# A Theory and Methods for Three Dimensional Free Form Shape Construction

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This paper describes a new mathematical theory for expressing three dimensional free forms of curves and surfaces, and also methods for application in computer aided design with the results obtained from the theory. First, generative expressions for shape segments of curves and surfaces are deduced from an assumption that a shape segment exists within the smallest convex hull spanned by the control points of the segment. These expressions have simple construction and are easy to be manipulated. Then, shape synthesis and control by partitioning, order raising of a segment and/or by smooth connection of segments, are explained with the use of control points. Tangents, osculating planes, torsions and curvatures at any points on curves or surfaces are simply expressed analytically as well as graphically by the local control points of the shape. As for the shape synthesis by connection, detailed discussion and design methodology are given, especially for non-regular patch connection. Equations for vector interpolation with smooth and natural transition are deduced. A curve constructed from the parametric *B*-spline curve segments is proved to be a special case of this interpolation. This theory and its application methods aid man's comprehension and visualization of three dimensional shapes and control of their geometric characteristics, and accordingly they are useful for interactive shape design and geometric model construction.

## 1. Introduction

In computer aided design and manufacturing of three dimensional shapes, a mathematical description of objects to be designed is required in order to store their data of shapes, to calculate their characteristic values and to display their shapes. When objects have free form shapes, their mathematical expressions usually become complicated owing to their esthetic and manufacturing requirements to be satisfied. Moreover, they must easily be manipulated and calculated in accordance with designers' intention or engineering practices.

So far, various approaches to this problem have been published [1], but none of them have met fully the various requirements in designing. For example, S. Coons' patch expressions [2] are famous, but it is difficult in shape control. In the Japanese car industries, Hosaka's approach [3] has been used because of its ability in good interpolation and smoothing. P. Bézier introduced the concept of a control polygon [4], edges of which are directly related to the shape of a curve to be designed. The generated shapes have good characteristics for practical use, but as his general expression is difficult to be manipulated, its uses are limited. R. Riesenfeld proposed the *B*-spline construction for shape expression [5] with its nodes taken as vertices of the control polygon. Though use of the parametric *B*-spline is good when the control vertices are given, it is not always satisfactory when passing points of a curve are specified in advance. As Bézier's or *B*-spline method does not have a general design methodology, they are generally used but only in design of ordinary shapes, and there are still unsolved problems even in practical applications [6, 7].

In recent years, owing to decrease of computing cost and increased competition in manufacturing industries, CAD and CAM have been considered even in small companies. And rationalization of design and manufacturing of free form shapes has been strongly hoped for. In order to respond to their demands, we re-examined problems in these fields and recognized the importance of establishing a consistent general design methodology of free form shapes for CAD and CAM. For this purpose, we developed a new theory of free forms, which promotes designers' understanding of three dimensional properties of shapes, and supports to establish various design methods of free forms. The features of our theory and methods are as follows.

(1) Simple forms of mathematical expressions.

As the old theories give complicated mathematical expressions for free form shapes, it is difficult for designers to manipulate and interpret them in a creative design. In order to grasp object properties visually, we use the concept of control points for manipulating a shape segment. Our new expressions have simple mathematical forms and are deduced from the practical assumption that the generated shape occupies inside the smallest convex hull which is spanned by the control points of the shape. It is proved that this simple expression for a curve segment is the same as the complicated Bézier's general curve expression. The control points correspond to the vertices of Bézier's polygon. These simple expres-

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sions are especially useful for description and construction of free form surfaces.

(2) Controllable geometric characteristic values.

Tangent, torsion, osculating plane, curvature of a space curve and tangential plane, Gaussian and mean curvature of a surface at any points of shapes can be simply expressed with local control points and easily visualized. Usually these values are difficult to be obtained, but with our method they can be drawn even graphically, so that man can easily grasp the geometric features of the shape and control these values.

(3) Various methods for design and control of shapes.

Old theories did not always provide a suitable application method, but ours have several control or synthesis methods, such as global, local and fine control, and connection of shapes. Control points of a segment can be increased by partitioning or order raising of the segment. The theory also gives methods of satisfying various externally specified design conditions.

