) \ll \mathsf{H}_{\mathrm{D}}(s)$.

(3) $s[\beta\sigma]_p \downarrow_{\mathsf{M}(R)} s$.

Proof

(1) Since s is a ground term, $\beta\sigma$ is a ground term by the definition of BudMap.

(2) By Lemma 29 (1), (3), it suffices to show that $\mathsf{height}_{\mathrm{D}}(\beta\sigma) < \mathsf{height}_{\mathrm{D}}(s_{|p})$. Let $f(s_1, \cdots, s_n)$ be $s_{|p}$ where $f \in D$, then

$$\begin{aligned} & \mathsf{height}_{\mathrm{D}}(\beta\sigma) \\ \le\ & \mathsf{height}(\beta) + \\ & \mathsf{max}\{\mathsf{height}_{\mathrm{D}}(x\sigma) \mid x \in \mathsf{V}(\beta)\} \\ \le\ & \mathsf{height}(\beta) + \\ & \mathsf{max}(\{\mathsf{height}_{\mathrm{D}}(s_i) \mid 1 \le i \le n\} \\ & \cup\{\mathsf{height}(\beta) \mid \alpha \to \beta \in \mathsf{M}(R)_{\mathrm{C}}\}) \\ <\ & 1 + \\ & 2\mathsf{max}\{\mathsf{height}(\beta) \mid \alpha \to \beta \in \mathsf{M}(R)\} \\ & +\mathsf{max}\{\mathsf{height}_{\mathrm{D}}(s_i) \mid 1 \le i \le n\} \\ =\ & \mathsf{height}_{\mathrm{D}}(f(s_1, \cdots, s_n)). \end{aligned}$$

(3) By Lemma 11 (3). □

Lemma 31 For any $s, r \in G, p \in \mathcal{O}_{D \setminus F_0}(s), \alpha \to \beta \in R_{\mathrm{nrg}}$, and θ, if $s \to^*_{R_{\mathrm{rg}}} r[\alpha\theta]_p \to_{R_{\mathrm{nrg}}} r[\beta\theta]_p$ then there exists $s' \in \mathsf{Aux}(s)$ such that $s' \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} r[\beta\theta]_p$.

Proof If $s \in F_0$ then we choose s' as s by Lemma 25 (3). So, we consider the case of $s \notin F_0$.

(a) Case of $p = \varepsilon$: By Lemmata 11 (2) and 25 (1), there exists $\sigma \in \mathsf{BudMap}_{\mathsf{M}(R)}(s, \alpha \to \beta)$ such that $\alpha'\sigma \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \alpha\theta$ and $\beta\sigma \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \beta\theta$. Hence, we choose $\beta\sigma$ as s'.

(b) Case of $p \neq \varepsilon$: if there exists $v \in \mathcal{R}(\gamma)$ such that $v < p$, then there exists $s_0 \in \mathsf{Bud}(R_{\mathrm{C}})$ such that $s \to^*_{R_{\mathrm{rg}}} r[s_0]_p$ and $s_0 \to^*_{R_{\mathrm{rg}}} \alpha\theta$ by Lemma 9. Since $\mathsf{root}(s_0) \in D$, s_0 must be a constant and $s_0 \to^*_R \beta\theta$, so that $s_0 \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \beta\theta$ by Lemma 25 (3). By Lemma 25 (1), $s \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} r[s_0]_p$ holds. Thus, we choose s as s'. If there does not exist $v \in \mathcal{R}(\gamma)$ such that $v < p$ then if $s_{|p} \in F_0$ then we choose s. Otherwise, by Lemmata 11 (2) and 25 (1), there exists $\sigma \in \mathsf{BudMap}_{\mathsf{M}(R)}(s_{|p}, \alpha \to \beta)$ such that $\alpha'\sigma \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \alpha\theta$ and $\beta\sigma \to^*_{\mathsf{M}(R)_{\mathrm{rg}}} \beta\theta$. Hence, we choose $s[\beta\sigma]_p$ as s'. □

Lemma 32 Aux is terminating.

Proof By Lemmata 11 (1) and 30 (2). □

Lemma 33 For any $s \in \mathsf{Aux}(t)$, $\mathsf{Aux}(s) \subseteq \mathsf{Aux}(t)$.

