Bandwidth of Convex Bipartite Graphs and Related Graph Classes

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It is known that the bandwidth problem is NP-complete for chordal bipartite graphs, while the problem can be solved in polynomial time for bipartite permutation graphs, which is a subclass of chordal bipartite graphs. This paper shows that the problem is NP-complete even for convex bipartite graphs, a subclass of chordal bipartite graphs and a superclass of bipartite permutation graphs. We provide polynomial-time approximation algorithms for convex bipartite graphs. We also provide a polynomial-time approximation algorithm for 2-directional orthogonal ray graphs which is a subclass of chordal bipartite graphs and a superclass of convex bipartite graphs.

1. Introduction

A linear layout of a graph G with vertex set $V(G)$ is a bijection $\pi : V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$. The bandwidth of π is defined as $b_\pi(G) = \max\{|\pi(u) - \pi(v)| \mid (u, v) \in E(G)\}$. The *bandwidth* of G is defined as $b(G) = \min b_\pi(G)$ where π ranges over all linear layouts of G. A layout π of G is said to be optimal if $b_\pi(G) = b(G)$. Given a graph G and an integer k, the bandwidth problem asks whether the bandwidth of G is at most k. Since the bandwidth of a graph is the maximum bandwidth over all its connected components, we shall consider only connected graphs.

Let G be a bipartite graph with bipartition (X, Y). The ordering $\prec$ of X is said to fulfill the adjacency property if for each $y \in Y$, the set of neighbors of y consists of vertices that are consecutive in the ordering $\prec$ of X. G is said to be *convex* if there is an ordering of X that fulfills the adjacency property. G is said to be *biconvex* if there is an ordering of X and an ordering of Y that fulfill the adjacency property. A graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$ is called a *permutation graph* if there exists a pair of permutations π_1 and π_2 on $N = \{1, 2, \ldots, n\}$ such that for all $i, j \in N$, $(v_i, v_j) \in E(G)$ if and only if $(\pi_1^{-1}(i) - \pi_1^{-1}(j))(\pi_2^{-1}(i) - \pi_2^{-1}(j)) < 0$. A bipartite graph which is also a permutation graph is called a *bipartite permutation graph.* A bipartite graph G is said to be *chordal* if G contains no induced cycles of length greater than 4. A tree is a chordal bipartite graph by definition. A bipartite graph G with bipartition (X, Y) is called a *2-directional orthogonal ray graph* if, in the xy-plane, there exist a family $\{R_a | a \in X\}$ of horizontal rays (half-lines) extending in the positive x-direction and a family $\{R_b | b \in Y\}$ of vertical rays extending in the positive y-direction, such that two rays R_a and R_b intersect if and only if a and b are adjacent in G. The following relationship between these classes of graphs is known[4),10)]: {Bipartite Permutation Graphs} $\subset$ {Biconvex Bipartite Graphs} $\subset$ {Convex Bipartite Graphs} $\subset$ {2-directional Orthogonal Ray Graphs} $\subset$ {Chordal Bipartite Graphs}.

Papadimitriou[9)] showed that the bandwidth problem is NP-complete for general graphs. Monien[8)] showed that it is NP-complete even for caterpillars of hair length at most 3, which are very special trees. This implies that it is also NP-complete for chordal bipartite graphs. On the other hand, Heggernes, Kratsch, and Meister[5)] recently showed that the bandwidth of bipartite permutation graphs can be computed in polynomial time. Uehara[13)] proposed a faster algorithm for the same problem. Polynomial-time algorithms are also known for chain graphs[6)], interval graphs[12)], and caterpillars of hair length at most 2[1)]. To the best of our knowledge, there are no prior results ascertaining the complexities of the bandwidth problem for 2-directional orthogonal ray graphs, convex bipartite graphs, or biconvex bipartite graphs. We show in Section 2.1 that the bandwidth problem is NP-complete even for convex trees and therefore for 2-directional orthogonal ray graphs. In Section 4, we show that the problem can be solved in polynomial time for biconvex trees .

