Information Processing in Japan Vol. 10, 1970

About the Three-Dimensional Hidden-Line Problem

HIDETAKA KAMIUCHI*

1. *Introduction*

Recently the field of computer graphics has played more and more important role in computer engineering and information processing engineering, since L. G. Roberts [1], I. E. Sutherland [2]. and et al. had the epoch making works. While many I/O devices have been developed in various directions, many fields have their relation eith each other through their development. Later many papers on the hidden-line problem appeared and discussed about the method on the base of visibility of polygons [5]~[7], and about the method of visibility computing of quadric surfaces [8] and etc. On one side, there exists the various researches as follows; computer aided design of surfaces, generation of surfaces, connection of surfaces, and polynominal approximation [9]~[13], and the representation of intersections [14]. Especially the formers are useful in the field of style design of airplanes and automobles, and in the field of parts design of machinary.

In this paper, we will introduce three new concepts; gazability, visibility and plottability. And we will investigate their meanings and relation among them.

Then an algorithm to get hidden-lines point-by-point is to be described. It will be shown with several examples that the problem to get concrete criterions of gazability and visibility is reduced to solving some simultaneous equations and some inequalitis. We will show some concrete criterions which we can compute numerically without so called "gradient method". And an experimental result on the computing time is given.

2. *Several Definitions and Their Nature*

For facility, we define the following symbols; P: a visual point, S: a surface of sight, Σ: the set of points which compose objects in the 3-D (three-dimensional) Euclidian space, K: the set of the points to be examined (in this paper, we suppose that K would be previously obtained independently of the set Σ, $\overrightarrow{PX}$: a spacial curve between P and X that does not intersect itself, $|\overrightarrow{PX}|$: the distance along $\overrightarrow{PX}$ between P and X, $\overrightarrow{PX}^*$: a half spacial curve that

This paper first appeared in Japanese in Joho-Shori (the Journal of the Information Processing Society of Japan), Vol. 11, No. 3 (1970), pp. 144-154.

* Faculty of Engineering, Kyoto University.

has P as the terminate, passes through X, and does not intersect in itself, $*\overrightarrow{PX}$: a half spacial curve that has P as the terminate and whose extension passes through X and that does not intersect in itself, Φ: an empty set.

Definition of the Obstruction Set

Objects in the space are considered as the subset of the set of points Σ. For example, there are three objects; A, B and C. Then Σ is represented as follows;

$$\Sigma=\{X \mid X\in A \text{ or } X\in B \text{ or } X\in C\}.$$

Now standing at a given visual point, let us look at X on Σ. The intersection of $\overrightarrow{PX}$ and Σ is called the obstruction set of X, denoted $\Sigma^*(X)$. The subset of Σ connected with $\Sigma^*(X)$ is called the suspected obstruction set of X, denoted $\Sigma^\circ(X)$.

We will set the following rule;

Rule 1 The point X and the point P are not the elements of $\Sigma^*(X)$.

Then we can describe $\Sigma^*(X)$ and $\Sigma^\circ(X)$ as follows;

$$\Sigma^*(X)=\overrightarrow{PX}\cap\{\Sigma-\{X\}-\{P\}\}=\overrightarrow{PX}\cap\Sigma^\circ(X)$$

$$\Sigma^\circ(X)=\{Y \mid Y\in\Sigma,\ \Sigma^\circ(X) \text{ is connected with } \Sigma^*(X)\}$$

$$\Sigma\ (X)=\Sigma-\{X\}-\{P\}\supset\Sigma^\circ(X)$$

$\Sigma(X)$ is often used, but has not a special name.

Definition of Gazability

When we look at objects through a window, this window determines a visual field. From its analogy, we can give the following definition.

Definition 1

If $\overrightarrow{PX}^*\cap S\neq\Phi$ for $X\in K$, X is called gazable. If $\overrightarrow{PX}^*\cap S=\Phi$ for $X\in K$, X is called ungazable.

Using the way how the structure of a camara or human eyes determines a visual field, we can define gazability as follows;

Definition 1′

If $*\overrightarrow{PX}\cap S\neq\Phi$ for $X\in K$, X is called gazable. If $*\overrightarrow{PX}\cap S=\Phi$ for $X\in K$, X is called ungazable.

