Intersection dimension of bipartite graphs

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Abstract: We introduce a concept of intersection dimension of a graph with respect to a graph class. This generalizes Ferrers dimension, boxicity, and poset dimension, and leads to interesting new problems. We focus in particular on bipartite graph classes defined as intersection graphs of two kinds of geometric objects. We relate well-known graph classes such as interval bigraphs, two-directional orthogonal ray graphs, chain graphs, and (unit) grid intersection graphs with respect to these dimensions. As an application of these graph-theoretic results, we show that the recognition problems for certain graph classes belong to NP.

Keywords: Ferrers dimension, Boxicity, Unit grid intersection graph, Segment-ray graph

1. Introduction

Given a family $\mathcal{F}$ of sets, the *intersection graph* of $\mathcal{F}$ is the graph in which each set in $\mathcal{F}$ is a vertex, and two vertices are adjacent if and only if the corresponding sets intersect. A typical example, when $\mathcal{F}$ is a family of intervals on a line, yields the well-known class of *interval graphs.* Interval graphs have linear-time recognition algorithms [2], [9], and nice forbidden structure characterizations. (For instance, the theorem of Lekkerkerker and Boland [21] characterizes interval graphs by the absence of induced cycles of length four and five, and the absence of asteroidal triples.)

It is natural to study a bipartite version of intersection graphs: given two families $\mathcal{F}$ and $\mathcal{F}'$ of sets, the *intersection bigraph* of $\mathcal{F}, \mathcal{F}'$ is the bipartite graph in which each set in $\mathcal{F}$ is a red vertex, each set in $\mathcal{F}'$ is a blue vertex, and a red vertex is adjacent to a blue vertex if and only if the corresponding sets intersect. When both $\mathcal{F}$ and $\mathcal{F}'$ are families of intervals on a line, we obtain *interval bigraphs* studied in [22], [27]. We denote the class of interval bigraphs by IBG. While the recognition of interval bigraphs is polynomial (in time $O(n^5m^6 \log n)$ [22]), there is no efficient algorithm known, and no characterization in terms of forbidden substructures. It turns out that there are better bipartite analogues of interval graphs. A *two-directional orthogonal ray graph*, or $\mathsf{2DOR}$ graph, is an intersection bigraph of a family $\mathcal{F}$ of upward rays, and a family $\mathcal{F}'$ of rightward rays, in the plane [29]. These graphs were introduced in connection with defect tolerance schemes for nano-programmable logic arrays [25], [33]. There are several reasons these $\mathsf{2DOR}$ graphs might be considered better bipartite analogues of interval graphs, including an ordering characterization [18], [29], and a Lekkerkerker-Boland type characterization [10], both analogous to the characterizations for interval graphs. Moreover, it follows from [10] that the class $\mathsf{2DOR}$ plays the same role for bigraphs as the class of interval graphs play for graphs, as far as polynomial solvability of certain constraint satisfaction problems is concerned. Other equivalent definitions, and forbidden structure characterizations of the class $\mathsf{2DOR}$ can be found in [10], [16], [17].

Several other graph classes can be defined as intersection bigraphs of two families $\mathcal{F}, \mathcal{F}'$. When both $\mathcal{F}$ and $\mathcal{F}'$ are inclusion-free families of intervals on a line, we obtain the class of *proper interval bigraphs* which turns out to be the same as the better known class BPG of *bipartite permutation graphs* [17], see below. When $\mathcal{F}$ is a family of points, and $\mathcal{F}'$ a family of rightward rays, in a line, we obtain the class CHAIN of *chain graphs* (cf. below). When $\mathcal{F}$ is a family of vertical segments, and $\mathcal{F}'$ a family of horizontal segments, in the plane, we obtain the class GIG of *grid intersection graphs.* Several other examples are included in the paper. We note that the following inclusions are well known or easy to derive

$$\mathsf{CHAIN} \subseteq \mathsf{BPG} \subseteq \mathsf{IBG} \subseteq \mathsf{2DOR} \subseteq \mathsf{GIG}.$$

We now introduce our concept of intersection dimension. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. The *intersection* $G \cap G'$ of G and G' is the graph $(V \cap V', E \cap E')$. For

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two graph classes $\mathcal{C}$ and $\mathcal{C}'$, we define the *pairwise intersection* of $\mathcal{C}$ and $\mathcal{C}'$ as $\mathcal{C} \otimes \mathcal{C}' = \{G \cap G' : G \in \mathcal{C}, G' \in \mathcal{C}'\}$. We also write $\mathcal{C}^k = \{G_1 \cap G_2 \cap \cdots \cap G_k : G_i \in \mathcal{C} \text{ for } 1 \le i \le k\}$. If both $\mathcal{C}$ and $\mathcal{C}'$ are closed under taking induced subgraphs, it is easy to check that $\mathcal{C} \otimes \mathcal{C}' = \{G \cap G' : G \in \mathcal{C}, G' \in \mathcal{C}', V(G) = V(G')\}$. Since every graph class in this paper is closed under taking induced subgraphs, we shall from now on use the latter equality, and assume that the vertex sets of the two graphs are the same, when defining the pairwise intersection of graph classes.