Several results regarding approximation algorithms to compute bandwidth are known for general and special graph classes. Unger[14)] showed that it is NP-hard to approximate the bandwidth of general graphs within some constant factor. Blache, Karpinski, and Wirtgen[2)] showed that it remains so even for trees (and

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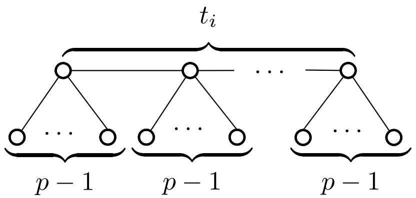
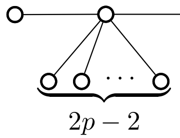
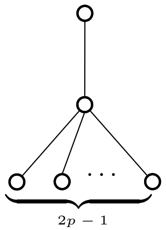


(a) T_i corresponding to task t_i.



(b) p-barrier.



(c) Turning point of height p.

Fig. 1 Components of caterpillar C.

therefore for chordal bipartite graphs). Constant-factor polynomial-time approximation algorithms are known for few special graph classes such as AT-free graphs and its subclasses as shown by Kloks, Kratsch, and Müller[7]. Convex graphs or 2-directional orthogonal ray graphs are not contained in any of these classes. We provide in Section 2.2 a linear-time 4-approximation algorithm and an $O(n \log n)$-time 2-approximation algorithm for convex bipartite graphs, and in Section 3 an $O(n^2 \log n)$-time 3-approximation algorithm for 2-directional orthogonal ray graphs, where n is the number of vertices of a graph.

2. Bandwidth of Convex Bipartite Graphs

2.1 NP-completeness Result

A *caterpillar* is a tree in which all the vertices of degree greater than one are contained in a single path called a *body*. An edge incident to a vertex of degree one is called a *hair*. A *generalized caterpillar* is a tree obtained from a caterpillar by replacing each hair by a path. A path replacing a hair is also called a hair.

Monien[8] showed the following:

Theorem 1. *The bandwidth problem is NP-complete for generalized caterpillars of hair length at most 3.* □

We can show the following by a simple modification of the proof of Theorem 1:

Theorem 2. *The bandwidth problem is NP-complete for convex trees.*

Proof. (Sketch.)As in the proof of Theorem 1, we reduce the multiple processor scheduling problem, which is known to be strongly NP-complete, to our problem. Given a set $T = \{t_1, t_2, \ldots, t_n\}$ of tasks (t_i being the execution time of task i), a deadline D, and the size m of a set $\{1, 2, \ldots, m\}$ of processors, the multiple processor schedule problem asks whether the tasks in T can be scheduled on the m processors satisfying the deadline D. Corresponding to an instance of this problem, a convex tree C is constructed as follows.

Each task t_i is represented by a caterpillar T_i shown in Figure 1(a). Each processor i is represented by a chain P_i of length $D-1$. Special components called "barrier" and "turning point" are constructed as shown in Figure 1(b) and Figure 1(c), respectively. C is constructed from these components as shown in Figure 2. Task caterpillars T_i and T_{i+1} are separated by a chain L_i of length Δ. Processor chains P_i and P_{i+1} are separated by a $(p+1)$-barrier B_i. A turning point of height $p+2n+1$ separates the upper task portion and the lower processor

