Display of the Solution Curves of a Differential Equation

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Abstract This paper discusses a method to display the solution curves of given differential equations. The solution curves are represented as sequences of points which are calculated successively by use of difference equations derived from the given differential equations. The characteristics of this method are summarized as follows; (1) the solution curves are displayed in two-dimensional forms, (2) the method is easily programmable, (3) it can deal with non-linear differential equations of implicit forms, (4) the approximated shapes of the solution curves can be obtained within a relatively short period.

1 Introduction

In the fields of science, physics and technology, we often encounter a non-linear differential equation. For the practical purpose of these cases, it is often sufficient only to obtain the approximate solution of the equation. This paper deals with a method to get the approximate solution curves of a given non-linear differential equation by relatively simple numerical calculations. This method is suitable for displaying the solution curves in two-dimensional forms with use of digital plotters or graphical terminals.

On solving a differential equation numerically, several methods are proposed. One of them is a method converting the differential equation into a corresponding difference equation. In this method, the solution of the given differential equation is represented as the limit of the solutions of the corresponding difference equations. The solutions of the difference equation can be obtained by reluxation method. This method has the characteristics that it can offer the solutions with any

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precision but it requires much calculation time because it contains recurrent processes.

Another well known method for solving the differential equation is Runge-Kutta method, where the solution curve is obtained from point to point by calculating the gradient at each point. Because of its simplicity in treatments, Runge-Kutta method is of use in practical cases. But it cannot be applied to non-linear equations directly.

The differential equation considered here is a non-linear differential equation given in an implicit form. In our method, the solution curves are extended successively from point to point as in Runge-Kutta method but instead of calculating the gradient of the curve, the solution curve is extended by calculating a "figure of merit" which is derived from the given differential equation.

The objectives of our method are to obtain the approximate solutions in relatively short period and to represent them in two-dimensional forms.

2 Display of the solution curves of a given differential equation

2.1 Differential equation and its corresponding difference equation

Difference equations play often important roles in solving a differential equation. Namely, the solutions of a differential equation can be represented as a limit of the solutions of its corresponding difference equation. Consider a differential equation

$$f(x, y, y', \ldots, y^{(k)}) = 0 \tag{1}$$

and assume that the solution of (1) is represented as a sequence of points such as

$$(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n) \tag{2}$$

Then the value

$$\frac{y_{i+1}-y_i}{x_{i+1}-x_i} \tag{3}$$

can be regarded as the differential coefficient y' of (1) at point (x_i, y_i) if the value $x_{i+1}-x_i$ is sufficiently small. Similar to the above consideration,

$$\frac{1}{x_{i+1}x_i}\left(\frac{y_{i+2}-y_{i+1}}{x_{i+2}-x_{i+1}}-\frac{y_{i+1}-y_i}{x_{i+1}-x_i}\right) \tag{4}$$

and

$$\frac{1}{d^m}\sum_{j=0}^{m}(-1)^j\binom{m}{j}y_{i+j} \qquad (5)$$

can be regarded as the differential coefficients y" and $y^{(m)}$ at point (x_i, y_i), respectively. Therefore, by replacing the $y', y'', \ldots, y^{(m)}$ of (1) with (3), (4) and (5) ,respectively, we can get the corresponding difference equation

$$F(x_i, y_i, y_{i+1}, y_{i+2}, \ldots\ldots, y_{i+k}) = 0 \qquad (6)$$

of (1). In other words, the solutions of n+1 difference equations (6) may also be regarded as the solutions of the differential equation (1). But in general, it is often difficult to solve the difference equation (6) especially in non-linear cases.

In this paper, we also use the above idea, but we don't solve the equation (6). And further we make the following replacements instead of (3), (4) and (5),

$$\left.\begin{array}{l} x \longrightarrow x_i\,, \qquad\qquad y \longrightarrow y_i \\ y' \longrightarrow \dfrac{\Delta y_i}{\Delta x_i} \triangleq D_i^{(1)} \\ y'' \longrightarrow \dfrac{1}{\Delta x_{i-1}}\left(\dfrac{\Delta y_i}{\Delta x_i} - \dfrac{\Delta y_{i-1}}{\Delta x_{i-1}}\right) \triangleq D_i^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y^{(m)} \longrightarrow \dfrac{1}{\Delta x_{i-1}}\left(D_i^{(m-1)} - D_{i-1}^{(m-1)}\right) \end{array}\right\} \qquad (7)$$