The *dimension* of a graph G with respect to the graph class $\mathcal{C}$ is the minimum k such that $G \in \mathcal{C}^k$. In the discussion below we shall point out how this definition generalizes Ferrers dimension, boxicity, cubicity, and poset dimension. We are particularly interested in expressing one graph class as a (subset of a) power of another graph class. It turns out that there are several natural statements of this kind. See the following section for a summary of results.

1.1 Our results

Among other results we will show that $\mathsf{2DOR} = \mathsf{CHAIN}^2, \mathsf{GIG} \subseteq \mathsf{CHAIN}^4$, and $\mathsf{UGIG} = \mathsf{BPG}^2$. We will also show that several of these inclusions are proper. See **Fig. 1** for the summary of our results. The characterizations we will present give compact representations for several graph classes, which implies that the recognition problems for those graph classes belong to NP. Combining with a recent result of Mustaţă and Pergel [23], we will conclude that the problems are NP-complete.

We also consider forbidden matrix characterizations of graph classes. It is known that some important classes of graphs such as CHAIN, 2DOR, and GIG have characterizations in terms of forbidden submatrices of their biadjacency matrix. We will show that a 2×3 forbidden matrix characterizes the class of intersection bigraphs of horizontal segments and upward rays.

Finally, we will show that two well-known concepts of graph dimension, boxicity and Ferrers dimension, are essentially the same for bipartite graphs.

2. Preliminaries

A graph $G = (V, E)$ is a *bipartite graph* (or a *bigraph* for short) with bipartition (X, Y) if V is partitioned into X and Y in such a way that each edge of G has one endpoint in X and the other in Y. We denote such a bigraph by $(X, Y; E)$. A *biadjacency matrix* M_B of a bigraph $B = (X, Y; E)$ is a 0-1 matrix with the rows indexed by the vertices of X and the columns indexed by the vertices of Y such that $\{x, y\} \in E$ if and only if the corresponding entry of M_B is 1. For $m \times n$ 0-1 matrices M' and M'', their intersection $M = M' \cap M''$ is the 0-1 matrix such that $M_{i,j} = 1$ if and only if $M'_{i,j} = M''_{i,j} = 1$. The *neighborhood* of a vertex v in a graph G, denoted $N_G(v)$, is the vertices adjacent to v in G.

2.1 Graph classes

Here we define the graph classes we deal with in this paper. We also introduce some important properties of them. For their inclusion relations and other known results for them, the readers can refer to the standard textbooks in this field [3], [12], [32].

For a graph class $\mathcal{C}$, the *recognition problem* of $\mathcal{C}$ is the problem deciding whether a given graph belongs to $\mathcal{C}$.

2.1.1 Chain graphs and Ferrers diagrams

A bipartite graph $B = (X, Y; E)$ is a *chain graph* if there is an ordering $(x_1, x_2, \ldots, x_p)$ on X such that $N_B(x_1) \supseteq N_B(x_2) \supseteq \cdots \supseteq N_B(x_p)$. It is easy to see that if there exists such an ordering on X, then there exists an ordering $(y_1, y_2, \ldots, y_q)$ on Y such that $N_B(y_1) \supseteq N_B(y_2) \supseteq \cdots \supseteq N_B(y_q)$. Chain graphs are also known as *difference graphs* and *Ferrers bigraphs*. It is known that chain graphs are exactly $2K_2$-free bigraphs [13]. The class of chain graphs is denoted by CHAIN.

A 0-1 matrix has the *Ferrers property* if its rows and columns can be reordered so that 1's in each row and column appear consecutively with the rows left-justified and the columns top-justified. The reordered matrix is called a *Ferrers diagram*. It is easy to see that a matrix has the Ferrers property if and only if it has none of the following 2×2 matrices as a submatrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (1)$$

Since chain graphs are exactly the $2K_2$-free bigraphs, it is easy to see that chain graphs are exactly the bigraphs whose biadjacency matrices have the Ferrers property.

2.1.2 Bipartite permutation graphs, convex graphs, biconvex graphs, interval bigraphs, and chordal bipartite graphs

A graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$ is a *permutation graph* if there is a permutation π over V such that $\{i, j\} \in E(G)$ if and only if $(i - j)(\pi(i) - \pi(j)) < 0$. A graph is a *bipartite permutation graph* if it is bipartite and a permutation graph. The class of bipartite permutation graphs is denoted by BPG. Several equivalent definitions of the class BPG are collected in [17].