We will use Definition 1 in the following discussion. If using Definition 1′, the following discussion would change scarecely.

Definition of Visibility $g(X, \Sigma(X))$

Definition 2

If $\Sigma^*(X)\neq\Phi$ for $X\in K$, X is called invisible, and denoted by $g(X, \Sigma(X))=0$. If by $\Sigma^*(X)=\Phi$ for $X\in K$, X is called visible, and denoted by $g(X, \Sigma(X))=1$.

Definition of Plottability $f(Z, \Sigma)$

For any gazable point X, $\overrightarrow{PX} \cap S \neq \Phi$. Then we can give the folloing definition for gazable points.

Definition 3

If $g(Y, \Sigma(Y))=1$ for at least one poiht $Y \in \overrightarrow{PX}^* \cap K$, Z is called plottable, and denoted by $f(Z, \Sigma)=1$. If $g(Y, \Sigma(Y))=0$. for all $Y \in PX^* \cap K$, Z is called implottable. and denoted by $f(Z, \Sigma)=0$, where $Z \in \overrightarrow{PX}^* \cap S$.

charactoristics of the Set of Visible points

Lemma 1

$\overrightarrow{PX}^* \cap K$ is partitioned into two sets Y_1 and Y_2 which don't intersect each other as follows;

$$\overrightarrow{PX}^* \cap K = Y_1 \cap Y_2$$

such that $g(Y, \Sigma(Y))=1$ for any $Y \in Y_1$, and $g(Y, \Sigma(Y))=0$ for any $Y \in Y_2$.

Otherwise if PX^* intersects in itself, $Y_1 \cap Y_2 = \Phi$ does't follow generally.

Theorem 1

If $K \cap \Sigma$, the number of the elements of the set Y_1 is not more than one. We get the following theorem, using Lemma 1 and Definition 3.

Theorem 2

If and only if $Y_1 = \Phi$, $f(Z, \Sigma)=1$. If and only if $Y_1 \neq \Phi$, $f(Z, \Sigma)=0$.

The subset Δ_i of Σ

Let $\Sigma(X) = \bigcup_{j=1}^{m} \Delta_j(X)$. Then $\Delta_j(X) = \Delta_j - \{X\} - \{P\}$, because $\Sigma = \bigcup_{j=1}^{m} \Delta_j$. Now let $\Delta_j^*(X) = \overrightarrow{PX}^* \cap \Delta_j(X)$. The next theorems follow;

Lemma 2

For $X \in K$, $\Delta_j^* \neq \Phi \leftrightarrow g(X, \Delta_j(X))=0$.

For $X \in K$, $\Delta_j^* = \Phi \leftrightarrow g(X, \Delta_j(X))=1$.

Theorem 3

$$g(X, \Sigma(X)) = \bigcap_{j=1}^{m} g(X, \Delta_j(X)).$$

Theorem 4

For $X_0 \in K$, $f(Z, \Sigma) = \bigcup_{X \in \overrightarrow{PX_0}^* \cap K} g(X, \Sigma(X)) = \bigcap_i f(Z, \Delta_i)$,

where $\overrightarrow{PX_0}^* \cap S \neq \Phi$.

Corollary 1

$$\bigcup_{X \in \overrightarrow{PX}^* \cap K} \bigcap_i g(X, \Delta_i(X)) = \bigcap_i \bigcup_{X \in \overrightarrow{PX}^* \cap K} g(X, \Delta_i(X)).$$

The Second Rule

When there are more than two elements in $\overrightarrow{PX}^* \cap K$, the point on $\overrightarrow{PX}^* \cap S$ corresponding to them is called that it degenerates. We will set the following rule.

Rule 2 The elements of the union set L of planes and lines containing the neighborhood of the visual point P are not the elements of the obstruction set of any $X \in K$.

Then we can rewrite the obstruction set $\Sigma^*(X)$ as follows;

$$\Sigma^*(X)=\overrightarrow{PX} \cap \Sigma(X), \quad \Sigma(X)=\Sigma-\{X\}-\{P\}-L.$$

So the next lemma follows for plottability.

Lemma 3

$$f(Z, \Sigma)=f(Z, \Sigma-L).$$

Now we will set the following condition COND.