Proof We prove by induction on $\mathsf{H}_{\mathrm{D}}(t)$. Basis: The proof is obvious by $\mathsf{Aux}(t) = \{t\}$, since t is a constant or a variable. Induction step: If $s = t$ then obvious. Otherwise, $s \in \mathsf{Aux}(t[\beta\sigma]_p) \subseteq \mathsf{Aux}(t)$ for some $p \in \mathcal{O}_{D \setminus F_0}(t), \alpha \to \beta \in \mathsf{M}(R)_{\mathrm{nrg}}$, and $\sigma \in \mathsf{BudMap}_{\mathsf{M}(R)}(t_{|p}, \alpha \to \beta)$ by the definition of Aux. By Lemma 30 (2) and the induction hypothesis, $\mathsf{Aux}(s) \subseteq \mathsf{Aux}(t[\beta\sigma]_p)$ holds. Thus, this lemma holds. □

Corollary 34 For any ground term s,

(1) For any $s' \in \mathsf{Aux}(s)$, s' is a ground term, $\mathsf{H}_{\mathrm{D}}(s') \leqq \mathsf{H}_{\mathrm{D}}(s)$, and $s' \downarrow_{\mathsf{M}(R)} s$.

(2) If $s \to_R^* t$ then there exists $s' \in \mathsf{Aux}(s)$ such that $s' \to_{\mathsf{M}(R)_{\mathrm{rg}}}^* t$.

Proof

(1) We show by induction on $\mathsf{H}_{\mathrm{D}}(s)$. Basis: The proof is obvious by $\mathsf{Aux}(s) = \{s\}$, since s is a constant. Induction step: If $s = s'$ then obvious. Otherwise, $s' \in \mathsf{Aux}(s[\beta\sigma]_p) \subseteq \mathsf{Aux}(s)$ for some $p \in \mathcal{O}_{D\setminus F_0}(s), \alpha \to \beta \in \mathsf{M}(R)_{\mathrm{nrg}}$, and $\sigma \in \mathsf{BudMap}_{\mathsf{M}(R)}(s_{|p}, \alpha \to \beta)$ by the definition of Aux. By Lemma 30, $s[\beta\sigma]_p$ is ground, $\mathsf{H}_{\mathrm{D}}(s[\beta\sigma]_p) \ll \mathsf{H}_{\mathrm{D}}(s)$, and $s[\beta\sigma]_p \downarrow_{\mathsf{M}(R)} s$. By the induction hypothesis, s' is ground, $\mathsf{H}_{\mathrm{D}}(s') \leqq \mathsf{H}_{\mathrm{D}}(s[\beta\sigma]_p)$, and $s' \downarrow_{\mathsf{M}(R)} s[\beta\sigma]_p$. Since $\mathsf{M}(R)$ is confluent by Corollary 21, $s' \downarrow_{\mathsf{M}(R)} s$.

(2) Let $\gamma : s \to_R^* t$. We show by induction on the number of applications of non-right-ground rules in γ. Basis: If $s \to_{R_{\mathrm{rg}}}^* t$ then $s \to_{\mathsf{M}(R)_{\mathrm{rg}}}^* t$ by Lemma 25(1). Induction step: Let $\gamma : s \to_R^* r[\alpha\theta]_p \to r[\beta\theta]_p \to_{R_{\mathrm{rg}}}^* t$ where $p \in \mathcal{O}(r), \alpha \to \beta \in R_{\mathrm{nrg}}$. By the induction hypothesis and Lemma 25 (1), there exists $s'' \in \mathsf{Aux}(s)$ such that $s'' \to_{\mathsf{M}(R)_{\mathrm{rg}}}^* r[\alpha\theta]_p \to r[\beta\theta]_p \to_{\mathsf{M}(R)_{\mathrm{rg}}}^* t$. By Lemma 31 and $\mathsf{M}(\mathsf{M}(R)) = \mathsf{M}(R)$, there exists $s''' \in \mathsf{Aux}(s'')$ such that $s''' \to_{\mathsf{M}(R)_{\mathrm{rg}}}^* r[\beta\theta]_p \to_{\mathsf{M}(R)_{\mathrm{rg}}}^* t$. By Lemma 33, $s''' \in \mathsf{Aux}(s)$. Thus, we can choose s''' as s'. □

We call s' in Corollary 34 (2) an *auxiliary term* of (s,t). This term will be used to transform non-right-ground rewrite sequences to right-ground rewrite sequences.