An ordering $<$ of X in a bipartite graph $B = (X, Y; E)$ has the *adjacency property* if for every vertex y in Y, $N(y)$ consists of vertices that are consecutive in the ordering $<$ of X. A bipartite graph $(X, Y; E)$ is *convex* if there is an ordering of X *or* Y that fulfills the adjacency property. A bipartite graph $(X, Y; E)$ is *biconvex* if there are orderings of X *and* Y that fulfill the adjacency property. We denote the classes of convex bipartite graphs and biconvex bipartite graphs by Convex and Biconvex, respectively.

A *bi-interval representation* of a bigraph $B = (U, V; E)$ is a pair $(\mathcal{I}_U, \mathcal{I}_V)$ of sets of closed intervals such that $\mathcal{I}_U = \{I_u = [\ell_u, r_u] : u \in U\}$ and $\mathcal{I}_V = \{I_v = [\ell_v, r_v] : v \in V\}$, and $\{u, v\} \in E$ for $u \in U$ and $v \in V$ if and only if $I_u \cap I_v \neq \emptyset$. A bi-interval representation $(\mathcal{I}_U, \mathcal{I}_V)$ is *unit* if for each interval $[\ell, r] \in \mathcal{I}_U \cup \mathcal{I}_V$, $r - \ell = 1$.

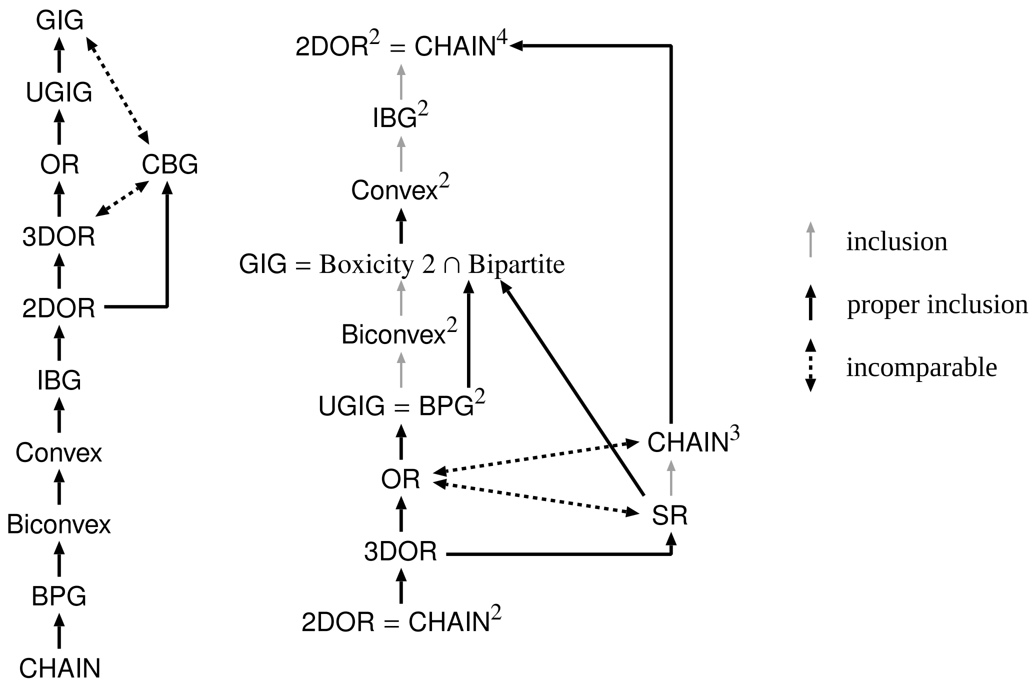


Fig. 1 (Left) Known hierarchy. (Right) New hierarchy based on intersection dimension.

A bigraph is a *chordal bipartite graph* if every induced cycle is of length four. The class of chordal bipartite graphs is denoted by CBG.

2.1.3 Orthogonal ray graphs

A bipartite graph $B = (X, Y; E)$ is an *orthogonal ray graph* if there is a pair $(\mathcal{R}_X, \mathcal{R}_Y)$ of families of rays (or half-lines) such that $\mathcal{R}_X = \{R_x : x \in X\}$ is a family of pairwise non-intersecting horizontal rays, $\mathcal{R}_Y = \{R_y : y \in Y\}$ is a family of pairwise non-intersecting vertical rays, and $\{x, y\} \in E$ if and only if R_x and R_y intersect. We call such a pair $(\mathcal{R}_X, \mathcal{R}_Y)$ an *orthogonal ray representation* of B. We denote the class of orthogonal ray graphs by OR.