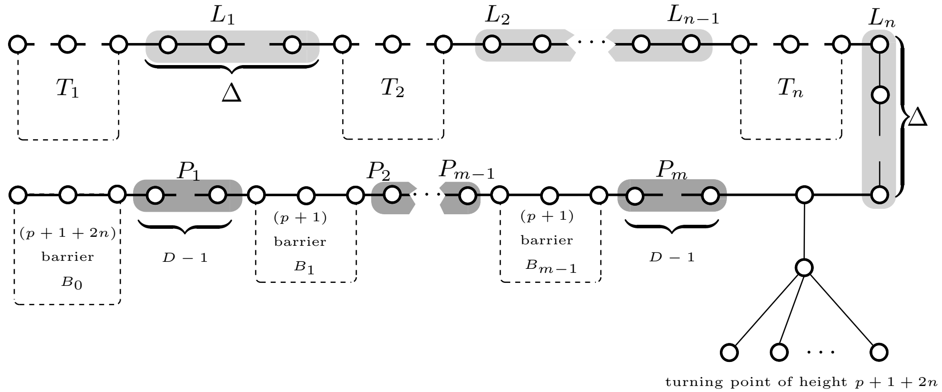


Fig. 2 Instance of bandwidth reduced from multiprocessor scheduling problem

portion. A $(p+2n+1)$-barrier B_0 is attached to the left of P_1.

If we remove from C the degree-1 vertices of the turning point, the remaining tree is a caterpillar. It is easy to see that a caterpillar is biconvex, and therefore both partitions of C have an ordering satisfying the adjacency property. If we restore the degree-1 vertices, irrespective of their position in the ordering of their partition, they do not disturb the adjacency property of the ordering of the other partition. Thus C is a convex tree.

If we set the values of Δ and p such that $\Delta = 2 \times (m(D+2)-2)$ and $p > 2n(D+4)$, it can be shown that the tasks in T can be scheduled on the m processors if and only if C has a bandwidth of $k = p+1+2n$. In fact, this proof is exactly the same as the proof of Theorem 1, except only for a slight difference in the structure of the turning point. Therefore, we shall only briefly describe the idea of the proof here. For a detailed treatment, we refer to Monien[8].

If there exists a scheduling of the tasks in T such that tasks $t_{i_1}, t_{i_2}, \ldots, t_{i_j}$ are assigned to processor i, then C has bandwidth k and an optimal layout can be achieved by

(a) laying out the vertices of the body of $T_{i_1}, T_{i_2}, \ldots, T_{i_j}$ between barriers B_{i-1} and B_i (betwen B_{m-1} and turning point, for $i = m$) and

(b) laying out the vertices of B_0 at the extreme left and those of the turning point at the extreme right.

Conversely, if C has bandwidth k, then in any optimal layout of C,

(a) the turning point must be laid out at one of the extreme ends, and barrier B_0 must be laid out at the other,

(b) all the vertices of the body of each T_j must be laid out between two barriers B_i and B_{i+1} for some i (or B_{m-1} and the turning point for $i = m-1$), and

(c) for each i, if between B_i and B_{i+1} (or between B_{m-1} and turning point for $i = m-1$), bodies of $T_{i_1}, T_{i_2}, \ldots, T_{i_j}$ are laid out, then $t_{i_1} + t_{i_2} + \cdots + t_{i_j} < D$.

This gives us a scheduling of the tasks in T.

□

2.2 Approximation Algorithms for Convex Bipartite Graphs

In this section, we present two algorithms that approximate the bandwidth of convex graphs with worst-case performance ratios of 2 and 4.

Let G be a convex bipartite graph with bipartition (X, Y) and an ordering $\prec$ of X satisfying the adjacency property with $X = \{x_1, x_2, \ldots, x_{|X|}\}$ and $x_1 \prec \ldots \prec x_{|X|}$. Assume $Y = \{1, 2, \ldots, |Y|\}$. Define mappings $s : Y \to \{1, 2, \ldots, n\}$ and $l : Y \to \{1, 2, \ldots, n\}$ such that for $y \in Y$, $x_{s(y)}$ and $x_{l(y)}$ are, respectively, the smallest and largest vertices in $\prec$ adjacenct to y. For each vertex $y \in Y$, let $m(y) = \lceil (s(y)+l(y))/2 \rceil$.