Example 35 For the rewrite sequence $\mathsf{not}(\mathsf{nand}(\mathsf{true},\mathsf{true})) \to_{\mathrm{rg}}^*$ $\mathsf{not}(\mathsf{nand}(\mathsf{not}(\mathsf{false}),\mathsf{not}(\mathsf{false}))) \to$ $\mathsf{not}(\mathsf{not}(\mathsf{and}(\mathsf{not}(\mathsf{false}),\mathsf{not}(\mathsf{false}))))$, we can choose $\mathsf{not}(\mathsf{not}(\mathsf{and}(\mathsf{true},\mathsf{true}))) \in$ $\mathsf{Aux}(\mathsf{not}(\mathsf{nand}(\mathsf{true},\mathsf{true})))$ and $\mathsf{not}(\mathsf{not}(\mathsf{and}(\mathsf{true},\mathsf{true}))) \to_{\mathrm{rg}}$ $\mathsf{not}(\mathsf{not}(\mathsf{and}(\mathsf{not}(\mathsf{false}),\mathsf{not}(\mathsf{false}))))$.

4.4 Joinability for Confluent Semi-constructor TRSs

Lemma 36 For any ground terms s and t, $s \downarrow_R t$ iff there exists $s' \in \mathsf{Aux}(s), t' \in \mathsf{Aux}(t)$ such that $s' \downarrow_{\mathsf{M}(R)_{\mathrm{rg}}} t'$.

Proof The only-if-part holds by Corollary 34 (2). Proof of the if-part. By Corollary 34 (1), $s \leftrightarrow_{\mathsf{M}(R)}^* t$ holds. By Lemma 25 (4), $s \leftrightarrow_R^* t$ holds. By confluence of R, $s \downarrow_R t$ holds. □

By Lemma 32, 36 and decidablity of $s' \downarrow_{\mathsf{M}(R)_{\mathrm{rg}}} t'$ [10], $s \downarrow_R t$ is decidable for ground terms s and t. If s or t is non-ground, $s \downarrow_R t$ is equivalent to $s\sigma \downarrow_R t\sigma$ where $\sigma : \mathsf{V}(s) \cup \mathsf{V}(t) \to F_0'$ is a bijection and F_0' is a set of constants which do not appear in R. Thus, we have the following theorem.

Theorem 37 Joinability for confluent semi-constructor TRSs is decidable.

By confluence, we have the following corollary too.

Corollary 38 The word problem for confluent semi-constructor TRSs is decidable.

5. Decidability of Joinability for Confluent Semi-monadic TRSs

Definition 39 A rewrite rule $\alpha \to \beta$ is *monadic* if $\mathsf{height}(\beta) \le 1$, *semi-monadic* if for every proper subterm β' of β, β' is ground or a variable.

We show that joinability for confluent semi-monadic TRSs is decidable. Semi-monadic TRSs can be transformed to monadic and standard TRSs using the technique described in Section 4.1. This transformation preserves confluence and joinability. Henceforth, we assume that TRS R is confluent, monadic, and standard.

Lemma 40 For any $\gamma : s \to_{R_{\mathrm{rg}}}^* t$ and $u \in \mathcal{O}(t)$, if there exists $v \in \mathcal{R}(\gamma)$ such that $v < u$, then there exists d such that $s \to_{R_{\mathrm{rg}}}^* t[d]_u$ and $d \to_{R_{\mathrm{rg}}}^* t_{|u}$.