Note that in a representation of an orthogonal ray graph horizontal rays can go rightward and leftward and vertical rays can go upward and downward. If we restrict horizontal rays to be only rightwards, then we have *3-directional orthogonal ray graphs*. Furthermore, if we restrict horizontal rays to be only rightwards and vertical rays to be only upwards, then we have *2-directional orthogonal ray graphs*. We denote the classes of 3-directional orthogonal ray graphs and 2-directional orthogonal ray graphs by 3DOR and 2DOR, respectively.

For the class 2DOR, several nice characterizations are known (see e.g. [10], [16], [18], [26], [27], [28], [29]). Among those characterizations, the followings are useful for our purpose. In this language they appear in [28], [29], in an equivalent graph theoretic form they are given in [16], [18].

Theorem 2.1. *For a bigraph B, the following conditions are equivalent:*

(1) B is a 2-directional orthogonal ray graph;

(2) B is γ-freeable; that is, the rows and columns of a biadjacency matrix of B can be independently permuted so that no 0 *has a* 1 *both below it and to its right;*

(3) B is of Ferrers dimension at most 2*. (The Ferrers dimension of a bigraph is defined in Section 2.1.8.)*

There are other equivalent characterizations of the class 2DOR, as suggested in the introduction. In particular, 2DOR is precisely the class of bigraphs whose complements are circular arc graphs [29]; because of the characterizations of the latter class in [10], [16], [17], one obtains several other forbidden structure characterizations of 2DOR, in terms of the absence of induced cycles and bipartite versions of asteroids, in terms of the so-called invertible pairs, and in other terms.

It is known that the recognition of 2DOR can be done in polynomial time [10], [29], while it is open for 3DOR and OR. Recently, Felsner, Mertzios, and Mustaţă [11] have shown that if the direction (right, left, up, or down) for each vertex is given, then it can be decided in polynomial time whether a given graph has an orthogonal ray representation in which each vertex has the given direction.

2.1.4 Grid intersection graphs

A bipartite graph $B = (X, Y; E)$ is a *grid intersection graph* if there is a pair $(\mathcal{S}_X, \mathcal{S}_Y)$ of families of segments such that $\mathcal{S}_X = \{S_x : x \in X\}$ is a family of pairwise non-intersecting horizontal segments, $\mathcal{S}_Y = \{S_y : y \in Y\}$ is a family of pairwise non-intersecting vertical segments, and $\{x, y\} \in E$ if and only if S_x and S_y intersect. We call such a pair $(\mathcal{S}_X, \mathcal{S}_Y)$ a *grid intersection representation* of B. A bipartite graph is a *unit grid intersection graph* if it has a grid intersection representation in which each segment if of length 1. We denote the classes of grid intersection graphs and unit grid intersection graphs by GIG and UGIG, respectively.

2.1.5 Recognition problems and inclusion relations

For the graph classes introduced above, the following relations are known [3], [24], [29]:

$$\begin{aligned}\mathsf{CHAIN} \subsetneq \mathsf{BPG} \subsetneq \mathsf{Biconvex} \subsetneq \mathsf{Convex} \subsetneq \mathsf{IBG}\\ \subsetneq \mathsf{2DOR} \subsetneq \mathsf{3DOR} \subsetneq \mathsf{OR} \subsetneq \mathsf{UGIG} \subsetneq \mathsf{GIG}.\end{aligned}$$

Also it is known that $\mathsf{2DOR} \subsetneq \mathsf{CBG}$ [29], and that CBG is incomparable to $\mathsf{3DOR}$ and GIG [24].

It is known that the recognition problems of CHAIN [15], BPG [30], $\mathsf{Biconvex}$ [32], Convex [32], IBG [22], $\mathsf{2DOR}$ [29], and CBG [31] can be solved in polynomial time. On the other hand, it is known that the recognition problems of GIG [20] and UGIG [23], [34] are NP-complete. The complexity of the recognition problems of $\mathsf{3DOR}$, OR, and SR is not known.

Note that even if three graph classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ and the recognition problems of $\mathcal{A}$ and $\mathcal{C}$ are both polynomial-time solvable (NP-hard), it does not mean the recognition problem of $\mathcal{B}$ is polynomial-time solvable (NP-hard, resp.).

2.1.6 Other graphs

The *d-dimensional hypercube* H_d is the graph with 2^d vertices in which the vertices corresponds to the subsets of $\{1, \dots, d\}$ and two vertices are adjacent if and only if the symmetric difference of the corresponding sets is of size 1.

Let $K_{a,b}$ denote the complete bipartite graph having a vertices in one side and b vertices in the other side. We denote by $K_{n,n} - nK_2$ the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$.

2.1.7 Boxicity and cubicity

An *interval graph* is the intersection graph of closed intervals on the real line. A *unit interval graph* is the intersection graph of closed unit intervals on the real line. We denote the classes of interval graphs and unit interval graphs by INT and UINT, respectively.