2.2.1 Algorithm 1

Our first algorithm is described in Figure 3. Algorithm 1 takes as input G along with the mappings s and l and outputs a linear layout π of G. The idea of the algorithm is to lay out the vertices of X in the same order as they appear in $\prec$ and insert the vertices of y between them, such that for each $y \in Y$, $\lfloor |N(y)|/2 \rfloor$ vertices of the set $N(y)$ of its neighbors are onto its left and the remaining to its right. Algorithm 1 starts by computing $m(y)$ for each vertex of Y and sorting the vertices according to their $m(i)$ values (Lines 1 and 2). It incrementally assigns labels to the vertices of X in the order in which they appear in $\prec$; stopping at each x_j to check whether there is a vertex in y with $m(y)$ value equal to j, in which case it assigns the current label to y. The process is repeated until all vertices have been labelled (Lines 3 through 8).

1 Compute $m(i)$ for each vertex $i \in Y$. Add a dummy vertex $|Y|+1$ to Y with $m(|Y|+1) = |X|+1$.
2 Let $\sigma(1), \ldots, \sigma(|Y+1|)$ be the vertices of Y sorted in the non-decreasing order of $m(i)$ value, where σ is a permutation on $\{1, \ldots, |Y|+1\}$.
3 Initialize $i \leftarrow 1$, $j \leftarrow 1$, $k \leftarrow 1$.
4 **while** $(j \leq |X|)$
5   **if** $j < m(\sigma(i))$
6         $\pi(x_j) = k$; $j \leftarrow j+1$; $k \leftarrow k+1$.
7   **else if** $j = m(\sigma(i))$
8         $\pi(\sigma(i)) = k$; $i \leftarrow i+1$; $k \leftarrow k+1$.
9 **return** π

Fig. 3 Algorithm 1.

Consider a layout π output by Algorithm 1. For a vertex $y \in Y$, let G_y be the subgraph of G induced by the vertices in

$$V_y = \{v|\pi(x_{s(y)}) \leq \pi(v) \leq \pi(y)\} \cup \{v|\pi(y) \leq \pi(v) \leq \pi(x_{l(y)})\}.$$

The diameter of a graph is the least integer k such that a shortest path between any pair of vertices of the graph is at most k.

Lemma 3. *The diameter of G_y is at most 4.*

Proof. Let u, v be a pair of vertices of V_y.

Consider first the case that both $u, v \in V_y \cap X$. Since π preserves the ordering $\prec$ of X, u must be x_i and v must be x_j for some $s(y) \leq i, j \leq l(y)$. Thus both u and v are adjacent to y. Hence the distance between u and v is 2.

Consider next the case that $u \in V_y \cap X$ and $v \in V_y \cap Y$. Vertex v must be adjacent to at least one vertex u' in $V_y \cap X$. If not, then it must be that v is connected to some x_j with $j < s(y)$ or $j > l(y)$, which means that $m(v) < s(y)$ or $m(v) > l(y)$, contradicting the assumption that Algorithm 1 placed v between $x_{s(y)}$ and y or between $x_{(l(y)}$ and y. If $u = u'$, then the distance between u and v is 2. Else, both u and u' are at distance 2 from the earlier case, and therefore u and v are at a distance 3.

Consider finally the case that both $u, v \in V_y \cap Y$. From the earlier case u must be adjacent to some vertex u', v must be adjacent to some vertex v'. Also u' and v' are both adjacent to y. Hence the distance between u and v is at most 4.

In all the above cases, the shortest path between any pair of vertices does not exceed 4, and thus we have the lemma. □

The following is a well-known lower bound for the bandwidth of a graph[1)].

Lemma 4. *For a graph G, $b(G) \geq \max\lceil (N'-1)/D' \rceil$, where the maximum is taken over all connected subgraphs G' of G, N' is the number of vertices of G', and D' is the diameter of G'.*

We are now ready to show the approximation ratio of Algorithm 1.

