Computation of Bessel Functions $K_n(z)$ with Complex Argument by τ-method

Toshio Yoshida* and Ichizo Ninomiya*

1. Introduction

The modified Bessel functions $K_n(z)$ are written as

$$K_n(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} f_n\left(\frac{1}{z}\right) \quad , \tag{1}$$

where $f_n(t)$ satisfies the following differential equation

$$t^2 f_n''(t) + 2(t+1) f_n'(t) - \left(n^2 - \frac{1}{4}\right) f_n(t) = 0 \tag{2}$$

In this paper, the τ-method[1,2] proposed by C.Lanczos is applied to the computation of $f_n(t)$. This method is useful for $\mathrm{Re}(t) > 0$ and has the advantage that the value of $f_n(t)$ is efficiently obtained outside the region of small $|t|$.

Now let us employ a finite expansion as an approximation $f_{nm}(t)$ to $f_n(t)$:

$$f_{nm}(t) = b_0 + b_1 t + b_2 t^2 + \cdots\cdots + b_m t^m \tag{3}$$

For this purpose, we consider the following differential equation with the free parameter β

$$t^2 f_n''(t) + 2(t+1) f_n'(t) - \left(n^2 - \frac{1}{4}\right) f_n(t) = \tau C_m^{*(\alpha)}\left(\frac{t}{\beta}\right) \quad , \tag{4}$$

where $C_m^{*(\alpha)}(t/\beta)$ in the additional term is the shifted Ultraspherical polynomial of degree m, which includes the shifted Chebyshev polynomial ($\alpha = 0$) generally used in the τ-method and the shifted Legendre polynomial ($\alpha = 0.5$), and is defined by

$$C_m^{*(\alpha)}(t) = \sum_{k=0}^{m} C_{mk}^{*(\alpha)} t^k = \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(2\alpha)} \sum_{k=0}^{m} (-1)^{m-k} \frac{1}{k!(m-k)!} \frac{\Gamma(2\alpha + m + k)}{\Gamma(\alpha + \frac{1}{2} + k)} t^k \quad . \tag{5}$$

Substituting Eq.(3) into Eq.(4), we have

$$f_{nm}(t) = \tau \sum_{k=0}^{m} \frac{-C_{mk}^{*(\alpha)} S_k(t)}{2(k+1) a_{k+1} \beta^k} \quad , \tag{6}$$

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* Department of Information Science, Faculty of Engineering, Nagoya University, Nagoya, Japan

where

$$a_0 = 1 \ , \qquad a_k = \frac{(4n^2-1)(4n^2-9)\cdots\cdots(4n^2-(2k-1)^2)}{k!\,8^k} \qquad (k \geqq 1) \ ,$$

$$S_k(t) = a_0 + a_1 t + a_2 t^2 + \cdots\cdots + a_k t^k \, .$$

We now satisfy the initial condition $f_{nm}(0)=1$ and then obtain

$$f_{nm}(t) = \sum_{k=0}^{m} \frac{C_{mk}^{*(\alpha)} S_k(t)}{(k+1)a_{k+1}\beta^k} \Big/ \sum_{k=0}^{m} \frac{C_{mk}^{*(\alpha)}}{(k+1)\,a_{k+1}\,\beta^k} \ . \tag{7}$$

The approximation $f_{nm}(t)$ has the free parameters α and β.

2. Error Analysis

We investigate how to determine the parameters α and β so that one can obtain a good approximation $f_{nm}(t)$.

Let $\eta_{nm}(t)$ be the absolute error of the approximation $f_{nm}(t)$:

$$f_n(t) = f_{nm}(t) + \eta_{nm}(t) \tag{8}$$

For $f_{nm}(t)$, the following equation holds

$$t^2 f_{nm}''(t) + 2(t+1) f_{nm}'(t) - \left(n^2 - \tfrac{1}{4}\right) f_{nm}(t) = E_{nm}(t) \ , \tag{9}$$

where

$$E_{nm}(t) = C_m^{*(\alpha)}\left(\frac{t}{\beta}\right) \Big/ \sum_{k=0}^{m} \frac{-C_{mk}^{*(\alpha)}}{2(k+1)\,a_{k+1}\,\beta^k} \ . \tag{10}$$