COND

Σ is a union set of planes and lines. K is a union set of curves containing any boundary line of Σ.

Then we can get the following theorem.

Theorem 5

If ψ ($\neq \Phi$) is the subset of Σ and consists of the neighberhood of the elements of $\overrightarrow{PX}^* \cap K$, then $f(Z, \psi)=1$ follows under the condition COND.

Corollary 2

$$f(X, \Sigma)=f(Z, \Sigma-\psi) \text{ where } \psi \neq \Phi.$$

3. *An Algorithm For Detecting Plottability* (Fig. 1)

Let $S(X)=S-\{X\}-\{P\}$. Then from Definition 1 and Definition 2, if $g(X, S(X))=0$, X is gazable. Now Σ is devided into simple objects represented in the form of the eq. (5.1), the eq. (5.3) or the eq. (5.6). We use a finite set of X_1, $X_2, \cdots\cdots, X_n$ instead of an infinite set K, such that a union set of the neighborhood of X_i with the same radius r covers the set K. Then plottability for X_i is detected as Fig. 1 shows. Then we will describe in the section 5 the definite method for computing gazability and visibility concretely.

4. *A Visual Point, A Line of Sight and A Surface of Sight*

The principle of Fermat tells us that light goes from X to P so that the value

$$[PX]=\int_X^P n(x, y, z)\, ds \tag{4.1}$$

should have the extreme value where ds is a line element along the line of light and $n(x, y, z)$ is the refractive index of a medium. Now if we consider the medium as homogeneous one with isotropy, we gain the straight line of sight. The equation of a straight line of sight is represented in the following three ways.

$$R=(P-X)s_1+X \qquad 0<s_1<1 \tag{4.2}$$

$$R=(X-P)s_2+P \qquad 0<s_2<1 \tag{4.3}$$

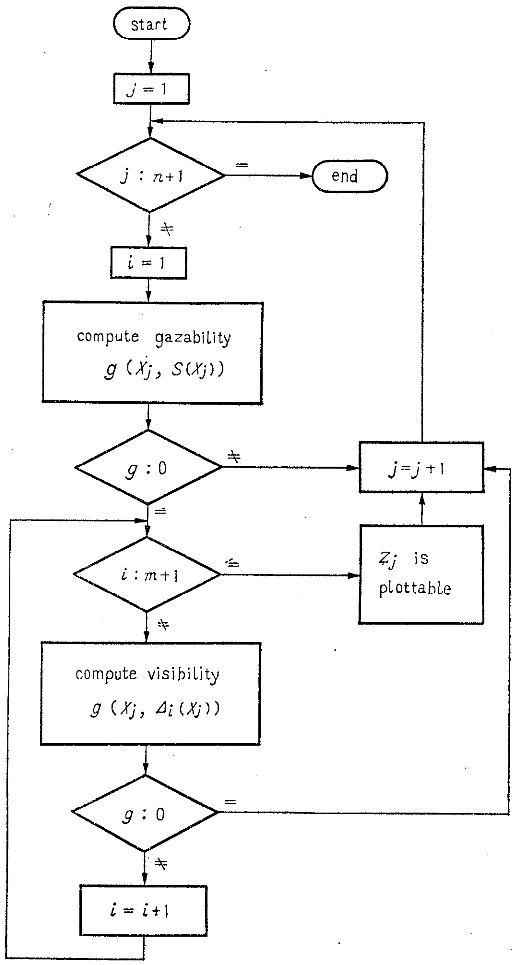

Fig. 1 An Algorithm for Detecting Plottability.

$$R=\Lambda s+X \qquad 0<s \tag{4.4}$$

where R, P and X are vectors, and where s_1, s_2 and s are scalar. Between the eq. (4.2) and the eq. (4.3), $s_1+s_2=1$ follows.