The *boxicity* of a graph G is the minimum integer k such that $G \in \mathsf{INT}^k$, and the *cubicity* of G is the minimum integer k such that $G \in \mathsf{UINT}^k$. It is known that given a graph, deciding whether its boxicity (or cubicity) is at most 2 is NP-complete [4], [20].

2.1.8 Ferrers dimension

The *Ferrers dimension* $\mathsf{fd}(B)$ of a bigraph B is the smallest number of Ferrers bigraphs whose intersection is B. That is, $\mathsf{fd}(B)$ is the minimum integer k such that $B \in \mathsf{CHAIN}^k$. As we will see in Section 3, if $B = (X, Y; E)$ and $\mathsf{fd}(B) = k$, then there are Ferrers bigraphs $B_i = (X, Y; E_i)$ for $1 \le i \le k$ such that $B = \bigcap_{1 \le i \le k} B_i$. That is, we can assume all the graphs B and B_i, $1 \le i \le k$, have the same bipartition.

A *Ferrers digraph* $D = (V, A)$ is a digraph whose adjacency matrix has the Ferrers property. The *Ferrers dimension* $\mathsf{fd}(D)$ *of a digraph* D is the smallest number of Ferrers digraphs whose intersection is D.

2.1.9 Poset dimension

The *poset dimension* $\mathsf{pd}(P)$ of a poset P is the minimum integer k such that there exist k linear extensions of P such that for any two elements x, y of P, $x < y$ in P if and only if $x < y$ in all the linear extensions. The *Ferrers dimension* $\mathsf{fd}(P)$ *of a poset* P is the Ferrers dimension of the digraph defined in such way that the vertices are the elements of P and there is an arc (u, v) if and only if $u < v$. Cogis [8] showed that for any poset P, $\mathsf{fd}(P) = \mathsf{pd}(P)$.

A poset is of *height 2* if every element is either a minimal element or a maximal element. The *underlying graph* of a height-2 poset is the bigraph $B = (X, Y; E)$ such that X is the set of minimal elements, Y is the set of maximal elements, and $\{x, y\} \in E$ if and only if $x < y$. It is easy to see that any bigraph is the underlying graph of some poset of height 2.

3. Bigraph intersection dimension

For two bipartite graph classes, if one of them is closed under disjoint union and taking induced subgraphs, we may assume that the bipartitions of G and G' are the same when taking their intersection. More precisely, we have the following lemma.

Lemma 3.1. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bipartite graph classes. If at least one of them is closed under disjoint union and taking induced subgraphs, then* $\mathcal{B} \bowtie \mathcal{B}' = \{(X, Y; E) \cap (X, Y; E') : (X, Y; E) \in \mathcal{B}, (X, Y; E') \in \mathcal{B}'\}$.

Unfortunately, CHAIN is not closed under disjoint union. For example, K_2 is a chain graph but $2K_2$ is not. It is the only exception in this paper. Fortunately, we have the following lemma for chain graphs.

Lemma 3.2. $\mathsf{CHAIN}^2 = \{(X, Y; E) \cap (X, Y; E') : (X, Y; E), (X, Y; E') \in \mathsf{CHAIN}\}$.

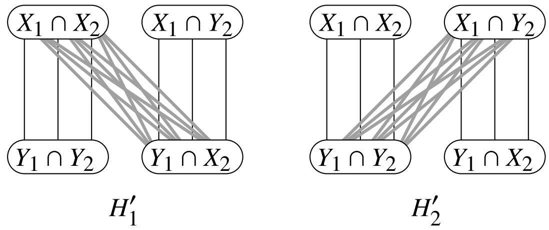


Fig. 2 Intersection of two chain graphs.

By Lemmas 3.1 and 3.2, we can assume that the bipartitions of two graphs are the same when we are defining the pairwise intersection of two graph classes, since, in this paper, either one of them is closed under disjoint union or both of them are the class of chain graphs.

4. $(P, Q; D)$-Bigraphs

We introduce the notion of $(P, Q; D)$-bigraphs, where a bigraph $B = (U, V, E)$ is said to be an $(P, Q; D)$-bigraph if and only if for some domain D (e.g., the real number line $\mathbb{R}$) each vertex in $u \in U$ can be represented as a type P subset P_u of D and each vertex $v \in V$ can be represented as a type Q subset Q_v of D such that for every $u \in U, v \in V$, $\{u, v\} \in E$ if and only if $P_u \cap Q_v \neq \emptyset$. For example, in this setting, interval bigraphs are (interval, interval, $\mathbb{R}$)-bigraphs. We will use $(P, Q; D)$ to denote the class of $(P, Q; D)$-bigraphs.