Accordingly, we obtain from Eqs.(2) and (8) the following equation,

$$t^2 \eta_{nm}''(t) + 2(t+1)\,\eta_{nm}'(t) - \left(n^2 - \tfrac{1}{4}\right)\eta_{nm}(t) = -E_{nm}(t) \ . \tag{11}$$

The general solution of Eq.(11) is expressed by

$$\eta_{nm}(t) = A f_n(t) + f_n(t) \int_0^t \frac{E_{nm}(x)\, g_n(x)}{x^2 \Delta}\, dx + B\, g_n(t) - g_n(t) \int_0^t \frac{E_{nm}(x)\, f_n(x)}{x^2 \Delta}\, dx \ , \tag{12}$$

where

$$\begin{aligned} f_n(t) &= \left(\frac{2}{\pi t}\right)^{\frac{1}{2}} e^{\frac{1}{t}} K_n\!\left(\frac{1}{t}\right) \ , \\ g_n(t) &= \left(\frac{2}{\pi t}\right)^{\frac{1}{2}} e^{\frac{1}{t}} I_n\!\left(\frac{1}{t}\right) \ , \end{aligned} \tag{13}$$

$$\Delta = f_n(x)\, g_n'(x) - f_n'(x)\, g_n(x) = -\frac{2}{\pi x^2}\, e^{\frac{2}{x}} \, .$$

Determining A and B from the following initial conditions

$$\begin{aligned} &\eta_{nm}(0) = 0 \ , \\ &\eta_{nm}'(0) = a_1 - a_1 \sum_{k=1}^{m} \frac{C_{mk}^{*(\alpha)}}{2(k+1)a_{k+1}\beta^k} \Big/ \sum_{k=0}^{m} \frac{C_{mk}^{*(\alpha)}}{2(k+1)a_{k+1}\beta^k} = \frac{1}{2} C_{m0}^{*(\alpha)} \Big/ \sum_{k=0}^{m} \frac{C_{mk}^{*(\alpha)}}{2(k+1)a_{k+1}\beta^k} \ , \end{aligned} \tag{14}$$

we have

$$\eta_{nm}(t)=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{r_{nm}}\left\{\left(\frac{2}{\pi t}\right)^{\frac{1}{2}}e^{\frac{1}{t}}\int_0^t S_{nm}(x)\left(\frac{1}{x}\right)^{\frac{1}{2}}C_m^{*(\alpha)}\left(\frac{x}{\beta}\right)dx\right\}$$
$$+\frac{1}{2}C_{m0}^{*(\alpha)}\frac{1}{r_{nm}}\left(\frac{1}{t}\right)^{\frac{1}{2}}e^{\frac{1}{t}}\lim_{x\to 0}S_{nm}(x)\left(\frac{1}{x}\right)^{-\frac{3}{2}}\quad, \tag{15}$$

where

$$r_{nm}=\sum_{k=0}^{m}\frac{C_{mk}^{*(\alpha)}}{2(k+1)a_{k+1}\beta^k} \tag{16}$$

and

$$S_{nm}(x)=\left(K_n\left(\frac{1}{t}\right)I_n\left(\frac{1}{x}\right)-I_n\left(\frac{1}{t}\right)K_n\left(\frac{1}{x}\right)\right)/e^{\frac{1}{x}}\,. \tag{17}$$

2.1 The Case of Re(t)>0

The second term of the right-hand side in Eq.(15) vanishes for Re(t)>0 and the relative error $\epsilon_{nm}(t)$ is given by

$$\epsilon_{nm}(t)=\eta_{nm}(t)/f_n(t)=p_{nm}\cdot q_{nm}(t)\quad, \tag{18}$$

where

$$p_{nm}=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{r_{nm}} \tag{19}$$

and

$$q_{nm}(t)=\int_0^t\left(I_n\left(\frac{1}{x}\right)-\frac{I_n\left(\frac{1}{t}\right)}{K_n\left(\frac{1}{t}\right)}K_n\left(\frac{1}{x}\right)\right)\frac{\left(\frac{1}{x}\right)^{\frac{1}{2}}}{e^{\frac{1}{x}}}C_m^{*(\alpha)}\left(\frac{x}{\beta}\right)dx\,. \tag{20}$$

We now investigate the behavior of the relative error for various values of α and β.

