Reflections on the Diagonal Theorem, and Related Topics

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A formula (function), that expresses a diagonal sequence generated by all finite length formulas (functions), is an essentially infinite length formula (function). The last report made it clear that the substitution function and the provability predicate in Gödel's paper are essentially infinitely long expressions. These indicate that Gödel's proof of the incompleteness theorems is incorrect. On the same basis this report shows that the diagonal theorem does not hold. Since Gödel's incompleteness theorems, Rosser's theorem, Tarski's theorem and the 1-incompleteness theorem are proved using the diagonal theorem. This does not mean that the theorems do not exist, but they must be reexamined. Furthermore, the recursion theorem and the fixed point theorem must be reviewed, because that these theorems are proved using essentially infinite length functions.

1. Introduction

In 1931 Gödel showed that there exists an unprovable and unrefutable finite length formula in an arithmetic. This astonishing discovery is called the first incompleteness theorem. A previous report[13] argued that the substitution function in Gödel's paper is an essentially infinitely long expression. This finding is based on the fact that a formula (function) to express a diagonal sequence defined by all finite length formulas (functions) is an essentially infinite length formulas (function). Another report[14] made it clear that the provability predicate that is the core concept in Gödel's paper is an essentially infinitely long expression. These results mean that Gödel's proof of the incompleteness theorems[4] is incorrect.

In this report we shall show that the diagonal theorem does not hold. Since the proof of Gödel's incompleteness theorems is founded on the diagonal theo-

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rem, the proof is incorrect. Some proofs of Rosser's theorem and Tarski's one are based on the diagonal theorem, thier proofs are also incorrect. At the moment we can not state that these theorems do not exist. In bounded arithmetic Π_1-incompleteness is known as one of the evidence of the incompleteness of Peano Arithmetic. However a proof of Π_1-incompleteness is based on the diagonal theorem, then the proof is wrong. An essentially infinite length function appears also in the proofs of the recursion theorems and the fixed point theorem. Their proofs must be reexamined. In Appendix some errors in the former reports are corrected.

2. Preliminaries

The predicate logic for an arithmetic with addition and multiplication follows Shoenfield[10], but the classification of symbols is slightly modified.

Definition 1. The symbols of the predicate logic for the arithmetic are defined as follows. (a1) individual constants $(a, b, c, \cdots)$, (a2) variables $(x, y, z, \cdots)$, (a3) function symbols $(+, *)$, (a4) a predicate symbol $(=)$, (a5) logical symbols 1 $(\neg, \vee)$, (a6) a logical symbol 2 $(\exists)$, (a7) subsidiary symbols $((,), \text{comma})$. $\exists$ is called an existential quantifier. In this paper let the basic symbols of the predicate logic for the arithmetic be the symbols of (a1) $\sim$ (a5) and (a7).

Let A and B be sets of collections of objects. A mapping from the set of n-tuples in A to B is called an n-ary function from A to B. A subset of the set of n-tuples in A is called an n-ary predicate in A. An occurrence of variable x in predicate A is bound in A, if it occurs in a part of A of the form $\exists x A$, otherwise it is free in A.

Definition 2. (b1) An individual constant is a term. (b2) A variable is a term. (b3) If $t_1, t_2, \cdots, t_n$ are terms and f^n is an n-ary function, $f^n(t_1, t_2, \cdots, t_n)$ is a term. (b4) Let $\{t_1, t_2, t_3, \cdots\}$ be an infinite set of terms. Define $T_1 = t_1 + t_2 + t_3 + \cdots$ and $T_2 = t_1 * t_2 * t_3 * \cdots$. T_1 and T_2 are terms.

Definition 3. (c1) If $t_1, t_2, \cdots, t_n$ are terms and P is an n-ary predicate,

$P(t_1, t_2, \cdots, t_n)$ is a formula. (c2) If A and B are formulas, $A \vee B$ and $\neg A$ are formulas. (c3) Let $\{A_1, A_2, A_3, \cdots\}$ be an infinite set of formulas. Define $R = A_1 \vee A_2 \vee A_3 \vee \cdots$. R is a formula. (c4) If $C(x)$ is a formula and x is a free variable, $\exists x C(x)$ is a formula.

