Journal of Information Processing Vol.20 No.3 707–708 (July 2012)

[DOI: 10.2197/ipsjjip.20.707]

Regular Paper

The Magnets Puzzle is NP-Complete

Jonas Kölker[1,a)]

Received: August 31, 2011, Accepted: March 2, 2012

Abstract: In a Magnets puzzle, one must pack magnets in a box subject to polarity and numeric constraints. We show that deciding solvability of Magnets instances is NP-complete.

Keywords: puzzles, combinatorics, computational complexity, NP-completeness

1. Introduction

Magnets is a type of puzzle of unsure origin. Simon Tatham publishes an implementation [4], citing Janko at Ref. [2] as the source, which claims the puzzle to be of unknown pedigree. The puzzle is also unmentioned in a survey paper by Erik Demaine [1] which mentions the NP-completeness of several other paper-and-pencil puzzles.

In a Magnets puzzle, one is given a $w \times h$ rectangle, a subdivision of this rectangle into dominos, and $2(w+h)$ integers denoted r_i^+, r_i^-, c_j^+ and c_j^- for $i = 0, \ldots, h-1$ and $j = 0, \ldots, w-1$.

The goal is to assign a value $\{-1, 0, +1\}$ to each square (representing positive or negative magnetic poles, or non-magnetic material) such that (1) the sum of the two values in the same domino is 0; (2) Each pair of horizontally or vertically adjacent numbers (x, y) satisfies $x = y = 0 \vee x \neq y$; and (3) in each row i, $+1$ occurs r_i^+ times and -1 occurs r_i^- times, and similarly for c_j^+ and c_j^- with respect to columns j, that is, the number of positive poles per row (column) equals the positive row (column) constraint, and likewise for negatives. For an example, see **Fig. 1**.

2. Reduction

Theorem 1. *The problem of deciding whether a given Magnets instance has a valid solution is NP-complete.*

Proof. It is easy to see that the problem is in NP: give a solution as a certificate. Its size is proportional to the input (problem) size and its verification is straightforward.

To show hardness, we give a reduction from monotone 1-in-3 boolean satisfiability, which is known to be NP-complete [3]; i.e., given n variables x_i and m clauses C_j, each containing at most three variables, find a set of true variables T such that $|T \cap C_j| = 1$ for every $j = 0, \ldots, m-1$.

The corresponding magnets instance has $w = 4n$ and $h = 2m$, and the rectangle will be divided into $n \cdot m$ gadgets, each of size 4×2.

The intuition of the gadgets are as follows: the leftmost column of each gadget insulates it from its left neighbor, its two rightmost columns connect it to all the other gadgets associated with the same variable, and its fourth column determines the truth value of its corresponding variable in those gadgets whose associated clause contains the gadget's associated variable.

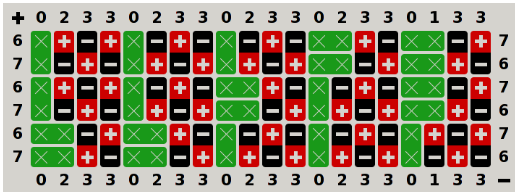

Fig. 1 Solution to example puzzle, where crossed green squares represent 0.

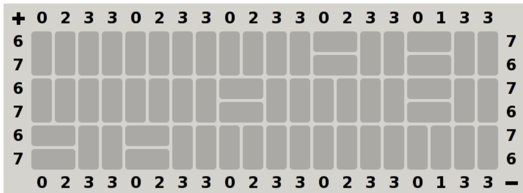

Fig. 2 Puzzle for the instance $\{(x_0, x_1, x_2), (x_0, x_1, x_3), (x_2, x_3, x_4)\}$.

In gadget (i, j), if $x_i \notin C_j$ the two leftmost dominos will be horizontal, otherwise vertical. The rightmost two dominos will always be vertical.

For $i = 0, \ldots, n-1$, let $c_{4i}^+ = c_{4i}^- = 0$, let c_{4i+1}^+ be the number of clauses containing x_i, and let $c_{4i+1}^- = c_{4i+1}^+$. Let $c_{4i+2}^+ = c_{4i+3}^+ = c_{4i+2}^- = c_{4i+3}^- = m$.

For $j = 0, \ldots, m-1$, set $r_{2j}^+ = n+1$ and $r_{2j}^- = n + |C_j| - 1$. Let $r_{2j+1}^+ = r_{2j}^-$ and $r_{2j+1}^- = r_{2j}^+$. For an example of the reduction applied, see **Fig. 2**.

We claim there is a one-to-one mapping between monotone 1-in-3 SAT solutions and Magnets solutions: $x_i \in T$ iff square $(4i+1, 2j)$ has value $+1$ for every j such that $x_i \in C_j$.

Observe that as $c_{4i}^+ = c_{4i}^- = 0$, no leftmost column of any gadget contains a non-zero value. As $c_{4i+2}^+ = c_{4i+3}^+ = m$, all values in columns $4i+2$ and $4i+3$ are non-zero. This contributes one $(+1)$ and one (-1) to each row.

Thus, for each j, row $2j$ contains n copies of $+1$ plus those in columns $4i+1$, and by r_{2j}^+ it must contain $n+1$ copies of $+1$. In other words, exactly one of the squares $(4i+1, 2j)_{i=1}^n$ must contain $+1$ (for each $j = 0, \ldots, m-1$).

1 Department of Computer Science, Aarhus University, Aarhus, Denmark

a) epona@cs.au.dk

Note that iff $x_i \notin C_j$ then square $(4i + 1, 2j)$ contains 0: in that case its horizontally adjacent domino buddy contains 0 by the above argument. Note also that in each connected region of non-zero squares, any two squares of even manhattan distance have equal polarity (i.e., value) and squares of odd distance have unequal polarity. Thus, if $(4i+1, 0) \neq 0$, then $(4i+1, 2j) = (4i+1, 0)$ for each $j = 0, \ldots, m-1$.

In other words, any assignment of values to squares is consistent with $x_i \in^? C_j$ according to the defined bijection, and well-defined (either x_i is or isn't in T, but not both), so this bijection maps Magnets solutions to monotone 1-in-3 SAT solutions as claimed. □

References

[1] Demaine, E.D. and Hearn, R.A.: Playing Games with Algorithms: Algorithmic Combinatorial Game Theory, *Games of No Chance 3*, Albert, M.H. and Nowakowski, R.J. (Eds.), Mathematical Sciences Research Institute Publications, Vol.56, pp.3–56, Cambridge University Press (2009).

[2] Janko, A. and Janko, O.: Beispiel mit ausführlicher Lösung (online), available from ⟨http://www.janko.at/Raetsel/Magnete/Beispiel.htm⟩ (accessed 2011-07-28).

[3] Schaefer, T.J.: The complexity of satisfiability problems, *Proc. 10th Annual ACM Symposium on Theory of Computing*, pp.216–226 (1978).

[4] Tatham, S.: Simon Tatham's Portable Puzzle Collection, (online), available from ⟨http://www.chiark.greenend.org.uk/~sgtatham/puzzles/⟩ (accessed 2011-07-28).

Jonas Kölker is a Ph.D. student at Aarhus University and a spare time free software developer.