Proof The proof is similar to that of Lemma 9. Consider the last v-reduction in γ such that $v < u$. That is, $\gamma : s \to_{R_{\mathrm{rg}}}^* r \xrightarrow{v}_{R_{\mathrm{rg}}} t[r']_u$ and $r' \to_{R_{\mathrm{rg}}}^* t_{|u}$ for some r, r'. Here, $r = r[\alpha\theta]_v$ and $t[r']_u = r[\beta]_v$ hold for some right-ground rule $\alpha \to \beta$ and θ. Since $v < u$, $\beta \notin F_0$ holds. Since R is monadic, $\mathsf{height}(\beta) = 1$ holds. Let $u = vw$, then $r' = \beta_{|w} \in F_0$. Thus, we can choose r' as d. □

Definition 41 Let $\alpha \to \beta \in R_{\mathrm{nrg}}$ and $\alpha' = \mathsf{linearize}(\alpha)$. Then, $\sigma : \mathsf{V}(\alpha') \to \mathsf{Psub}(s) \cup F_0$ is called a constant substitution for s and $\alpha \to \beta$ if $s \to_{R_{\mathrm{rg}}}^* \alpha'\sigma$ and σ is joinability preserving under relation $\equiv$ for R_{rg}. Let $\mathsf{ConstantMap}_R(s, \alpha \to \beta)$ be the set of such constant substitutions.

Lemma 42 Let $\alpha \to \beta \in R_{\mathrm{nrg}}$ and $s \to_{R_{\mathrm{rg}}}^* \alpha\theta$. Then, there exists $\sigma \in \mathsf{ConstantMap}_R(s, \alpha \to \beta)$ such that $\alpha'\sigma \to_{R_{\mathrm{rg}}}^* \alpha\theta$, and $\beta\sigma \to_{R_{\mathrm{rg}}}^* \beta\theta$.

Proof This proof is similar to that of Lemma 11 (2). Let $\gamma: s \rightarrow^*_{R_{\mathrm{rg}}} \alpha\theta$. For any $u \in \mathcal{O}_X(\alpha)$, if there exists $v \in \mathcal{R}(\gamma)$ such that $v < u$, then there exists $d \in F_0$ such that $s \rightarrow^*_{R_{\mathrm{rg}}} \alpha\theta[d]_u$ and $d \rightarrow^*_{R_{\mathrm{rg}}} \alpha\theta_{|u}$ by Lemma 40. Otherwise, $s_{|u} \rightarrow^*_{R_{\mathrm{rg}}} \alpha\theta_{|u}$. Hence, there exists $\sigma : \mathsf{V}(\alpha') \rightarrow \mathsf{Psub}(s) \cup F_0$ such that $s \rightarrow^*_{R_{\mathrm{rg}}} \alpha'\sigma \rightarrow^*_{R_{\mathrm{rg}}} \alpha\theta$ and $\beta\sigma \rightarrow^*_{R_{\mathrm{rg}}} \beta\theta$. For any $x \in \mathsf{V}(\alpha)$ and $x' \in \mathsf{V}(\alpha')$, if $x' \equiv x$ then $x\sigma \rightarrow^*_{R_{\mathrm{rg}}} x\theta$ and $x'\sigma \rightarrow^*_{R_{\mathrm{rg}}} x\theta$, so that σ is a constant substitution. □

We can define a function similar to $\mathsf{AddShortcut}$ in the algorithm M to produce new rewrite rules of form $d \rightarrow \beta\sigma$ where $\sigma \in \mathsf{ConstantMap}_R(d, \alpha \rightarrow \beta)$. For every shortcut rule $d \rightarrow \beta\sigma$ produced by this new $\mathsf{AddShortcut}$, the height of $\beta\sigma$ is at most 1 by the definition of $\mathsf{ConstantMap}_R$. So, the number of shortcut rules is finite. Thus, we do not need to apply $\mathsf{Determinize}$ in the case of monadic TRSs. Let $R' = \mathsf{AddShortcut}(R)$. Then, we have the following lemma.

Lemma 43 $c \downarrow_R d$ iff $c \downarrow_{R'_{\mathrm{rg}}} d$.

The proof is similar to that of Section 4.2, so omitted. By this lemma, joinability of two constants for confluent monadic and standard TRSs reduces to that of confluent right-ground TRSs which is decidable [10]. In Section 4.3, we have described how to extend the joinability checking algorithm for two constants to that for arbitrary two terms. The same technique can be applied to this case. There is an alternative method for this lifting. That is, joinability checking for two ground terms s and t is reducible to that for two new constants c and d by simply adding new ground rules $c \rightarrow s$ and $d \rightarrow t$, which are semi-monadic. Thus, we have the following theorem.