Our discussion will focus on the cases when P, Q are the following subsets of $\mathbb{R}$: points, rays, unit-intervals, and intervals; and the following axis-aligned subsets of $\mathbb{R}^2$: points, rays, unit-segments, segments, squares, and rectangles. Note that, for rays, we will use $\rightarrow, \downarrow, \leftarrow$, and $\uparrow$ to denote the *rightward*, *downward*, *leftward*, and *upward* rays respectively. Moreover, when we refer to a ray r (rather than using a specific arrow), r can be any axis-aligned ray from the domain.

4.1 $(P, Q; \mathbb{R})$-Bigraphs

We begin with some easy observations characterizing CHAIN, Convex, and Biconvex bigraphs as $(P, Q; D)$-bigraphs (see Proposition 4.1). This is followed by a couple essential lemmas that we will use to relate $(P, Q; \mathbb{R})$-bigraphs to $(P', Q'; \mathbb{R}^2)$-bigraphs.

Proposition 4.1. *For a bigraph $B = (X, Y, E)$:*
(1) B is CHAIN *if and only if B is (point, $\rightarrow$; $\mathbb{R}$).*
(2) B is Convex *if and only if B is (point, interval; $\mathbb{R}$).*
(3) B is Biconvex *if and only if B is both (point, interval; $\mathbb{R}$) and (interval, point; $\mathbb{R}$).*

It is also known that a bigraph is a bipartite permutation graph (BPG) if and only if it is a unit-interval bigraph [17]; i.e., BPG = (unit-interval, unit-interval; $\mathbb{R}$). Interestingly, we observe that (unit-interval, unit-interval; $\mathbb{R}$)-bigraphs actually have a simpler representation. Specifically, (unit-interval, unit-interval; $\mathbb{R}$) = (point, unit-interval; $\mathbb{R}$) and we prove this via the following more general lemma.

Lemma 4.2. *For a bigraph $B = (U, V; E)$ and any $Q \in \{\rightarrow$, ray, unit-interval, interval$\}$, $B \in$ (unit-interval, Q; $\mathbb{R}$) if and only if $B \in$ (point, Q; $\mathbb{R}$).*

Lemma 4.2 allows us to equate several $(P, Q; \mathbb{R})$ classes. These are given in the following two corollaries.

Corollary 4.3. *For each $Q \in \{\rightarrow$, ray, unit-interval, interval$\}$, the following classes of bigraphs are the same: (point, Q; $\mathbb{R}$), ($\rightarrow$, Q; $\mathbb{R}$), (ray, Q; $\mathbb{R}$), (unit-interval, Q; $\mathbb{R}$).*

Corollary 4.4. *For each $P, Q \in \{$point, $\rightarrow$, $\leftarrow$, unit-interval$\}$, a bigraph B is (P, Q; $\mathbb{R}$) if and only if B is (Q, P; $\mathbb{R}$).*

Notice that the statement of Corollary 4.4 does not allow either of P or Q to be ray-type sets. This is because Lemma 4.2 cannot be used to give us the desired "biconvexity-like" result when rays are allowed for a given set. However, by Lemma 4.2, we can transform any (ray, ray; $\mathbb{R}$) representation into a (point, ray; $\mathbb{R}$) representation. Thus, (ray, ray; $\mathbb{R}$) is a subset of the bigraphs which are both (point, ray; $\mathbb{R}$) and (ray, point; $\mathbb{R}$). One open question would be whether these are the same.

Moreover, the graph (P_7) given in Figure 3 is (point, ray; $\mathbb{R}$) but not both (point, ray; $\mathbb{R}$) and (ray, point; $\mathbb{R}$). This is easy to see since no three vertices in the same partition (say, X) can have pairwise incomparable neighborhoods; i.e., two of the three must be represented by rays in the same direction and thus must have nested neighborhoods. Moreover, the graph in Figure 3 has $a, b, c \in X$ such that their neighborhoods are pairwise incomparable. This is formalized in the following proposition.

Proposition 4.5. *If a bigraph $B = (X, Y; E)$ is (ray, point; $\mathbb{R}$) where each $x \in X$ is a ray then for every $\{x, x', x''\} \subseteq X$ and every $y \in Y$, there exists $x^* \in \{x, x', x''\}$ and $x^{**} \in \{x, x', x''\} \setminus \{x^*\}$ such that $N(x^*) \subseteq N(x^{**})$ or $N(x) \subseteq N(x'')$.*

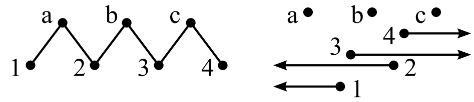


Fig. 3 The path on seven vertices (P_7) and a (point, ray; $\mathbb{R}$) representation of it. Note: P_7 is not both (point, ray; $\mathbb{R}$) and (ray, point; $\mathbb{R}$) since the neighborhoods of a, b, and c are pairwise incomparable.