Lemma 5. *For layout π returned by Algorithm 1, $b_\pi(G) \leq 4 \times b(G)$.*

Proof. Let (x, y), $x \in X, y \in Y$ be an edge of G such that $|\pi(x) - \pi(y)| = b_\pi(G)$. Let V'_y be the set of vertices v such that v lies between x and y in π. Then $b_\pi(G) = |V'_y| - 1$. On the other hand, from Lemmas 3 and 4, we get $b(G) \geq \lceil (|V_y| - 1)/4 \rceil$. Thus we have:

$$\frac{b_\pi(G)}{b(G)} \leq \frac{|V'_y| - 1}{(|V_y| - 1)/4}$$

Since the order of X in $\prec$ is preserved in π, x must be $x_{s(y)}$ or $x_{l(y)}$, and therefore $V'_y \subseteq V_y$. Thus we get:

$$\frac{b_\pi(G)}{b(G)} \leq 4.$$

□

There exist graphs for which this ratio is asymptotically equal to 4. Figure 4(a) shows an example of such a graph. Let us assume that the mappings s and l provided to Algorithm 1 are based on the left to right ordering of the vertices of the upper partition as shown in Figure 4(a). The layout π returned by Algorithm 1 will lay out between y and x_{n+1} all the vertices $x_i, x'_i, y_{ij}, y'_{ij} (1 \leq i \leq n, 1 \leq j \leq 2n-1)$. Thus $b_\pi(G) = 2n^2 + 2n + 1$. On the other hand, the diameter of this graph is 4, and so from Lemma 4,

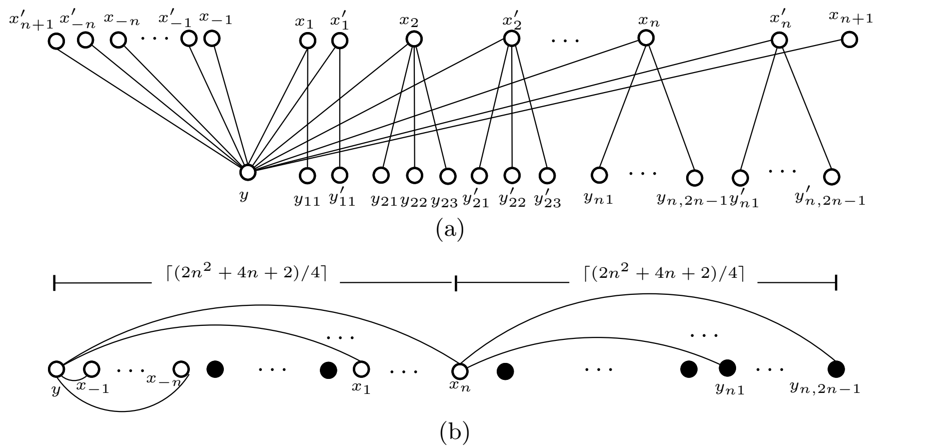


Fig. 4 (a)An example for which the approximation ratio of Algorithm 1 is asymptotically equal to 4. (b) A layout with bandwidth $\lceil (2n^2 + 4n + 2)/4 \rceil$. Only the half right of y is shown as the left half contains the primed counterparts in a symmetric layout. The regions denoted by black vertices denote the vertices y_{ij}, which can be laid out within the same bandwidth.

$b(G) \geq \lceil (2n^2+4n+2)/4 \rceil$. In fact, for large values of n, there is a layout of bandwidth $\lceil (2n^2+4n+2)/4 \rceil$, as shown in Figure 4(b). Thus the approximation ratio $b_\pi(G)/b(G)$ is aymptotically equal to 4.

Algorithm 1 can be implemented to run in $O(|X|+|Y|)$ time. So it follows from Lemma 5 that:

Theorem 6. *Algorithm 1 computes a linear layout of a convex graph G with bipartition (X,Y) in $O(|X|+|Y|)$ time such that $b_\pi(G) \leq 4 \times b(G)$.* □

If only G, and not s and l, is given, we can compute an ordering satisfying the adjacency property (and thus s and l) in time linear to the number of vertices and edges of the graph, as shown by Booth and Lueker[3]. In that case, the time complexity would be $O(|X|+|Y|+|E|)$, where E is the edge set of G. In the next subsection, we show a different algorithm that runs slower but improves the approximation ratio to 2.