Theorem 44 Joinability for confluent semi-monadic TRSs is decidable.

By confluence, we have the following corollary too.

Corollary 45 The word problem for confluent semi-monadic TRSs is decidable.

Note that joinability is undecidable for flat TRSs which are included in the class of monadic TRSs [4]. Thus, the confluence condition in this theorem can not be removed.

6. Undecidability of Reachability for Confluent Monadic TRSs

In this section, we show that reachability for confluent monadic TRSs is undecidable whereas the joinability is decidable.

Theorem 46 Reachability for confluent monadic TRSs are undecidable.

Proof [sketch] The proof is by a reduction from the PCP. Let $P = \{\langle u_i, v_i\rangle \in \Sigma^* \times \Sigma^* \mid 1 \leq i \leq k\}$ be an instance of the PCP. The corresponding TRS R_P is constructed as follows: Let $F = F_0 \cup F_1 \cup F_2$ where $F_0 = \{\mathsf{b}, \mathsf{c}, \mathsf{d}, \$\}$, $F_1 = \Sigma$, $F_2 = \{\mathsf{f}, \mathsf{g}\}$, and $R_P = \{e \rightarrow a(e),\ e \rightarrow a(\$) \mid e \in \{\mathsf{c}, d\},\ a \in \Sigma\} \cup \{\mathsf{f}(x, x) \rightarrow \mathsf{g}(x, x)\} \cup \{\mathsf{g}(u_i(x), v_i(y)) \rightarrow \mathsf{g}(x, y) \mid 1 \leq i \leq k\} \cup \{h(x_1, \cdots, x_n) \rightarrow \mathsf{b} \mid h \in F,\ x_1, \cdots, x_n \textit{ are pairwise distinct variables and } n = \mathsf{ar}(h)\}$. Here, $u(x)$ is an abbreviation for $a_1(a_2(\cdots a_k(x)))$ where $u = a_1 a_2 \cdots a_k \in \Sigma^*$. By the last rules, R_P is confluent since every non-variable term can reach b and every right-hand side in R_P is not a variable. For R_P, $\mathsf{f}(\mathsf{c}, \mathsf{d}) \rightarrow^* \mathsf{g}(\$, \$)$ iff there exists a sequence $a_1 \cdots a_n \in \Sigma^+$ such that $\mathsf{f}(\mathsf{c}, \mathsf{d}) \rightarrow^{2n+2} \mathsf{f}(a_1 \cdots a_n(\$), a_1 \cdots a_n(\$)) \rightarrow \mathsf{g}(a_1 \cdots a_n(\$), a_1 \cdots a_n(\$)) \rightarrow^+ \mathsf{g}(\$, \$)$ where $a_1 \cdots a_n = u_{i_1} \cdots u_{i_m} = v_{i_1} \cdots v_{i_m}$ for some sequence of indexes $i_1 \cdots i_m \in \{1, \cdots, k\}^+$. Thus, $\mathsf{f}(\mathsf{c}, \mathsf{d}) \rightarrow^* \mathsf{g}(\$, \$)$ iff P has a solution. Hence, this theorem holds. □

7. Concluding Remarks

In this paper, we have shown that joinability is undecidable for linear semi-constructor TRSs, but it is decidable both for confluent semi-constructor TRSs and for confluent semi-monadic TRSs. The latter result shows the decidability of joinability for possibly non-right-linear TRSs. To our knowledge, such attempts were very few so far. Moreover, we have shown that reachability is undecidable for confluent monadic TRSs. Quite recently, we obtained the undecidability result of reachability for confluent semi-constructor TRSs [7]. Borders between decidable and undecidable classes of joinability, reachability, and the word problems are shown in **Fig. 1**, **Fig. 2** and **Fig. 3**, respectively. Using the decidability result of joinability for confluent semi-constructor TRSs, our forthcoming paper shows that this unification problem is decidable [5]. However, unification for confluent monadic TRSs has been shown to be undecidable [6].