Formulas $(A \rightarrow B), (A \wedge B), (A \leftrightarrow B)$ and $\forall x A(x)$ are the abbreviations of $(\neg A \vee B), \neg(A \rightarrow \neg B), ((A \rightarrow B) \wedge (A \leftarrow B))$ and $\neg \exists \neg A(x)$, respectively. Symbol $\forall$ is a universal quantifier. A formula without free variables is called a sentence. We sometimes define function symbols and predicate symbols that are not in the basic symbols. In this paper we assume that defined function symbols and defined predicate symbols are rewritten using the basic symbols.

A proof is a finite sequence of one or more formulas such that each formula of the sequence is either an axiom or an immediate consequence of preceding formulas of the sequence. If A is the last formula in a proof P, P is said to be a proof of A. A is said to be provable or to be a theorem. Sometimes $\exists x C(x)$ represents finite numbers of Cs such that $C(x)(x = 1, 2, \cdots, n)$. We call such $\exists x C(x)$ a finite existential formula. If $\exists x C(x)$ represents an infinite number of Cs such as $C(x)(x = 1, 2, \cdots)$, we call $\exists x C(x)$ an infinite existential formula.

Definition 4. The length of a function is defined as the number of the basic symbols in the function. If a function consists of infinitely many basic symbols, it is called an infinite length function. If a function is not an infinite length function, it is a finite length function. The lengths of a term, a formula and a predicate are similarly defined.

Terms T_1 and T_2 in Definition 2 are infinite length terms. Formula R in Definition 3 is an infinite length formula. Let Q be a finite length formula without infinite existential formulas. Assume that $\exists x C(x)$ is an infinite existential formula and there is an axiom or a theorem such that $\exists x C(x) \rightarrow Q$. $\exists x C(x)$ can be converted to a finite length formula.

Definition 5. If an infinite length function can not be transformed into a finite one with any efforts, the function is called an essentially infinite length function. An essentially infinite length formula and that predicate are defined in the similar way.

Many types of Gödel numbering have been proposed. We do not specify a particular numbering.

3. Diagonal Sequences

The set $\tilde{A}$ of finite length formulas with one free variable is a countably infinite set. Enumerate all finite length formulas with one free variable u.

$$A_1(u), A_2(u), A_3(u), \cdots \tag{1}$$

Consider formulas $I(u)$ and $J(u)$ such as

$$I(k) = \neg A_k(k) \quad (k = 1, 2, 3, \cdots). \tag{2}$$
$$J(k) = A_k(k) \quad (k = 1, 2, 3, \cdots). \tag{3}$$

(2) and (3) are called the antidiagonal sequence and the diagonal sequence of (1), respectively. If $I(u)$ is in $\tilde{A}$, it is a finite length formula. Since $I(k) \neq A_k(k)(k = 1, 2, 3, \cdots)$, it is easy to prove by the diagonal method that $I(u)$ is not in $\tilde{A}$. Since $I(u)$ is not in $\tilde{A}$, it is not a finite length formula. $I(u)$ can not be transformed to a finite length formula. Therefore, it is an essentially infinite length formula. Furthermore, we have

$$J(u) = \neg I(u). \tag{4}$$

Since $I(u)$ is an essentially infinite length formula, so is $\neg I(u)$. That is, $J(u)$ is an essentially infinite length formula.

Change the order of formulas in $\tilde{A}$. Let it be as follows.

$$A'_1(u), A'_2(u), A'_3(u), \cdots \tag{5}$$

Let $I'(u)$ and $J'(u)$ be the antidiagonal sequence and the diagonal sequence of (5), respectively. $I'(u)$ and $J'(u)$ are also essentially infinite length formulas. Note that there are infinitely many different sequences defined by all finite length formulas with one free variable. For each sequence, there are an antidiagonal sequence and a diagonal sequence. Both of them are essentially infinite length sequences. If an antidiagonal sequence and a diagonal sequence are defined based

on all finite length formulas with one free variable, we need not pay attension to the order of formulas.

Let regard $A_k(u)(k = 1, 2, 3, \cdots)$ as functions. Define a modified diagonal sequence $K(u)$ and the diagonal sequence $L(u)$ such as

$$K(k) = A_k(k) + 1 \quad (k = 1, 2, 3, \cdots). \tag{6}$$

$$L(k) = A_k(k) \quad (k = 1, 2, 3, \cdots). \tag{7}$$

It is easy to see that $K(u)$ is an essentially infinite length function. Note that

$$K(u) = L(u) + 1. \tag{8}$$

From (8) the diagonal sequence $L(u)$ is also an essentially infinite length function.