where

$$\left.\begin{array}{l} \Delta x_i \triangleq x_{i+1} - x_i \\ \Delta y_i \triangleq y_{i+1} - y_i \end{array}\right\} \qquad (8)$$

$$\Delta x_i^2 + \Delta y_i^2 = \delta^2 \qquad (9).$$

The meaning of (7) is similar to the one stated before. But the distance between the successive two points (x_i, y_i) and (x_{i+1}, y_{i+1}) is always constant δ along the solution curve (see (9)). Here we define a "figure of merit" g of the approximate solution for equation (1) as follows;

$$g = F(x_i, y_i, \Delta x_i, \Delta y_i, \ldots\ldots, \Delta x_{i-k+1}, \Delta y_{i-k+1}) \qquad (10).$$

The "figure of merit" has the following characteristics. If the k+1 points $(x_{i-k}, y_{i-k}), \ldots\ldots, (x_i, y_i)$ are the solutions of the differential equation (1), the value of (10) will be zero under the condition that the distance between the succesive two points is sufficiently small. Therefore, if we want to extend the solution curve $(x_1, y_1), \ldots\ldots, (x_i, y_i)$ to the next new point (x_{i+1}, y_{i+1}), the curve should be extended to the direction that the "figure of merit" at point (x_i, y_i) satisfies to be zero. In our practical programs, the next point is chosen from the 64($\sim$ 128) points around the considering point (x_i, y_i) (see Fig. 1).

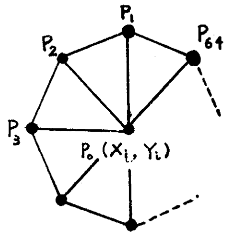

Fig. 1 Searching directions

2.2 Extension of the solution curve

Consider the case that the given differential equation is of k-th order and shown as in the form (1). And further assume that the solution curve has been determined to the i-th step; $(x_1, y_1), \ldots, (x_i, y_i)$. The "figure of merit" at this step is represented as (10) by using the latest k+1 points. But the equation (10) can be reduced to a two variable function

$$g = G(\Delta x_i, \Delta y_i) \tag{11}$$

because of the above assumption. Then the next point (x_{i+1}, y_{i+1}) of the solution curve should be chosen from the set of points around the point (x_i, y_i) so that the "figure of merit" g may satisfies to be zero. In the practical calculations, the point having the smallest value of "figure of merit" is chosen to be the next point. Executing these procedures successively, we can get the whole solution curves of the given differential equation.

2.3 Determination of the initial condition

In the previous section, we have assumed that the k points on the solution curve have been determined in advance. Then, at first we must determine the initial k points in order to execute the previous procedures. On the other hand, the solutions of a given differential equation cannot be determined uniquely without initial conditions. In general, k-th order differential equation requires k initial conditions. The initial conditions of a k-th order differential equation are as follows;

$$\left.\begin{array}{lll} \text{The coordinates of the initial point} & \cdots & p_0 = (x_0, y_0) \\ \text{the first order differential coefficient at } p_0 & \cdots & \left(\frac{dy}{dx}\right)_{p_0} \\ \cdots\cdots\cdots\cdots & & \\ \text{the k-th order differential coefficient at } p_0 & \cdots & \left(\frac{d^k y}{dx^k}\right)_{p_0} \end{array}\right\} \quad (12).$$

In order to obtain the first k-1 points of the solution curve, we use the initial condition (12). The relation between the points and the differential coefficients are given in (7). Then we can get the first k-1 points of the solution curve by solving the simultaneous equation (7) inversely. The first k-1 points of the solution curve are given as follows.

$$\left.\begin{array}{l} \Delta y_{i-1} = \beta\left(\frac{dy}{dx}\right)_{p_0} \\ \Delta y_{i-2} = \beta\left(\frac{dy}{dx}\right)_{p_0} - \beta^2\left(\frac{d^2y}{dx^2}\right)_{p_0} \\ \cdots\cdots\cdots\cdots \\ \Delta y_{i-k+1} = \sum_{j=1}^{k-1} (-\beta)^j \left(\frac{d^jy}{dx^j}\right)_{p_0} \end{array}\right\} \quad (13)$$

,where we assume

$$\Delta x_{i-1} = \Delta x_{i-2} = , \cdots, = \Delta x_{i-k+1} = \beta \qquad (14).$$

Using these initial values (13) and (14), we can execute the procedures stated before.