Generally speaking, a surface of sight is represented with the non-linear simultaeous equations as follows;

$$\begin{cases} S_1(x, y, z, p, q)=0 \\ S_2(x, y, z, p, q)=0 \\ S_3(x, y, z, p, q)=0 \end{cases} \tag{4.5}$$

Especially, we represent a surface of sight with a plane of sight as follows;

$$S=Ap+Bq+C, \tag{4.6}$$

where the condition deciding its form is $0\leq p, q\leq 1$. By choising parameters A, B and C in proper ways, we can pay a proper regard to rotation, expansion and contraction, and parallel move. Gazability $g(X, S(X))$ has the inverse nature which visibility $g(X, \Sigma(X))$ has. Now let the plane of sight denoted by S. We can define gazability by using the visibility function from Definition 1 and Definition 2 as follows; if for $X \in K$, $g(X, S(X))=1$, then X is ungazable. If for $X \in K$, $g(X, S(X))=0$, then X is gazable. Ultimately, we can compute gazability with the algorithm of computig visibility.

5. *Representation of Δ_i And The Methods of Computing Visibility*

5.1 *Basic Idea*

There are three methods to represent Δ_i as follows;

Method 1

We represent any surface in the 3-D space by using three direct variables x, y and z. Its nature is represented with

$$Q(x, y, z)=0, \tag{5.1}$$

where its boundary condition is represented with

$$q_i(x, y, z) \leq 0, \qquad i=1, 2, \cdots\cdots, n. \tag{5.2}$$

Method 2

We represent any surface in the 3-D space by using two indirect parameters u and t as follows;

$$R=F(u, t), \tag{5.3}$$

where its boundary condition is generally represented with

$$q_i(u, t) \leq 0 \qquad i=1, 2, \cdots\cdots, m. \tag{5.4}$$

For normalization and simplicity, we use the following representation.

$$0 \leq u, \ t \leq 1. \tag{5.5}$$

Method 3

We represent any body in the 3-D space by using three indirect parameters u, v and t. Its nature is represented with

$$R=F(u, v, t), \tag{5.6}$$

where its boundary condition is generally represented with

$$q_i(u, v, t) \leq 0 \qquad i=1, 2, \cdots\cdots, m. \tag{5.7}$$

For normalization and simplicity, we use the following representation;

$$0 \leq u, \ v, \ t \leq 1. \tag{5.8}$$

We will give the method of computing visibility for each case. Now we use the eq. (4.5) as the line of sight, and obtain the following simultaneous equations using the eq. (4.5) and the equations in each method.

$$\begin{cases} Q(\Lambda_1 s+X_1, \ \Lambda_2 s+X_2, \ \Lambda_3 s+X_3)=0 \\ q_i(\Lambda_1 s+X_1, \ \Lambda_2 s+X_2, \ \Lambda_3 s+X_3) \leq 0 \qquad i=1, 2, \cdots\cdots, r \end{cases} \tag{5.9}$$

$$\Lambda s+X=F(u, t) \qquad 0 \leq u, \ t \leq 1 \tag{5.10}$$

$$\Lambda s+X=F(u, v, t) \qquad 0 \leq u, \ v, \ t \leq 1. \tag{5.11}$$

Now let H_{i}, H_{ii} and H_{iii} be each solution set of s satisfying each eq. (5.9), (5.10) and (5. 11). And let R_+ be the set of positive real numbers. Then we obtain the following criterion;

Criterion

If $H_r \cap R_+ = \Phi$, then the point X is visible. If $H_r \cap R_+ \neq \Phi$, then the point X is invisible where r=i, ii, iii.

Otherwise, when we use the eq. (4. 3), as a line of sight let R_+ as be

$$R_+ = \{x \mid 0 < x < 1\}.$$

Then we can use the same criterion described in the above.

The case that s in the eq. (5. 9) has an explicit solution generally is when the former of the eq. (5. 9) is reduced to polynominals of s of less than order 4. The case that s in the eq. (5. 10) has an explicit solution is when the eq. (5. 10) is linear for u and t, or when it has a special quadric form. About the eq. (5. 11), we need to investigate the nature of the solution of some simultaneous inequalities.

