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Hoffman パズル解の列挙と一般化に関する研究

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大きさ $(a+b+c)\times(a+b+c)\times(a+b+c)$ の箱に大きさ $a\times b\times c$ の直方体のブロックを 27 個詰め込むパズルは Hoffman パズルと呼ばれている．これは D. G. Hoffman が 1978 年に提案したパズルで，非常に難しいパズルとして知られていて，21 通りの解を持つことがわかっている．ホフマンパズルでは辺の長さに $(a+b+c)/4<a<b<c$ という条件をおいている．2004 年に D. E. Knuth はこの条件を緩めて $(a+b+c)/4=a<b<c$ とし，$(a,b,c)=(3,4,5)$ のとき，28 個目のブロックが入ることを示した．しかしその詳細はよくわかっていない．本研究では Knuth の拡張したパズルを解析し，28 個目のブロックが入るための条件を明らかにした．また 28 個目のピースが入る場合のすべての解を示した．さらに 29 個以上は入らないことを証明した．

On the Hoffman Puzzle and its generalization

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The packing problem of 27 blocks of size $a\times b\times c$ into a box of size $(a+b+c)\times(a+b+c)\times(a+b+c)$ is called the *Hoffman puzzle*. This puzzle was proposed by D. G. Hoffman in 1978, and it is well known as a "difficult" puzzle that has 21 solutions. In the Hoffman puzzle, the lengths should satisfy the condition $(a+b+c)/4<a<b<c$. In 2004, D. E. Knuth loosened the condition to $(a+b+c)/4=a<b<c$, and showed that we can pack the 28th block in the case $(a,b,c)=(3,4,5)$. However, more details are not known. In this paper, we analyze this extended Hoffman-Knuth puzzle, and investigate the condition that the 28th block can be packed. We also show all solutions of this puzzle, and prove that we cannot pack the 29th block.

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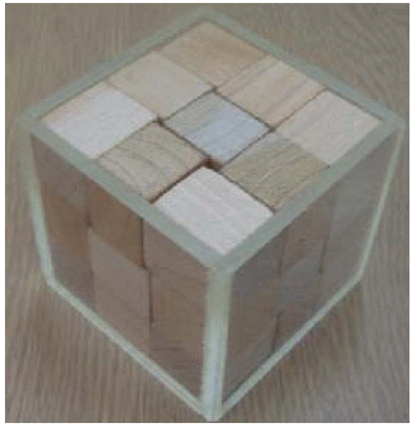

Figure 1 The Hoffman puzzle.

1. Introduction

The *Hoffman puzzle* is one of classic and well-known difficult puzzles in the puzzle society[1)]. We are given 27 congruent *blocks* of size $a\times b\times c$ and one *box* of size $(a+b+c)\times(a+b+c)\times(a+b+c)$. We aim at packing these 27 blocks into the box (see Figure 1). It was proposed by Hoffman in 1978, and Conway and Cutler found all of its 21 solutions in 1981. In the puzzle, the condition $(a+b+c)/4<a<b<c$ is required to inhibit to stack four blocks in a line. In other words, this condition is necessary and sufficient condition that the Hoffman puzzle has 21 solutions.

In 2004, Knuth loosened this condition to $(a+b+c)/4=a<b<c$. He investigated the case $(a,b,c)=(3,4,5)$ and obtained three other solutions that pack the 28th block into the box. We name this extended puzzle *Hoffman-Knuth puzzle* (HK puzzle, for short). This HK puzzle was distributed at a local (and closed) puzzle meeting called IPP (International Puzzle Party) by George Miller in 2005[3)]. Recently, Ishino investigate the case $(a,b,c)=(4,5,7)$ and obtain all possible patterns for packing 28 blocks into the box[2)]. However, as far as the authors know, the details of the HK puzzle have not been investigated up to now.

In this paper, we investigate the HK puzzle and characterize the condition that the

28th block can be packed into the box. More precisely, we show that we can pack the 28th block in the HK puzzle if and only if $a < b \leq 4a/3$ and $5a/3 \leq c < 2a$. That is, even if we have $(a+b+c)/4 = a < b < c$, not always we can pack the 28th block. For example, we can pack the 28th block if $(a,b,c) = (3,4,5),(4,5,7),(5,6,9)$, but we cannot pack it if $(a,b,c) = (5,7,8),(7,10,11),(8,11,13)$. We can generate infinitely many examples in both cases. We also check all possible 20 solutions of the HK puzzle with the 28 blocks. In a sense, these seem to be three essentially different solutions and their variations on local swapping and rotations. Hence this may be a reconfirmation of the solutions by Knuth.