2.2.2 Algorithm 2

Let G be a convex bipartite graph with bipartition (X,Y) and an ordering $\prec$ of X satisfying the adjacency property with $X = \{x_1, x_2, \ldots, x_{|X|}\}$ and $x_1 \prec \ldots \prec x_{|X|}$. Let s and l be mappings defined in Section 2.2.1. Let G_I be a graph obtained from G by adding to it an edge (y_1, y_2) for each pair $y_1,y_2 \in Y$ having a common neighbor. A graph is said to be an *interval graph* if for every vertex of the graph, there exists an interval on the real line, such that two intervals intersect if and only if their corresponding vertices are adjacent.

Lemma 7. *G_I is an interval graph.*

Proof. We can see that G_I is an interval graph by defining interval $[i,i]$ for each vertex $x_i \in X$, and interval $[s(y), l(y)]$ for each vertex $y \in Y$. □

Lemma 8. $b(G_I) \leq 2b(G)$

Proof. Let π be an optimal layout of G. Consider the bandwidth of the same layout of graph G_I. For edge $(u,v) \in E(G) \cap E(G_I), \pi(u) - \pi(v) \leq b(G)$. For edge $(u,v) \in E(G_I) \setminus E(G)$, there exists a common neighbor of u and v in G, and therefore $\pi(u) - \pi(v) \leq 2b(G)$. Thus $b_\pi(G_I) \leq 2b(G)$. Since $b(G_I) \leq b_\pi(G_I)$, we get $b(G_I) \leq 2b_\pi(G)$. □

Sprague[12] showed the following about interval graphs.

Lemma 9. *For an interval graph with n vertices, the bandwidth problem can be solved in $O(n \log n)$ time if the interval model is provided.*

Given a convex bipartite graph G and mappings s and l, Algorithm 2 simply constructs the interval model of G_I and applies the algorithm for interval graphs. The interval model of G_I can be constructed from s and l in time linear to the number of vertices in G, and therefore we have from Lemmas 8 and 9 the following theorem:

Theorem 10. *Algorithm 2 computes a linear layout π of a convex graph G with n vertices in $O(n \log n)$ time such that $b_\pi(G) \leq 2 \times b(G)$.* □

For a path of length 3, whose bandwidth is 1, Algorithm 2 may return a layout of bandwidth 2. Therefore the above-mentioned bound is tight.

3. Bandwidth of 2-directional Orthogonal Ray Graphs

Since the set of convex bipartite graphs is a proper subset of the set of two-directional orthogonal ray graphs, the bandwidth problem is NP-complete for 2-directional orthogonal ray graphs, by Theorem 2. In this section, we show a 3-approximation algorithm for 2-directional orthogonal ray graphs.

Let G be a bipartite graph with bipartition (X,Y), and let $(\prec_X, \prec_Y)$ be a pair of orderings of X and Y, respectively. Two edges (x,y) and (x',y') of G are said to *cross* in $(\prec_X, \prec_Y)$ if $x' \prec_X x$ and $y \prec_Y y'$. If for every pair (x,y) and (x',y') that cross, (x',y) is also an edge of G, then $(\prec_X, \prec_Y)$ is said to be a *weak ordering* of G. If for every pair (x,y) and (x',y') of crossing edges, both (x,y') and (x',y) are edges of G, then $(\prec_X, \prec_Y)$ is said to be a *strong ordering* of G.

Spinrad, Brandstädt, and Stewart[11] gave the following characterization of bipartite permutation graphs.

Lemma 11. *A graph G is a bipartite permutation graph if and only if G has a strong ordering.* □

In an earlier work[10], we showed the following characterization of 2-directional orthogonal ray graphs.

Lemma 12. *A graph G is a 2-directional orthogonal ray graph if and only if G has a weak ordering.* □

Given a 2-directional orthogonal ray graph G with bipartition (X,Y), edge set E, and a weak ordering $(\prec_X, \prec_Y)$ of G, we can construct a graph G_{BP} having vertex set $V_{BP} = X \cup Y$ and edge set $E_{BP} = E \cup E'$, where E' is the set consisting of an edge (x, y') for every pair of edges (x, y) and (x', y') that cross in $(\prec_X, \prec_Y)$.