Lemma 1. There is no finite length formula (function) that has the values of any diagonal sequences defined by all finite length formulas (functions) with one free variable.

4. Diagonal Theorem

4.1 Diagonal theorem

We shall quote Smorynsky's explanation[11] (p.827) of the diagonal theorem. Let **T** be some fixed, but unspecified, consistent formal theory. Assume that the encoding is done in some fixed formal theory **S** and that **T** contains **S**. Define function $sub(\alpha, \beta)$ as follows.

$$sub(\lceil \varphi(x) \rceil, \lceil t \rceil) = \lceil \varphi(t) \rceil. \tag{9}$$

where φ is any formula with one free variable, and $\lceil \varphi(x) \rceil$ is the Gödel number for $\varphi(x)$.

Diagonal theorem Let $\varphi(x)$ in the language of **T** have only the free variable indicated. Then there is a sentence ψ such that

$$\mathbf{S} \vdash \psi \leftrightarrow \varphi(\lceil \psi \rceil). \tag{10}$$

Proof. Given $\varphi(x)$, let

$$\theta(x) \leftrightarrow \varphi(sub(x, x)) \tag{11}$$

be the diagonalization of φ. Let

$$m = \lceil \theta(x) \rceil, \quad \psi = \theta(m). \tag{12}$$

Then we claim

$$\mathbf{S} \vdash \psi \leftrightarrow \varphi(\lceil \psi \rceil). \tag{13}$$

For, in **S**, we see that

$$\psi \leftrightarrow \theta(m) \leftrightarrow \varphi(sub(m, m)) \tag{14}$$

$$\leftrightarrow \varphi(sub(\lceil \theta(x) \rceil, m)) \tag{15}$$

$$\leftrightarrow \varphi(\lceil \theta(m) \rceil) \leftrightarrow \varphi(\lceil \psi \rceil). \quad qed \tag{16}$$

4.2 Qustions about the diagonal theorem

(1) The incorrectness of the proof

Let F be the set of all finite length formulas with one free variable. From the following equation

$$sub(\lceil A_k(x) \rceil, \lceil k \rceil) = \lceil A_k(k) \rceil, \quad (k = 1, 2, \cdots) \tag{17}$$

we know that $sub(x, x)$ is a diagonal sequence fefined by F. Let $k = \lceil \varphi(y) \rceil$ and change the notation from $\varphi(y)$ to $\varphi_k(y)$.

$$sub(k, k) = sub(\lceil \varphi_k(y) \rceil, \lceil \varphi_k(y) \rceil) \tag{18}$$

$$= \lceil \varphi_k(\lceil \varphi_k(y) \rceil) \rceil = \lceil \varphi_k(k) \rceil. \tag{19}$$

[Note] From Lemma 1 $\varphi_x(x)$ is an essentially infinite length function, and its Gödel number $\lceil \varphi_x(x) \rceil$ is infinite. Therefore $\varphi(sub(x, x))$ is not defined. So is $\theta(x)$. Furthermore, "infinite" is not a natural number. The diagonal theorem has not been proved as it was intended in the beginning, and the proof violates the finitistism.

(2) A fixed point

Assume that the diagonal theorem holds. Though "infinity" is not a natural number, let inf be infinity for covenience. Since $sub(x, x) = inf$, from (11) we have

$$\theta(x) \leftrightarrow \varphi_k(inf). \tag{20}$$

Then $\theta(x)$ must be a sentence. Let $\theta(x) = C$, where C is a sentence. From (20), we have

$$C \leftrightarrow \varphi_k(inf). \tag{21}$$

From (12), we know that

$$m = \lceil \theta(x) \rceil = \lceil C \rceil, \quad \psi = \theta(m) = C. \tag{22}$$

From (13) and (22), the following is obtained.

$$C \leftrightarrow \varphi_k(\lceil C \rceil). \tag{23}$$

Let a be a sequence of symbols. Define $a = \lfloor b \rfloor$, if $b = \lceil a \rceil$. From (21) and (23), we conclude that

$$\lfloor inf \rfloor \leftrightarrow \varphi_k(inf). \tag{24}$$

Note that we can not reconstruct a formula from $\lfloor inf \rfloor$. φ_k is an arbitrary formula with one free variable. Let φ_h be an arbitrary another formula in F, where $h \neq k$. φ_k and φ_h have the same fixed point. That is,

$$\lfloor inf \rfloor \leftrightarrow \varphi_h(inf). \tag{25}$$

[Note] All formulas have at least one same fixed point in common. This indicates that the diagonal theorem does not hold.