We also consider the problem whether the 29th block can be packed into the box. We prove that it is impossible. Hence we conclude that the HK puzzle has the above solutions, and we have no other solution.

2. Preliminaries

We are given *blocks* of size $a \times b \times c$ and one *box* of size $(a+b+c) \times (a+b+c) \times (a+b+c)$. The *Hoffman condition* is the sequence of the inequations

$$\frac{a+b+c}{4} < a < b < c,$$

and the *Hoffman puzzle* aims at packing the 27 blocks with the condition into the box.

The *Hoffman-Knuth puzzle* (HK puzzle, for short) aims at packing the 28 blocks into the box under the following condition, which is called the *Hoffman-Knuth condition* (HK condition, for short):

$$\frac{a+b+c}{4} = a < b < c.$$

We first show two simple observations:

Observation1 (1) Under the HK condition, there exists an ϵ with $0 < \epsilon < a/2$ such that $a < b = 3a/2 - \epsilon < 3a/2 < c = 3a/2 + \epsilon < 2a$.
(2) Under the HK condition, we cannot pack the 32 blocks into the box.

Proof. (1) By the condition, we have $a+b+c = 4a$ and hence $b+c = 3a$. Since $a < b$, we have $c < 2a$. Thus the observation follows.

(2) Clearly, the total volumes of the blocks has to be less than or equal to the capacity of the box. Hence $(a+b+c)^3 \geq k(abc)$ holds, where k is the number of blocks. By the HK condition, we have $a+b+c = 4a$. By (1), we also have $bc = (3a/2-\epsilon)(3a/2+\epsilon) = 9a^2/4 - \epsilon^2$ with $0 < \epsilon < a/2$. Thus, $k \leq (a+b+c)^3/(abc) = (4a)^3/(a(9a^2/4-\epsilon^2)) < 64a^2/(9a^2/4-(a/2)^2) = 64a^2/(9a^2/4-a^2/4) = 256/(9-1) = 32$. ∎

We here introduce the coordinates to represent the arrangements of the blocks in the box. We fix one corner of the box as $(0,0,0)$. The other corners can be represented as (α,β,γ), where $\alpha,\beta,\gamma \in \{0, a+b+c\}$. For each $1 \leq i,j,k \leq 3$, we define *center points* $p_{i,j,k} = (\alpha_i, \alpha_j, \alpha_k)$ by $\alpha_1 = \frac{a+b+c}{6}$, $\alpha_2 = 3\alpha_1$, and $\alpha_3 = 5\alpha_1$. Roughly speaking, each center point indicates the center of a block. We say that a center point p is included in a box b if p is in b or p is on b. (That is, p is included in b even if it is on the surface of b.)

For any fixed i and j with $1 \leq i,j \leq 3$, a *segment* is a set of blocks that includes one of three centers on a line. More precisely, a segment $S_{i,j,*}$ is a set of blocks that includes $p_{i,j,1}$, $p_{i,j,2}$ or $p_{i,j,3}$. The other segments $S_{i,*,k}$ and $S_{*,j,k}$ are defined in the same manner. In an arrangement of the blocks in the box, a segment S is said to be q-segment if S contains q blocks. In an arrangement of the blocks in the box, we consider the smallest box B that includes all blocks in a segment S. Then the *length* of S is defined by the length of the longest side of B.

In the Hoffman puzzle, we have 27 blocks and 27 segments. Since $(a+b+c)/4 < a$, each segment can contain at most 3 blocks, and of length at most $(a+b+c)$. Counting the lengths of the blocks, $27(a+b+c)$ is the lower bound of the total length of the segments. Hence, the average length of a segment is at least $(a+b+c)$. If some segment have a length less than $(a+b+c)$, there exists a segment of length greater than $(a+b+c)$, which cannot be packed into the box. This implies that each segment has a length exactly $(a+b+c)$. Since $a < b < c$, we can observe that all segments are 3-segment, and each length is given by three different lengths. Thus in each solution of the Hoffman puzzle, each block cannot be slid.

3. Analysis of the HK puzzle

Hereafter, we suppose that the HK condition $\frac{a+b+c}{4} = a < b < c$ holds.

3.1 Extension of the Hoffman puzzle

Under the HK condition, we have $4a = a+b+c$, that is, we may use 4-segment and

Vol.2011-AL-134 No.18
2011/3/7

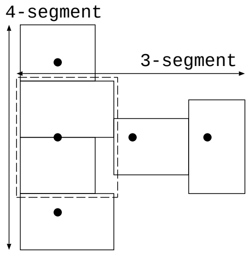

Figure 2 4-segment and 3-segment.

pack the 28th block into the box. In the case, we have to slightly change the notion of 3-segment (Figure 2). A 4-segment is a set of blocks that are aligned along a line with the sides of length a. Then, the other segment are considered as 3-segment. Especially, there can be a set of 4 blocks that horizontally contains three center points in Figure 2. But we consider this set of blocks as a 3-segment that consists of two right blocks and one left bigger virtual block that is the minimum box containing two vertically aligned blocks that share the center point.