4.2 $(P, Q; \mathbb{R}^2)$-Bigraphs

In this subsection we consider the domain $\mathbb{R}^2$ and describe several classes of bigraphs as the intersection of one dimensional bigraph classes (i.e., as $(P, Q; \mathbb{R}) \bowtie (P', Q'; \mathbb{R})$). Notice that, for $P, Q \in \{$point, unit-interval, interval$\}$, $(P, Q; \mathbb{R})$ is hereditary and closed under disjoint union. Thus, by Lemma 3.1, for $P, Q \in \{$point, unit-interval, interval$\}$ and any choices of P' and Q', $B = (X, Y; E)$ is $(P, Q; \mathbb{R}) \bowtie (P', Q'; \mathbb{R})$ if and only if $B = (X, Y; E \cap E')$ for $(X, Y; E) \in (P, Q; \mathbb{R})$ and $(X, Y; E'') \in (P', Q'; \mathbb{R})$.

Theorem 4.6. UGIG $=$ BPG^2 $=$*(point, unit-interval; $\mathbb{R}$)^2.*

Using Theorem 4.6 and Corollary 4.4 the following is immediate.

Corollary 4.7. *(unit-square, unit-square; $\mathbb{R}^2$) = (point, unit-interval; $\mathbb{R}$)^2 =* UGIG.

The corollary above implies that a bipartite graph of cubicity 2 is UGIG. It is easy to see that the star $K_{1,5}$ is UGIG, but its cubicity is more than 2. Therefore, we have the following corollary, which is a nice complement to the fact Boxicity-2 $\cap$ Bipartite = GIG [1].

Corollary 4.8. *Cubicity-2 $\cap$ Bipartite $\subsetneq$* UGIG.

The proof of the following theorem is an easy modification of the proof of Theorem 4.6. The relation GIG $\neq$ Convex^2 is shown by **Fig. 5**.

Theorem 4.9. Biconvex^2 $\subseteq$ (Biconvex $\bowtie$ Convex) $\subseteq$ GIG $\subsetneq$ Convex^2.

Since Convex $\subset$ 2DOR, it holds that GIG $\subseteq$ 2DOR^2 $=$ CHAIN^4. Therefore, every grid intersection graph has Ferrers dimension at most 4.

Corollary 4.10. *The recognition problems of* BPG^2*,* Biconvex^2*, and* Biconvex $\bowtie$ Convex *are NP-complete.*

5. Segment-ray graphs

A bipartite graph $B = (X, Y; E)$ is a *segment-ray graph* if it belongs to the class SR = (horizontal-segments, $\uparrow$; $\mathbb{R}^2$). We will find a forbidden matrix characterization of SR.

Let F be a matrix with entries $0, 1, *$, where $*$ means "don't care." A matrix M is F-*free* if M does not have F as a submatrix ignoring $*$-entries. A bipartite graph is F-*freeable* if it has a F-free biadjacency matrix.

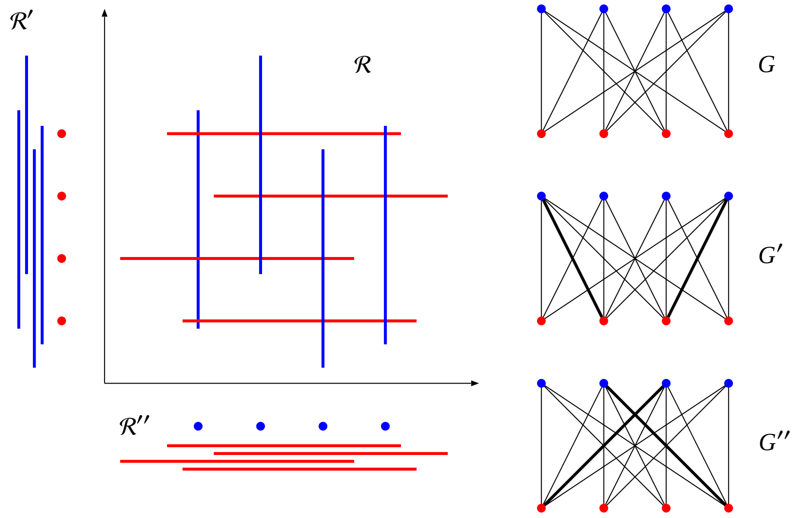


Fig. 4 UGIG = BPG^2.

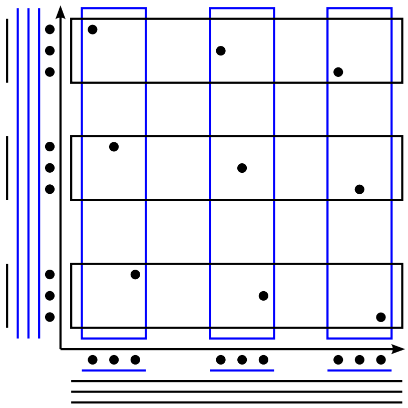


Fig. 5 A (point, interval; $\mathbb{R}$)^2 representation of the full subdivision H of $K_{3,3}$; i.e., $H \in \mathsf{Convex}^2$. On the other hand, $H \notin \mathsf{GIG}$, since it is the full subdivision of a non-planar graph, and thus not a string graph.