Quite recently, we found that confluence is undecidable for semi-constructor TRSs [7], but some sufficient conditions to ensure confluence of semi-constructor TRSs are known: the class of semi-constructor TRSs is a subclass

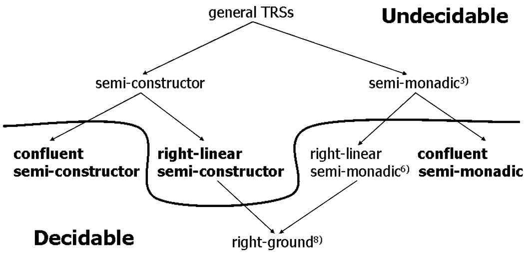

Fig. 1 Border between decidable and undecidable classes of joinability.

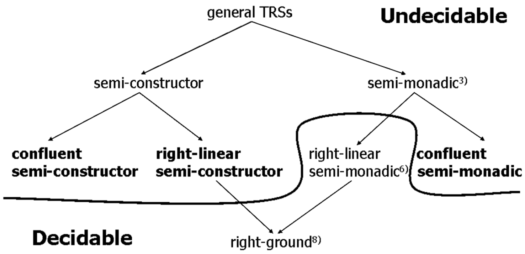

Fig. 2 Border between decidable and undecidable classes of reachability.

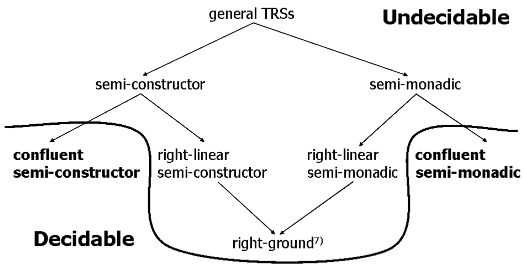

Fig. 3 Border between decidable and undecidable classes of the word problem.

of strongly weight-preserving TRSs, for which some sufficient conditions to ensure confluence are given in Ref. 3).

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Appendix

A.1 Proof of Lemma 6

To show Lemma 6, we need the following definition and lemma.

Definition 47 Let $s[t_i/d_1, \cdots, t_m/d_m]$ be the term obtained from s by replacing all occurrences of d_i by t_i where $i \in \{1, \cdots, m\}$.

Lemma 48 Let $R_{k+1} = (R_k \setminus \{\alpha \to \beta\}) \cup \{\alpha \to \beta[d_1, \cdots, d_m]_{(u_1, \cdots, u_m)}\} \cup \{d_i \to \beta_{|u_i} \mid 1 \leq i \leq m\}$ where $d_1, \cdots, d_m$ are new pairwise distinct constants which do not appear in R_k or T, and $\{u_1, \cdots, u_n\} \subseteq \mathcal{O}_G(\beta)$. Then, the following propositions hold.

(1) $\rightarrow_{R_k} \subseteq \rightarrow^+_{R_{k+1}}$.
(2) Let $\Theta = [\beta_{|u_1}/d_1, \cdots, \beta_{|u_m}/d_m]$. If $s \rightarrow^*_{R_{k+1}} t$ then $s\Theta \rightarrow^*_{R_k} t\Theta$.
(3) If R_k is confluent and semi-constructor then R_{k+1} is confluent and semi-constructor.