3.2 Solutions of the HK puzzle

Recently, Ishino obtains all the solutions of the HK puzzle by computer experiments[2]. Three typical solutions are given in Figure 3. In the first solution, we have two kinds of 3-segments and 4-segments. Considering the combinations and rotations of the segments, we can obtain 6 different solutions from this one. In the second solution, the top half is simple three 4-segments. The bottom half consists of the combination of 4 blocks shown in Figure 4. Considering the combinations and rotations, we obtain 10 solutions. The third solution in Figure 3 contains two sets of 4 blocks in Figure 4, and the other blocks are fixed. Hence we have 4 solutions. Therefore, in total, we have 20 different solutions that can be partitioned to three groups.

3.3 Characterizations of the HK puzzle

We here improve Observation 1 and obtain the main theorem:

Theorem2 The Hoffman Knuth puzzle aims for packing 28 blocks into the box under the HK condition.

(1) The Hoffman Knuth puzzle have 20 solutions if and only if there exists an ϵ with $a/6 < \epsilon < a/2$ such that $a < b = 3a/2 - \epsilon < 3a/2 < c = 3a/2 + \epsilon < 2a$.

(2) For the puzzle, we cannot pack the 29th block into the box.

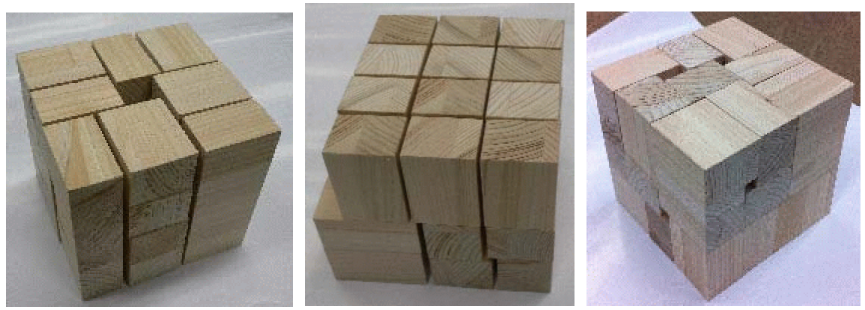

Figure 3 Three typical solutions of the HK puzzle.

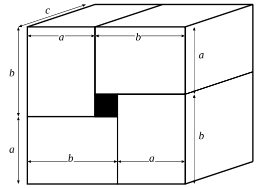

Figure 4 A combination of 4 blocks.

Proof. In all the 20 solutions in section 3.2, we have at least one 3-segment of length $3b$. Moreover, we can observe that these solutions are valid if and only if $3b \leq a+b+c$, or equivalently, $2b \leq a + c$.

(1) By Observation 1, we have an ϵ with $0 < \epsilon < a/2$ such that $a < b = 3a/2 - \epsilon < 3a/2 < c = 3a/2 + \epsilon < 2a$. Then $2b \leq a + c$ implies that $3a - 2\epsilon < a + 3a/2 + \epsilon$, or equivalently, $\epsilon > a/6$.

(2) To derive a contradiction, we assume that the HK puzzle has a solution for 29 blocks. Then we have at least one 4-segment S in the solution. From S, we remove one block. Then the resultant arrangement of 28 blocks forms a solution of the HK puzzle with 28 blocks, and there is at least one 3-segment S' of length $3a$. However, there is no solution with the segment S' in Section 3.2. Since all the solutions with 28 blocks are listed in Section 3.2, this is a contradiction. Hence we cannot pack the 29th block

into the box. ∎

In the Hoffman puzzle, any 3-tuple of (a, b, c) with the Hoffman condition has 21 solutions (we have checked all of them). On the other hand, by Theorem 2, we can partition 3-tuples of (a, b, c) with the HK condition into two groups. For example, we have 20 solutions of the HK puzzle if

$$(a, b, c) = (3, 4, 5), (4, 5, 7), (5, 6, 9), \ldots,$$

but there are no solution if

$$(a, b, c) = (5, 7, 8), (7, 10, 11), (8, 11, 13), \ldots.$$

Intuitively, we have the solutions if we have $a \sim b \ll c \sim 2a$.

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