3 Programs and examples

The procedures mentioned in the previous chapter were programmed with FORTRAN language and the number of the whole steps was about 300. The calculations were executed by computer NEAC 2200/500 and the solution curves were displayed with digital plotter N244A-1.

Example Solve the Bessel's differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - \left(1 - \frac{k^2}{x^2}\right) y = 0 \qquad (15)$$

under the following initial conditions;

$$\text{for } k = 0 \qquad p_0 = (0.1, \quad 0.9975) \qquad \left(\frac{dy}{dx}\right)_{p_0} = -0.0499$$

$$\left.\begin{array}{ll}\text{for } k = 1 & p_0 = (\,0.1,\ \ 0.05) \\ & \left(\frac{dy}{dx}\right)_{p_0} = 0.4981\end{array}\right\} \qquad (16).$$

<u>Answer</u> For the given equation (15), the "figure of merit" is of the form

$$g = \frac{1}{\Delta x_{i-1}}\left(\frac{\Delta y_i}{\Delta x_i} - \frac{\Delta y_{i-1}}{\Delta x_{i-1}}\right) + \frac{1}{x}\,\frac{\Delta y_i}{\Delta x_i} + \left(1 - \frac{k^2}{x_i^2}\right) y_i \qquad (17).$$

The solution curve obtained are shown in Fig. 2. The total number of points taken in the solution curves is about 6000 and the computation time required is about 3 minutes including the drawing.

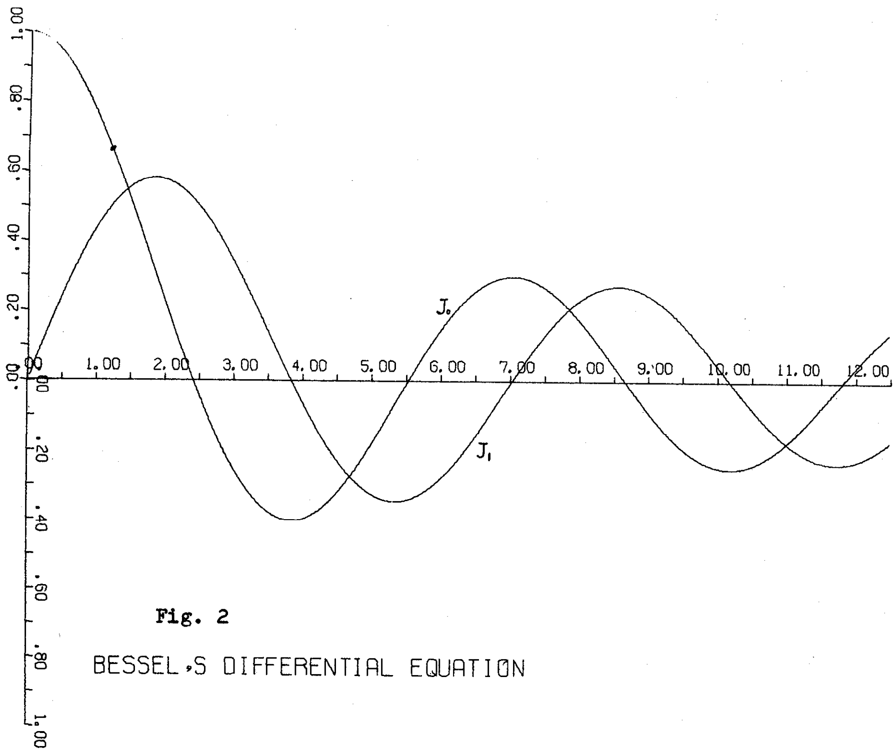

Fig. 2

BESSEL.S DIFFERENTIAL EQUATION

4 Conclusion

The numerical method for solving a differential equation is mentioned. This method can be applied directly to non-linear differential equations of implicit form. This method does not necessarily lead to the exact solutions but it has the following characteristics; (1) it requires little computation time and (2) it can be easily programmable.

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