4.3 Does the diagonal theorem exist ?

Maehara[7] (pp.132-133) explains the substance of the diagonal theorem in the following way. Enumerate all formulas with one free variable x.

$$A_0(x), A_1(x), A_2(x), \cdots \tag{26}$$

Consider the diagonal values.

$$A_0(0), A_1(1), A_2(2), \cdots, A_k(k), \cdots \tag{27}$$

Assume that an arbitrary univalent correspondence Φ between formulas is given. Let the following be the sequence of formulas obtained by (27) and Φ.

$$\Phi(A_k(k)) \quad (k = 0, 1, 2, \cdots). \tag{28}$$

Assume that a formula $G(x)$ represents (28). That is

$$\Phi(A_k(k)) = G(k) \quad (k = 0, 1, 2, \cdots). \tag{29}$$

$G(x)$ is sure to be in (26). Let it be $A_n(x)$.

$$\Phi(A_k(k)) = A_n(k) \quad (k = 0, 1, 2, \cdots). \tag{30}$$

Put $k = n$. We obtain the following.

$$\Phi(A_n(n)) = A_n(n). \tag{31}$$

Let $A_n(n)$ be A.

$$\Phi(A) = A. \tag{32}$$

This is the substance of the diagonal theorem. In the case that $\Phi(x)$ is not a formula but a function, it is enough to rewrite $\Phi(A)$ with $\Phi(\lceil A \rceil)$ in the above discussion.

[Note] Maehara did not put any restrictions on formulas in (26). However each formula in (26) must be a finite length formula. As we have seen in Section 2, there is no finite length formula (function) that represents the diagonal sequence defined by all finite length formulas (functions). Since $A_x(x)$ is an essentially infinite length formula, $\Phi(A_x(x))$ is not defined. Hence, in general, there is no finite length $G(x)$ such as (29) for any Φ. That is, we can not find n of $A_n(x)$ in (30). This means the non-existence of the diagonal theorem.

5. Related topics

5.1 Gödel and Rosser's theorems

Gödel's theorem and Rosser's one are from Maehara[7] (p.134, p.142). Some notations are changed.

Gödel's first incompleteness theorem: If a set K of formulas is representable and ω consistent, there is a sentence A that can be neither proved nor refuted from K.

A rough sketch of the proof without mention of ω consistent is as follows: Let $Prov_K(x, y)$ be the predicate that x is a proof of a formula y. Define the provability predicate $Pr_K(y)$ such that

$$Pr_K(y) = \exists x Prov_K(x, y). \tag{33}$$

Let $\varphi(y)$ in (10) be $\neg Pr_K(y)$ and apply the diagonal theorem to $\neg Pr_K(y)$. Assume that we have

$$A = \neg Pr_K(\lceil A \rceil), \tag{34}$$

where A is a sentence. Assume that A is provable from K.

$$K \vdash A \rightarrow K \vdash Pr_K(\lceil A \rceil) \rightarrow K \vdash \neg A. \tag{35}$$

This is a contradiction. A is not provable. Assume that $\neg A$ is provable from K.

$$K \vdash \neg A \rightarrow K \vdash Pr_K(\lceil A \rceil) \rightarrow K \vdash A. \tag{36}$$

This is also a contradiction. $\neg A$ is not provable. That is, A is neither provable nor refutable.

[Note] The proof of the first incompleteness theorem is carried out using the diagonal theorem. Since the diagonal theorem (32) does not hold, the proof is not correct. It is no need to say that the proof of the second incompleteness theorem is also incorrect, because it is based on the proof of the first theorem.

Rosser's theorem: If a set of formulas K is representable and consistent, there is a sentence A that can be neither proved nor refuted from K.

[Note] We shall omit the proof. Since the proof of Rosser's theorem is based on the diagonal theorem or Gödel's first incompleteness theorem, the proof is incorrect.

5.2 Tarski's theorem

We shall refer Maehara's explanation[7] (pp.138-139).