Lemma 13. *G_{BP} is a bipartite permutation graph.*

Proof. We will show that G_{BP} is a bipartite permutation graph by showing that $(\prec_X, \prec_Y)$ is a strong ordering of G_{BP}.

Let $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ be two edges of G_{BP} that cross in $(\prec_X, \prec_Y)$. We distinguish three cases: **Case 1**. both $e_1, e_2 \in E$, **Case 2**. one each of e_1, e_2 is in $E' \setminus E$ and E, and **Case 3**. both $e_1, e_2 \in E' \setminus E$.

Case 1: Since $(\prec_X, \prec_Y)$ is a weak ordering of G, $(x_2, y_1) \in E$. By definition of E', $(x_1, y_2) \in E'$. Hence both $(x_2, y_1), (x_1, y_2) \in E_{BP}$.

Case 2: Without loss of generality, assume $e_1 \in E' \setminus E$ and $e_2 \in E$. By definition of E', $e_1 \in E' \setminus E$ implies that there exist $y'_1 \prec_Y y_1$ and $x'_1 \prec_X x_1$ such that (x_1, y'_1), $(x'_1, y_1) \in E$ and they cross. Since (x_1, y'_1) and (x_2, y_2) also cross, (x_1, y_2) must be in E' and therefore in E_{BP}. To see that $(x_2, y_1) \in E_{BP}$, we further distinguish three cases depending on the order of x'_1 and x_2 in $\prec_X$.

Case 2.1. $x'_1 = x_2$**:** $(x_2, y_1) = (x'_1, y_1)$ and hence $(x_2, y_1) \in E \subseteq E_{BP}$.

Case 2.2. $x_2 \prec_X x'_1$**:** since (x'_1, y_1) and (x_2, y_2) cross, $(x_2, y_1) \in E \subseteq E_{BP}$.

Case 2.3. $x'_1 \prec_X x_2$**:** since (x_1, y'_1) and (x_2, y_2) cross, $(x_2, y'_1) \in E$; Moreover, (x_2, y'_1) and (x'_1, y_1) cross, implying that $(x_2, y_1) \in E' \subseteq E_{BP}$.

In all the above subcases of Case 2, we have shown that $(x_2, y_1) \in E_{BP}$, and hence both $(x_2, y_1), (x_1, y_2) \in E_{BP}$.

Case 3: By definition of E', $e_1 \in E' \setminus E$ implies that there exist $y'_1 \prec_Y y_1$ and $x'_1 \prec_X x_1$ such that $(x_1, y'_1), (x'_1, y_1) \in E$ and they cross. Again by definition of E', $e_2 \in E' \setminus E$ implies that there exist $y'_2 \prec_Y y_2$ and $x'_2 \prec_X x_2$ such that (x_2, y'_2),$(x'_2, y_2) \in E$ and they cross. Since (x_1, y'_1) and (x'_2, y_2) also cross, (x_1, y_2) must be in E' and therefore in E_{BP}. To see that $(x_2, y_1) \in E_{BP}$, we further distinguish three cases depending on the order of x'_1 and x_2 in $\prec_X$.

Case 3.1. $x'_1 = x_2$**:** since $(x_2, y_1) = (x'_1, y_1)$, we have $(x_2, y_1) \in E \subseteq E_{BP}$.

Case 3.2. $x_2 \prec_X x'_1$**:** since $(x'_1, y_1) \in E$ and $(x_2, y_2) \in E' \setminus E$ cross, we have $(x_2, y_1) \in E_{BP}$ from Case 2.

Case 3.3. $x'_1 \prec_X x_2$**:** we further distinguish three cases, depending on the order of y'_2 and y_1 in $\prec_Y$.