Example 1 We consider the case t=0.5 as an example of real positive t. For m=8 and n=0,1, the relative errors ϵ_{nm} and the values of the integration q_{nm} are shown in Table 1. It is found from this table that for each α the relative error attains a minimum when β is equal to t, and for α=0.5 the minimum is smaller than the others. It is also seen that the minima of $q_{0,8}$ and $q_{1,8}$ occur at the same time as $\epsilon_{0,8}$ and $\epsilon_{1,8}$. This suggests that the function p_{nm} changes very slowly comparing with q_{nm}. It is sufficient to consider only q_{nm} for studying the behavior of the error ϵ_{nm}.

For a polynomial $F(x)$ of degree m-1, we have

$$\int_0^t F(x)P_m^*\left(\frac{x}{t}\right)dx=0 \tag{21}$$

from the orthogonality of the shifted Legendre polynomial. The above result for q_{nm} suggests that the integrand, with the exception of $C_m^{*(\alpha)}(x/\beta)$, of Eq.(20) is well approximated by a polynomial of degree m-1.

Table 1 Relative errors $\epsilon_{0,8}$ and $\epsilon_{1,8}$ (t = 0.5)

		n = 0		n = 1	
α	t/β	$q_{0,8}$	$\epsilon_{0,8}$	$q_{1,8}$	$\epsilon_{1,8}$
0	0.8	-6.85E-04	4.18E-09	-6.69E-04	-4.91E-09
	0.9	-7.53E-04	2.63E-09	-7.02E-04	-2.93E-09
	1.0	-3.55E-04	7.39E-10	-3.65E-04	-9.05E-10
	1.1	9.99E-03	-1.29E-08	8.07E-03	1.24E-08
0.4	0.8	-5.62E-04	3.27E-09	-5.03E-04	-3.52E-09
	0.9	-2.77E-04	9.24E-10	-2.61E-04	-1.04E-09
	1.0	-1.49E-04	2.97E-10	-1.52E-04	-3.61E-10
	1.1	1.35E-02	-1.67E-08	1.09E-04	1.61E-08
0.49	0.8	-5.92E-04	2.32E-09	-4.97E-04	-2.34E-09
	0.9	-1.16E-04	2.61E-10	-1.03E-04	-2.76E-10
	1.0	-2.55E-05	3.42E-11	-2.61E-05	-4.18E-11
	1.1	2.12E-03	-1.77E-08	1.73E-02	1.71E-08
0.5	0.8	-5.84E-04	2.19E-09	-4.84E-04	-2.19E-09
	0.9	-8.41E-05	1.81E-10	-7.10E-05	-1.83E-10
	1.0	-8.82E-07	1.14E-12	-1.11E-06	-1.71E-12
	1.1	2.22E-02	-1.78E-08	1.81E-02	1.72E-08
0.51	0.8	-5.73E-04	2.07E-09	-4.69E-04	-2.03E-09
	0.9	-4.83E-05	9.98E-11	-3.57E-05	-8.84E-11
	1.0	2.65E-05	-3.28E-11	2.66E-05	3.93E-11
	1.1	2.33E-02	-1.79E-08	1.90E-02	1.73E-08
0.6	0.8	-3.10E-04	7.90E-10	-1.56E-04	-4.78E-10
	0.9	4.72E-04	-6.89E-10	4.76E-04	8.32E-10
	1.0	4.27E-04	-3.73E-10	4.32E-04	4.51E-10
	1.1	3.49E-02	-1.90E-08	2.85E-02	1.84E-08

Example 2 Here we consider the case of $|t|$=0.5 and arg t =10°(10°)80° as an example of complex t. For m=8 and n=0, the absolute value of the relative error, $|\epsilon_{nm}|$, is shown in Table 2. We confine ourselves to the case where β is set equal to t, since this choice may lead to higher accuracy by the analogy of example 1. The value of α, for which $|\epsilon_{0,8}|$ takes a minimum value, depends on arg t : when arg t approaches $\pi/2$ from 0, it becomes graduately larger than 0.5 and then the relative error increases, which means that the integrand, with the exception of $C_m^{*(\alpha)}(x/\beta)$, of Eq.(20) departs from a function which is well approximated by a polynomial of degree m-1. Moreover, in the vicinity of arg t =π/2, $|\epsilon_{0,8}|$ does not vary much with α.