Tarski's theorem: If a set of formulas K is consistent, there does not exist a unary formula $T(x)$ such that the following is proved from K

$$A \leftrightarrow T([A]), \qquad (37)$$

where $[A]$ is the term corresponding to the Gödel number $\lceil A \rceil$ of a formula A.

Proof: Assume that there exists such $T(x)$ and show that K is inconsistent. Applying the diagonal theorem to a unary formula $\neg T(x)$, we can prove that there exists a sentence B such that

$$B \leftrightarrow \neg T([B]). \qquad (38)$$

However if we substitute B for A in (37), the following is derived.

$$B \leftrightarrow T([B]). \qquad (39)$$

(38) and (39) contradict. Then K must be inconsistent.   qed

[Note] Since the diagonal theorem is applied in the proof, the proof is incorrect.

5.3 Peano arithmetic: Π_1-incompleteness

The basic notions on bounded arithmetic are from Hajek[5] (p.13, pp.30-31). $(\exists x < y)\varphi$ is an abbreviation for $(\exists x)(x \leq y \wedge \varphi)\varphi$ and $(\forall x \leq y)\varphi$ is an abbreviation for $(\forall x)(x \leq y \rightarrow \varphi)$. By convention, x and y must be distinct variables. An L_0-formula is bounded if all quantifiers occuring in it are bounded, i.e. occur in a context as above. Futhermore, $(\forall x \leq y)\varphi$ is an abbreviation for $(\forall x \leq y)(x \neq y \rightarrow \varphi)$ and similarly for $(\forall x \leq y)$; $x \neq y$ is the same as $\neg(x = y)$. Σ_0-formulas $= \Pi_0$-formulas $=$ bounded formulas; Σ_{n+1} -formulas have the form $(\exists x)\varphi$ where φ is Π_n, Π_{n+1}-formulas have the form $(\forall x)\varphi$ where φ is Σ_n. Thus a Σ_n-formula has a block of n alternating quantifiers, the first one being existential, and this block is followed by a bounded formula. Similarly for Π_n.

The axioms of Robinson's arithmetic Q are as follows.
(A1) $S(x) \neq \overline{0}$

(A2) $S(x) = S(y) \rightarrow x = y$
(A3) $x \neq \overline{0} \rightarrow (\exists y)(x = S(y))$
(A4) $x + \overline{0} = x$
(A5) $x + S(y) = S(x + y)$
(A6) $x * \overline{0} = \overline{0}$
(A7) $x * S(y) = (x * y) + x$
(A8) $x \leq y \equiv (\exists)(z + x = y)$

Peano arithmetic (PA) is defined from Q by adding the induction schema (A9), where $\varphi(x)$ is a formula in an ordered ring.
(A9) $\varphi(0) \& \forall(\varphi(x) \rightarrow \varphi(x + 1)) \rightarrow (\forall x)\varphi(x)$
Arithmetic $Q_<$ is defined by the axioms (A1) $\sim$ (A6) and the following (A10).
(A10)$\forall(x \neq 0 \rightarrow \exists y(y + 1 = x))$

[Q] The followings are known on Q. Kashima[16] (pp.67-80)
(Q1) A total recursive function is representable by a Σ_0-formula or a Π_1-formula in Q.
(Q2) [Σ_1-completeness of Q]. Let $\varphi(x)$ be a Σ_0-formula with the only free variable x and let $N \models (\exists x)\varphi(x)$. Then $Q \vdash (\exists x)\varphi(x)$.
(Q3) From (Q2), PA is Σ_1-complete.

[$Q_<$] The followings are known on $Q_<$. Tanaka[16] (pp.114-128)
($Q_<$1) A total recursive function is strongly representable as a Σ_1-formula in $Q_<$.
($Q_<$2) (Σ_1-completeness of $Q_<$)If φ is a Σ_1-sentence and $N \models \phi$, $Q_< \vdash \varphi$.
($Q_<$3) From ($Q_<$2), PA is Σ_1-complete.

[PA] The followings are known on PA. Kashima[17] (pp.59-86)
(PA1) A computable function and a computable predicate are expressed by a Σ_1-formula or a Π_1-formula over PA.
(PA2) [Σ_1-completeness of PA] If φ is a Σ_1-sentence and $N \models \phi$, PA $\vdash \varphi$.
(PA3) If a sentence G in PA is derived by the diagonal theorem, G is a Π_1-sentence that is neither provable nor refutable in PA.