5.2 *Concrete Criterion*

If a given surface has the following representation, we can get an explicit solution on visibility.

$$Q(x, y, z) = a_1x + a_2y + a_3z + a_4 = 0 \tag{5. 12}$$

$$Q(x, y, z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5zy + a_6zx + a_7x + a_8y + a_9z + a_{10} = 0 \tag{5. 13}$$

$$Q(x, y, z) = a_1x^3 + a_2y^3 + a_3z^3 + a_4xy^2 + a_5xz^2 + a_6yx^2 + a_7yz^2 + a_8zx^2 + a_9zy^2 + a_{10}xyz + a_{11}x^2 + a_{12}y^2 + a_{13}z^2 + a_{14}xy + a_{15}yz + a_{16}xz + a_{17}x + a_{18}y + a_{19}z + a_{20} = 0 \tag{5. 14}$$

$$R = Au + Bt + C \qquad 0 \leq u,\ t \leq 1 \tag{5. 15}$$

$$R = Awt + Bt + C \qquad 0 \leq w,\ t \leq 1 \tag{5. 16}$$

$$R = Aut + Bu + Ct + D \qquad 0 \leq u,\ t \leq 1 \tag{5. 17}$$

$$R = Au + Bv + Ct + D \qquad 0 \leq u,\ v,\ t \leq 1 \tag{5. 18}$$

$$R = Auvt + Bvt + Ct + D \qquad 0 \leq u,\ v,\ t \leq 1 \tag{5. 19}$$

The eq. (5. 12) means a plane, and the eq. (5. 13) means a quadric surface. The eq. (5. 14) means a cubic surface. The eq. (5. 15) means a plane. The eq. (5. 16) means a triangular. The eq. (5. 17) means a quadric surface, consisted of lines. The eq. (5. 18) means a parallelepiped hexahedron. The eq. (5. 19) means a triangular pyramid. The method of computing visibility using the eq. (5. 13) was investigated by R. A. Weiss [8] in 1966.

We think that it is very convenient to compute visibility by using a triangular because every surface can be partitioned into triangulars with arbitrary given resolution. Here we give the criterion for such a triangular. It is very simple. For example, we get the following solution of the simultaneous equations of the eq. (4. 5) and the eq. (5. 16) if $|J| \neq 0$.

$$s=\frac{1}{|J|}\begin{vmatrix} C_1-X_1 & -A_1 & -B_1 \\ C_2-X_2 & -A_2 & -B_2 \\ C_3-X_3 & -A_3 & -B_3 \end{vmatrix},\quad t=\frac{1}{|J|}\begin{vmatrix} A_1 & -A_1 & -B_1 \\ A_2 & -A_2 & -B_2 \\ A_3 & -A_3 & -B_3 \end{vmatrix}$$

$$w=\frac{1}{|J|}\begin{vmatrix} A_1 & C_1-X_1 & -B_1 \\ A_2 & C_2-X_2 & -B_2 \\ A_3 & C_3-X_3 & -B_3 \end{vmatrix}\begin{vmatrix} A_1 & -A_1 & C_1-X_1 \\ A_2 & -A_2 & C_2-X_2 \\ A_3 & -A_3 & C_3-X_3 \end{vmatrix}, \tag{5.20}$$

where

$$|J|=\begin{vmatrix} A_1 & -A_1 & -B_1 \\ A_2 & -A_2 & -B_2 \\ A_3 & -A_3 & -B_3 \end{vmatrix}.$$

Then the criterion is given as follows;

1. If $t=0$ and $s>0$, the point X is invisible.
2. If $t\neq 0$, $s>0$ and $0\leq w$, $t\leq 1$, the point X is invisible.
3. In the case except the above 1. and 2., the point X is visible.

When $|J|=0$, if s satisfies the following inequalities

$$\begin{cases} (Ds-E)(Fs-G)\geq 0 \\ \{(F-D)s+E-G\}(Fs-G)\geq 0 \\ |J_k|G\leq |J_k|Fs\leq |J_k|^2+|J_k|G \\ s>0, \end{cases} \tag{5.21}$$

the point X is invisible, and otherwise the point X is visible, where each constant is represented as follows;

$$D=\begin{vmatrix} A_i & B_i \\ A_j & B_j \end{vmatrix},\quad E=\begin{vmatrix} C_i-X_i & B_i \\ C_j-X_j & B_j \end{vmatrix},\quad F=\begin{vmatrix} A_i & A_j \\ A_i & A_j \end{vmatrix}, \tag{5.22}$$

$$G=\begin{vmatrix} A_i & A_j \\ C_i-X_i & C_j-X_j \end{vmatrix},\quad |J_k|=\begin{vmatrix} A_i & B_i \\ A_j & B_j \end{vmatrix},\quad |J_k|\neq 0,\quad \begin{matrix} i\neq j,\ i\neq k, \\ j\neq k, \\ i,\ j,\ k=1,\ 2,\ 3 \end{matrix}$$

In our almost case, $|J|\neq 0$ is satisfied. The case that we need to consider about $|J|=0$ is the case that we compute hidden-lines for N/C machines.