It is known that a bipartite graph is a chordal bipartite graph if and only if it is Γ-freeable (see [19]), a 2-directional orthogonal ray graph if and only if it is γ-freeable [29], and a grid intersection graph if and only if it is cross-freeable [14], where the forbidden matrices are defined as follows:

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \qquad \mathsf{cross} = \begin{pmatrix} * & 1 & * \\ 1 & 0 & 1 \\ * & 1 & * \end{pmatrix}.$$

In this section, using the following matrix V, we characterize segment-ray graphs:

$$\mathsf{V} = \begin{pmatrix} 1 & 0 & 1 \\ * & 1 & * \end{pmatrix}.$$

Obviously, a matrix is cross-free if it is V-free, and V-free if it is γ-free.

The proof of the following theorem is similar to the proofs of the cross-free characterization of GIG [14] and the γ-free characterization of 2DOR [29].

Theorem 5.1. *A bipartite graph is a segment-ray graph if and only if it is* V*-freeable.*

Now we show that every segment-ray graph has Ferrers dimension at most 3. To this end, we need the following simple fact.

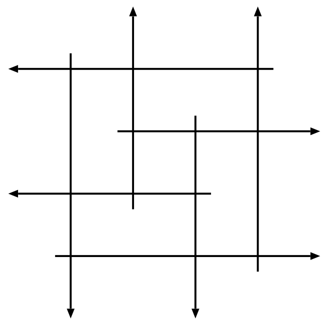
Lemma 5.2. *An $m \times n$ 0-1 matrix M is* V*-free if and only if for each entry (i, j) with $M_{i,j} = 0$ at least one of the following holds:*
(1) $M_{i,k} = 0$ for all $1 \le k \le j$;
(2) $M_{i,k} = 0$ for all $j \le k \le n$;
(3) $M_{k,j} = 0$ for all $i \le k \le m$.

Theorem 5.3. *Every segment-ray graph has Ferrers dimension at most 3.*

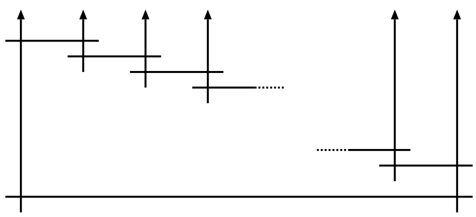
Note that the upper bounds of the Ferrers dimension for GIG (≤ 4) and 2DOR (≤ 2) can be shown in similar ways by using the forbidden submatrix characterizations.

Corollary 5.4. OR *is incomparable to both* CHAIN^3 *and* SR.

Corollary 5.5. SR *is a proper subset of* GIG.



(a) $H_3 \in$ OR.



(b) $C_{2n} \in$ SR.

Fig. 6 Examples showing incomparability.

6. Boxicity and Ferrers dimension

Chatterjee and Ghosh [7] presented some relations between the boxicity of undirected graphs and the Ferrers dimension of the directed graphs obtained somehow from the undirected graphs. Here we present a similar but more direct relation between the boxicity and the Ferrers dimension of bigraphs.

If $\mathsf{fd}(B) = 1$, then $\mathsf{box}(B) \leq 2$. This is because, $\mathsf{fd}(B) = 1$ implies that B is a chain graph, and thus B is a grid intersection graph [24]. This bound is tight since $\mathsf{fd}(K_{n,n}) = 1$ and $\mathsf{box}(K_{n,n}) = 2$ for every $n \geq 2$.

Theorem 6.1. *Let B be a bigraph with $\mathsf{fd}(B) \geq 2$. It holds that*

$$\mathsf{box}(B) \leq \mathsf{fd}(B) \leq 2\mathsf{box}(B).$$

The upper bound in Theorem 6.1 is tight. It is known that $\mathsf{box}(K_{n,n} - nK_2) = \lceil n/2 \rceil$ [5] and $\mathsf{fd}(K_{n,n} - nK_2) = n$ [35], [36].

Bellatoni, Hartman, Przytycka, and Whitesides [1] showed that the grid intersection graphs are exactly the bigraphs of boxicity at most 2. This implies that the Ferrers dimension of a grid intersection graph is at most 4. We show that the converse is not true.

Theorem 6.2. $\mathsf{GIG} \subsetneq \mathsf{CHAIN}^4$.

Chandran, Francis, and Mathew [6] showed that boxicity is unbounded for chordal bipartite graphs. Thus we have the following.

Corollary 6.3. *Ferrers dimension is unbounded for chordal bipartite graphs.*

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