Case 3.3.1. $y'_2 = y_1$**:** since $(x_2, y_1) = (x_2, y'_2)$, we have $(x_2, y_1) \in E \subseteq E_{BP}$

Case 3.3.2. $y'_2 \prec_Y y_1$**:** since $(x_2, y'_2) \in E$ and $(x'_1, y_1) \in E$ cross, $(x_2, y_1) \in E' \subseteq E_{BP}$.

Case 3.3.3. $y_1 \prec_Y y'_2$**:** since $(x_1, y_1) \in E' \setminus E$ and $(x_2, y'_2) \in E$ cross, we have $(x_2, y_1) \in E_{BP}$ from Case 2.

In all the above subcases of Case 3, we have shown that $(x_1, y_1) \in E_{BP}$, and hence both $(x_2, y_1), (x_1, y_2) \in E_{BP}$.

Thus we have shown that for every $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ of G_{BP} that cross in $(\prec_X, \prec_Y)$, both (x_2, y_1) and (x_1, y_2) are also edges of G_{BP}; and therefore from Lemma 11, G_{BP} is a bipartite permutation graph. □

Lemma 14. $b(G_{BP}) \leq 3 \times b(G)$.

Proof. Let π be an optimal layout of G. Consider the same layout of G_{BP}. For an edge (x, y) of $G \cap G_{BP}$, $|\pi(x) - \pi(y)| \leq b(G)$. For an edge (x, y) of $G_{BP} \setminus G$, there exist vertices $x' \in X$ and $y' \in Y$ such that $(y, x'), (x', y), (y', x)$ are edges of G, and therefore $|\pi(x) - \pi(y)| \leq 3 \times b(G)$. Thus we have $b_\pi(G_{BP}) \leq 3b(G)$. Since $b(G_{BP}) \leq b_\pi(G_{BP})$, we get $b(G_{BP}) \leq 3 \times b(G)$. □

We shall assume that along with a 2-directional orthogonal ray graph G, a weak ordering $(\prec_X, \prec_Y)$ is also provided as input. If not, then such an ordering as be computed in $O(n^2)$ time, where n is the number of vertices of G[10]. We can construct G_{BP} from G in $O(n^2)$ time. This can be done by first remembering for each $x \in X$, its smallest neighbor y_x in $\prec_Y$ and for each $y \in Y$, its smallest neighbor x_y in $\prec_X$, and then adding to G an edge (x, y) for each pair x, y for which $y_x \prec y$ and $x_y \prec x$. Uehara[13] showed that an optimal layout of a n-vertex bipartite permutation graph having bandwidth k can be computed in $O(n^2 \log k)$ time. Then it follows from Lemma 14 that:

Theorem 15. *There is an $O(n^2 \log n)$-time algorithm which computes a linear layout π of an n-vertex 2-directional orthogonal ray graph G such that $b_\pi(G) \leq 3 \times b(G)$.* □

4. Bandwidth of Biconvex Trees

The *2-claw* is a graph obtained from the complete bipartite graph K_{13} by replacing each edge by a path of length 2. The following lemma can be quickly verified.

Lemma 16. *The 2-claw is not a biconvex tree.* □

Biconvex trees can be characterized as follows:

Lemma 17. *A tree T is biconvex if and only if T is a caterpillar.*

Proof. The sufficiency is easy. To prove the necessity, suppose T is a biconvex graph. Let P be a longest path in T. If the length of P is less than five, T is trivially a caterpillar, and so we assume that it is greater than five. Suppose there exists a vertex not in P having degree greater than 1. This implies that T contains the 2-claw as a subtree, contradicting the assumption that T is biconvex graph. Therefore T is a caterpillar. □

Assmann, Peck, Sysło, and Zak showed the following:

Lemma 18. *The bandwidth of generalized caterpillars of hair length at most two can be computed in linear time.* □

From Lemma 17 and Lemma 18, we have:

Theorem 19. *The bandwidth of biconvex trees can be computed in linear time.* □

We conclude this paper with a note that the complexity of bandwidth problem for biconvex graphs is open.

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