It is very easy to compute the solutions of the inequalities (5.21) and of the eq. (5.18) and the eq. (5.19). But we don't discuss about them in detail.

6. *Conclusion*

In a system, surfaces are often represented in the form of hign order polynominals with normarized boundary conditions [9], [11]. Then the eq. (5.10) for computing visibility are non-linear simultaneous equations. So we cannot generally get the explicit solution of s, u and t. This fact requires repitition computing of so called "gradient method". This requirement gives many faults. Now we suggest that it is very convenient that we partition any surface into triangulars and use only such triangulars as the objects in the 3-D space.

We investigated the modification time of its method in an experiment by cooperation of Mr. Sakanashi. We had neighboring 32 triangulars as Σ and 280 points on Σ as the elements of K. It took 78 seconds on FACOM 230-10 FORTRAN and about 0.1 seconds on HITAC 5020 FORTRAN to compute visibility of one point. It took about 26 seconds on HITAC 5020 FORTRAN to compute visibility of the whole points in that picture. This fact gives us a fore cast that its method may satisfy the requirements of I. E. Sutherland [3].

It is difficult to compute K from arbitrary surfaces automatically, because K consists of outlines and intersections and arbitrary patterns on surfaces. In this paper, we discussed based on the assumption that K is given explicitly.

The author wishes to express his gratitude to Mr. Sakanashi for his coding of FORTRAN programs, Mr. Uehara and Mr. Fusaoka for their kind advices and Professor Hagiwara and his staffs for their discussions.

Reference

[1] Robert, L. C., Machine Perception of Three-Dimensional Solid, Lincoln Laboratory, *Tech. Rep. 315*, 22 May (1963).
[2] Sutherland, I. E., SKETCHPAD Man-Machine Graphical Comminucation System, *Proc. SJCC*, 329-346 (1963).
[3] Sutherland, I. E., Computer Graphics—Ten Unsolved Problems—, *Datamation*, *12*, 5, 22-27, May (1966).
[4] IIT Research Institute, APT Part Programming, McGraw-Hill (1967).
[5] Wylie, C. et al., Half-Tone Drawings by Computer, *Proc. FJCC*, 49-58 (1967).
[6] Chapman, G. A. et al., VISTA—Computed Motion Pictures for Space Research, *Proc. FJCC*, 59-63 (1967).
[7] Freeman, H. et al., An Algorithm for the Solution of the Two-Dimensional "Hidden-Line" Problem, *IEEE Trans*, *EC-16*, 6, 784-790, Dec. (1967).
[8] Weiss, R. A., BE VISION—A Package of IBM 7090 FORTRAN Programs to Draw Orthographi Views of Combinations of planes and Quadric Surface, *J. of ACM*, *13*, 2, 194-204 (1966).
[9] Hosaka, M. et al., On Generation, Storage and Processing of Curve-Connected Patterns, *Joho-Shori*, *6*, 3, 129-138, May (1965) (in Japanese).
[10] Hosaka, M., Theory and Design of Free-Formed Surface, *Joho-Shori*, *8*, 2, 65-72, March (1967) (in Japanese).
[11] Bésier, Piere, Procédé de Définition Numérique des Courbe et Surfaces Non Mathématiques, *Automatisme-TOM XII*, 5, 189-196, Mai (1968).
[12] Shasen, S. H., et al., Experience in the Application of Interactive Graphics, Emerging Concepts in Computer Graphics 1967, *University of Illinois Conference*, W. A. Benjamin Inc. (1968).
[13] Herzog, B., Computer Graphics for Designers, Emerging Concepts in Computer Graphics 1967, *University of Illinois Conference*, W. A. Benjamin Inc. (1968).
[14] Comba, P. G., A Procedure for Detecting Intersection of Three-Dimensional Objects, *J. of ACM*, *15*, 3, 354-366, July (1968).