[Note](1) (Q2) and (PA3) should be reexamined, since they are obtained using the diagonal theorem. The incompleteness of PA is described in Hajek[5] (pp.160-161). The proof is also based on the diagonal theorem.

(2) From $(Q_{<}1) \sim (Q_{<}3)$, a Σ_1-formula in $Q_{<}$ can express a total recursive function, and $Q_{<}$ is Σ_1-complete. So is PA.

6. Functions proved by infinite length functions

6.1 Recursion theorem

Many versions of recursion theorems are proposed. We shall quote Davis[3] (pp.98-99).

Recursion theorem: Let $g(e, x_1, \cdots, x_m)$ be a partially computable function of $m + 1$ variables. Then there is a number e such that

$$\Phi_e^{(m)}(x_1, \cdots, x_m) = g(e, x_1, \cdots, x_m). \tag{40}$$

e is the Gödel number of $\Phi_e^{(m)}$.

Note that usually $\stackrel{\sim}{=}$ is used to show "equality" between partial functions, but Davis uses $=$.

Proof. Consider the partially computable function

$$g(S_m^1(v, v), x_1, \cdots, x_m), \tag{41}$$

where S_m^1 is the function that occurs in the parameter theorem. Then we have for some number z_0,

$$g(S_m^1(v, v), x_1, \cdots, x_m) = \Phi^{(m+1)}(e, x_1, \cdots, x_m, v, z_0) \tag{42}$$
$$= \Phi^{(m)}(e, x_1, \cdots, x_m, S_m^1(v, z_0)), \tag{43}$$

where we have used the parameter theorem. Setting $v = z_0$ and $e = S_m^1(z_0, z_0)$, we have

$$g(e, x_1, \cdots, x_m) = \Phi^{(m)}(e, x_1, \cdots, x_m, e) \tag{44}$$
$$= \Phi_e^{(m)}(x_1, \cdots, x_m). \tag{45}$$

[Note] S_m^n is a primitive recursive function that appears in the parameter theorem (which has been called the iteration theorem and s-m-n theorem)[3] (p.85) such that

$$\Phi^{(m+n)}(x_1, \cdots, x_m, u_1, \cdots, u_n, y) = \Phi^{(m)}(x_1, \cdots, x_m, S_m^n(u_1, \cdots, u_n, y)). \tag{46}$$

Put $n = 1$. $S_m^1(v, v)$ is the diagonal sequence of all finite length functions of type $S_m^1(\alpha, \beta)(\alpha, \beta = 1, 2, \cdots)$. Therefore $S_m^1(v, v)$ is an essentially infinite function. So the proof is incorrect.

6.2 Fixed point theorem

We shall quote Rogers[8] (p.21, p.180). P_x is the set of instructions associated with the integer x in the fixed listing of all sets of instructions. x is called the Gödel number of P_x. $\varphi_x^{(k)}$ is the partial function of k variables determined by P_x. x is called Gödel number of $\varphi_x^{(k)}$. We shall drop the subscript (k) when its value is clear from context or when $k = 1$.

The fixed point theorem: Let f be any recursive function; then there exists an n such that

$$\varphi_n = \varphi_{f(n)}. \tag{47}$$

(We call n is a fixed-point value for f.)

Proof. Let any u be given. Define a recursive function Ψ by the following instructions: to compute $\Psi(x)$, first use P_u with input u; if and when this terminates and gives an output w, use P_w with input x; if and when this terminates, take its output as $\Psi(x)$. We summarize this:

$$\Psi(x) = \begin{cases} \varphi_{\varphi_u(u)}(x), & \text{if } \varphi_u(u) \text{ is } convergent; \\ divergent, & \text{if } \phi_u(u) \text{ is } divergent. \end{cases}$$

The instructions for Ψ depend uniformly on u. Take $\tilde{g}$ to be the recursive function which yields, from u, the Gödel number for these instructions for Ψ. Thus

$$\varphi_{\tilde{g}(u)}(x) = \begin{cases} \varphi_{\varphi_u(u)}(x), & \text{if } \varphi_u(u) \text{ is } convergent; \\ divergent, & \text{if } \varphi_u(u) \text{ is } divergent. \end{cases}$$