Table 2 Relative error $|\epsilon_{0,8}|$ $(|t| = 0.5)$

	$\|\epsilon_{0,8}\|$ arg t							
α	10°	20°	30°	40°	50°	60°	70°	80°
0	7.62E-10	8.37E-10	1.00E-09	1.39E-09	2.35E-09	4.90E-09	1.17E-08	3.01E-08
0.4	3.06E-10	3.40E-10	4.24E-10	6.46E-10	1.26E-09	2.94E-09	7.56E-09	2.07E-08
0.49	4.20E-11	7.41E-11	1.62E-10	3.89E-10	9.74E-10	2.52E-09	6.78E-09	1.90E-08
0.5	*2.02E-11	*5.82E-11	1.48E-10	3.73E-10	9.51E-10	2.48E-09	6.70E-09	1.88E-08
0.51	3.74E-11	6.28E-11	*1.44E-10	*3.62E-10	9.31E-10	2.45E-09	6.62E-09	1.86E-08
0.55	1.79E-10	1.91E-10	2.33E-10	3.89E-10	*8.92E-10	2.32E-09	6.32E-09	1.79E-08
0.6	3.80E-10	4.02E-10	4.47E-10	5.63E-10	9.59E-10	*2.24E-09	6.00E-09	1.71E-08
0.7	8.44E-10	8.94E-10	9.84E-10	1.14E-09	1.46E-09	2.43E-09	*5.64E-09	1.57E-08
0.8	1.39E-09	1.47E-09	1.62E-09	1.86E-09	2.26E-09	3.12E-09	5.79E-09	1.47E-08
0.9	2.01E-09	2.14E-09	2.36E-09	2.71E-09	3.25E-09	4.19E-09	6.54E-09	*1.44E-08
1.0	2.72E-09	2.89E-09	3.19E-09	3.67E-09	4.39E-09	5.54E-09	7.86E-09	1.48E-08

From the above discussion, we conclude that the choice of $\alpha=0.5$ and $\beta=t$ leads to the accurate and efficient computation of $f_n(t)$. Thus for the approximation $f_{nm}(t)$, we have

$$f_{nm}(t) = \sum_{k=0}^{m} \frac{P_{mk}^{*}\, S_k(t)}{(k+1)a_{k+1}\, t^k} \Bigg/ \sum_{k=0}^{m} \frac{P_{mk}^{*}}{(k+1)\, a_{k+1}\, t^k}\ , \tag{22}$$

where P_{mk}^{*} is the coefficient of the k-th power in the shifted Legendre polynomial.

As an example of the relative error of Eq.(22), the case of $m=7$ for

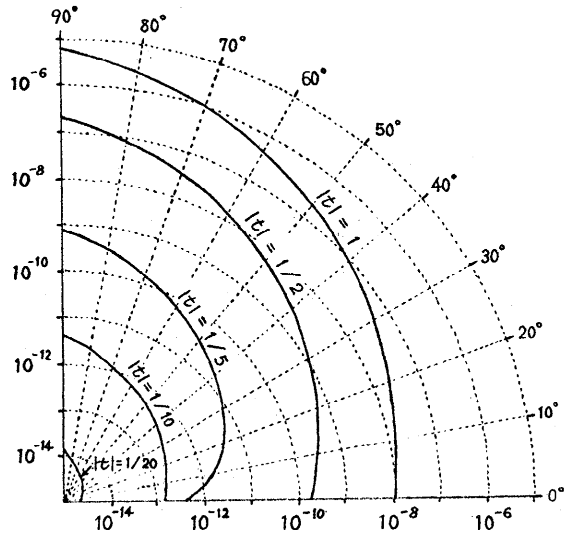

Fig.1 Relative error $|\epsilon_{0,7}|$

$n=0$ is shown in Fig.1. In the region of small $|t|$ near the imaginary axis, a larger value of m is necessary to obtain higher accuracy.

2.2 The Case of $\mathrm{Re}(t) \leqq 0$

The second term of the right-hand side in Eq.(15) diverges for $\mathrm{Re}(t) \leqq 0$ and therefore ϵ_{nm} may not be expressed by such a simple form as Eq.(18). The value of ϵ_{nm} is larger relative to the case of $\mathrm{Re}(t) > 0$ and it does not become smaller with increasing value of m.

3. Conclusion

This method is valid for $\mathrm{Re}(t) > 0$ and leads to the efficient computation of $f_n(t)$ outside the region of small $|t|$. This method has great advantage that, for the same amount of arithmetic operation, the value of $K_n(z)$ is obtained more accurately by 2~4 significant figures than the method using the continued fraction algorithm[3].

References

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