Proof
(1) It suffices to show that $\alpha \rightarrow^+_{R_{k+1}} \beta$. The proof is obvious by $\alpha \rightarrow_{R_{k+1}} \beta[d_1, \cdots, d_m]_{(u_1,\cdots,u_m)} \rightarrow^m_{R_{k+1}} \beta$.
(2) It suffices to show that if $s \overset{u}{\rightarrow}_{R_{k+1}} t$ then $s\Theta \rightarrow_{R_k} t\Theta$ or $s\Theta = t\Theta$. If $s \rightarrow_{R_k} t$ then $s\Theta \rightarrow_{R_k} t\Theta$ holds obviously. We assume $s_{|u} = \alpha\sigma$ for some σ or $s_{|u} = d_i$ for some $i \in \{1, \cdots, m\}$. If $s_{|u} = \alpha\sigma$ and $t_{|u} = \beta[d_1, \cdots, d_m]_{(u_1,\cdots,u_m)}\sigma$ then $s\Theta \rightarrow_{R_k} t\Theta$, since $\alpha\sigma \rightarrow_{R_{k+1}} \beta[d_1, \cdots, d_m]_{(u_1,\cdots,u_m)}\sigma$ implies that $\alpha\sigma\Theta \rightarrow_{R_k} \beta[d_1, \cdots, d_m]_{(u_1,\cdots,u_m)}\sigma\Theta(= \beta\sigma\Theta)$. If $s_{|u} = d_i$ and $t_{|u} = \beta_{|u_i}$ then $s\Theta = t\Theta$, since $d_i\Theta = \beta_{|u_i}\Theta = \beta_{|u_i}$. Thus, (2) holds.
(3) Obviously, R_{k+1} is semi-constructor. Assume $r \rightarrow^*_{R_{k+1}} s$ and $r \rightarrow^*_{R_{k+1}} t$. By (2), $r\Theta \rightarrow^*_{R_k} s\Theta$ and $r\Theta \rightarrow^*_{R_k} t\Theta$ hold, so that $s\Theta \downarrow_{R_k} t\Theta$ holds by confluence of R_k. Hence, $s\Theta \downarrow_{R_{k+1}} t\Theta$ by (1). Since $s \rightarrow^*_{R_{k+1}} s\Theta$ and $t \rightarrow^*_{R_{k+1}} t\Theta$, we have $s \downarrow_{R_{k+1}} t$. □

Now, we show Lemma 6.

Proof (Lemma 6)
By the definition of S, $\mathsf{S}(R_0) = R_k$ for some $k \geq 0$. By Lemma 48 (3), proposition (1) holds. By Lemma 48 (1), Only-If-Part of Proposition (2) holds. By Lemma 48 (2), If-Part of Proposition (2) holds, since s, t do not contain new constants. □

A.2 Existence of function τ

Definition 49 Let m be $|F|$. Using an injection function $\phi : F \rightarrow \{1, \cdots, m\}$, we define function $\tau : G \rightarrow \{1, \cdots, m\}^+$, as follows: $\tau(f(s_1, \cdots, s_n)) = \tau(s_1) \cdots \tau(s_n)\phi(f)$. Here, we assume that $\tau(t)$ is a base $m + 1$ number. In order to compare $\tau(s)$ and $\tau(t)$, we use the usual relation $<$ on $\mathbf{N}$.

Lemma 50
(1) The function τ is injective.
(2) For any r, s, t and position $u \in \mathcal{O}(r)$, if $\tau(s) < \tau(t)$ then $\tau(r[s]_u) < \tau(r[t]_u)$.

Proof
(1) Terms can be considered as trees. The postorder of a term is defined as follows: $\mathbf{p}ostorder(f(s_1, \cdots, s_n)) = \mathbf{p}ostorder(s_1) \cdots \mathbf{p}ostorder(s_n)f$ [1], pp.561–562]. It is known that $\mathbf{p}ostorder$ is injective. Since ϕ is injective, so is τ.
(2) Let $\hat{s} = f(r_1, \cdots, r_{i-1}, s, r_{i+1}, \cdots, r_n)$ and $\hat{t} = f(r_1, \cdots, r_{i-1}, t, r_{i+1}, \cdots, r_n)$ where $f \in F_n$ and $i \in \{1, \cdots, n\}$. It suffices to show that $\tau(\hat{s}) < \tau(\hat{t})$, that is, $\tau(r_1) \cdots \tau(r_{i-1})\tau(s)\tau(r_{i+1}) \cdots \tau(r_n)\phi(f) < \tau(r_1) \cdots \tau(r_{i-1})\tau(t)\tau(r_{i+1}) \cdots \tau(r_n)\phi(f)$. If $|s| < |t|$ then $\tau(\hat{s}) < \tau(\hat{t})$ holds by $|\hat{s}| < |\hat{t}|$. If $|s| = |t|$ then $|\hat{s}| = |\hat{t}|$. Thus, $\tau(\hat{s}) < \tau(\hat{t})$ holds by $\tau(s) < \tau(t)$. □

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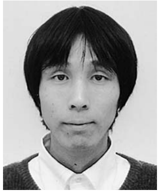

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