Now let any recursive function f be given. Then $f\tilde{g}$ is a recursive function. Let v be a Gödel number for $f\tilde{g}$. Since $\varphi_v = f\tilde{g}$ is total, $\varphi_v(v)$ is convergent. Hence putting v for u in the definition of $\tilde{g}$, we have

$$\varphi_{\tilde{g}(u)} = \varphi_{\varphi_v(v)} = \varphi_{f\tilde{g}(v)}. \tag{48}$$

Thus $n = \tilde{g}(v)$ is a fixed-point value, as desired.　　　　qed

[Note] u is the Gödel number of $\varphi_u(x)$. $\varphi_u(u)$ is the diagonal sequence of all finite length functions $\varphi_\alpha(x)(\alpha = 1, 2, \cdots)$. Therefore $\varphi_u(u)$ is an essentially infinite function. $\varphi_{\varphi_u(u)}(x)$ can not be defined.

7. Concluding Remarks

This report has discussed with the diagonal theorem. The findings are as follows. (1) The diagonal theorem does not hold. This indicates that Gödel's proof of the incompleteness theorems is incorrect. Furthermore the proof of Π_1-incompleteness is also incorrect. Besides them we should note that a total recursive function can be expressed by a Σ_1-formula in $Q_<$ and $Q_<$ is Σ_1-complete. (2) There are other theorems, such as Rosser's theorem and Tarski's theorem, that are derived using the diagonal theorem. This fact does not necessary mean that the theorems do not hold. However the theorems must be reexamined. (3) In the theory of computation an essentially infinite length function is applied to derive the recursion theorem and the fixed point theorem. Those theorems must be reviewed.

Including the topics that we do not deal with the author feels that the fundamentals of metamathematics and those of the theory of computaion need more reezamination.

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Appendix

(1) Reflections on the incompleteness theorems [12]

The following was distributed at the meeting (March 5, 2009).

"Errata and supplementary explanations

p.2 line 10 ↓

The definition of the length of a formula shoud be changed to the following. "The predicate logic can be expressed using the following symbols. These are, (1a) individual constants, (2a) variables, (3a) function symbols, (4a) predicate symbols, (a5) logical symbols ($\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \forall, \exists$), (6a) subsidiary symbols (parenthses, comma). The set of symbols (1a) $\sim$ (6a) is called the basic symbols. We assume that defined functions and defined predicates which are not in the basic symbols are rewitten using the basic symbols. Since $\forall x P(x)$ is the abbreviation for $P(1) \vee P(2) \vee P(3) \vee \cdots$, $\forall x P(x)$ represents infinite symbols. However if there exists an axiom such that $\forall x P(x) \rightarrow Q$(that consists of basic symbols), $\forall x P(x)$ is converted to basic symbols. It is the same with $\exists x P$. Taking

the above observations into consideration we define the length of the formula as the smallest number of symbols in it.

p.3 line 3 ↑
"find" $\Longrightarrow$ "found"

Replace all the sentences in Appendix with the followings.
There are two possibilities to construct a complete formal system.
(1) We can not manipulate infinitely long formulas, since a proof is a finite sequence of formulas. Therefore the first measure is exclude infinitely long formulas from the object of our studies.
(2) Another possibility to avoid the incompleteness of a formal system is to change the definition of a proof. That is, change "the finite sequence of formulas" to "the sequence of formulas". In other words, we admit an infinitely long sequence of formulas in a proof."

(2) Reflections on Gödel and Turing [13]

$Sb(x, \lceil z \rceil, Z(y))$ was defined as a predicate in the report, but it must be a function. The correct definition of Sb is as follows: Consider function $Sb(x, \lceil z \rceil, Z(y))$, where x, $\lceil z \rceil$ and $Z(y)$ are the Gödel number of formula **x** that has free variable z, that of z and that of term y, respectively. $Sb(x, \lceil z \rceil, Z(y))$ indicates the Gödel number of the formula obtained by substituting y for z in **x**. Consider $Sb(\alpha, \beta, \gamma)$ for arbitrary α, β and γ. If α is not the Gödel number of a formula, or β is not the Gödel number of a free variable in formula α, or γ is not the Gödel number of a term, $Sb(\alpha, \beta, \gamma) = 0$. In spite of the confusing usage of Sb, the discussions in the report are correct.
Note that Gödel defined Sb as a function, but he always treat Sb as a predicate. For instance, $\neg Sb$.

(3) Gödel and Turing from the Viewpoint of the Theory of Computation [14]

Delete the subsection (3) of Section 3.