(4) Creation of forms with characteristic features.

The main purpose of using free forms in design of shapes is to make the shapes attractive. Old theories are not suitable for expressing these characteristic shapes. In them, neighbouring patch shapes cannot be quite different for smooth connection, and shape modification does not correspond to designer's intuitive methods, and unwanted undulation is apt to occur. Our methods permit non-regular patch connection for constructing shapes with characteristic features.

(5) Use of effective interpolation formulas.

In our methods, we do not use values which are difficult to give, such as derivatives with respect to parameters. Instead, we use geometrically defined values, such as control point positions, tangent directions or radii of curvatures. When these values are not explicitly specified, they can be estimated by solving the vector interpolation equations which are deduced from the theory. Accordingly, the curve synthesis by connection with given passing points does not produce unwanted inflexions, and values of torsion of a surface at nodes can be estimated for smooth suface segment connection.

(6) Completeness as geometric models.

As the older theory and methods can be applied only to regions where shapes are simple, special shape regions are left to manual processing. Accordingly, it is difficult to treat automatically various processes which are required in design and manufacturing. With our methods, whole shapes can be expressed in consistent manner and can be stored as geometric models of the corresponding real objects, so various data which are required in interference problems, cutter path generation, engineering drawings, or assembling, can be produced by processing the stored models.

As described above, our theory and methods are suitable not only for computer applications, but also for manual processing of free form design. Even with the limited space and the simple explanation figures, rich contents of this paper are due to simpleness of our expressions and to ease of their manipulation.

## 2. Control Points and Shape Equations

The simplest shape segment —a line segment— is defined by its end points $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$. We can consider that these two points are the control points of the shape segment, and the line segment can be expressed as

$$\boldsymbol{R}_0(t\colon 1)=\boldsymbol{P}_0\cdot(1-t)+\boldsymbol{P}_1\cdot t, \tag{1}$$

where $t$ takes a value between 0 and 1. $1-t$ and $t$ can be considered as influence functions of the control points. The subscript of $\boldsymbol{R}_0$ means that $\boldsymbol{P}_0$ is the first control point of $\boldsymbol{R}_0$. In the parenthsis of $\boldsymbol{R}_0$, $t$ is the parameter variable, and 1 means the highest degree of the variable. In general, with $n+1$ control points $\{\boldsymbol{P}_0, \boldsymbol{P}_1, \cdots, \boldsymbol{P}_n\}$ or $\{\boldsymbol{P}_i\}_0^n$ in short and their influence functions $\{f_0(t), f_1(t), \cdots, f_n(t)\}$ or $\{f_i(t)\}_0^n$, we can express a space curve segment $\boldsymbol{R}_0(t\colon n)$ as

$$\boldsymbol{R}_0(t\colon n)=\sum_{i=0}^{n}\boldsymbol{P}_i\cdot f_i(t),\ t\in[0,1]. \tag{2}$$

In the similar way, a surface segment can be expressed by a set of control points $\{\boldsymbol{P}_{ij}\}_{0,0}^{m,n}$ with their associated influence functions $\{f_{ij}(u, v)\}_{0,0}^{m,n}$.

$$\boldsymbol{S}_{00}(u, v\colon m, n)=\sum_{i=0}^{m}\sum_{j=0}^{n}\boldsymbol{P}_{ij}\cdot f_{ij}(u, v),\ u, v\in[0,1]. \tag{3}$$

We postulate that $\boldsymbol{R}$ or $\boldsymbol{S}$ is within a convex hull which is spanned by their control points. When the control points are coplanar, colinear, or converged to a point, then the associated shape segment is on the same plane, the line or the point. Then the following relations must hold.

$$f_i(t)\geq 0,\ \sum_{i=0}^{n}f_i(t)\equiv 1,$$

$$f_{ij}(u, v)\geq 0,\ \sum_{i=0}^{m}\sum_{j=0}^{n}f_{ij}(u, v)\equiv 1. \tag{4}$$

For a shape segment of a line, eq. (1) gives

$$f_0(t)=1-t,\quad f_1(t)=t. \tag{5}$$

Next we define order and degree of a shape segment. A line segment has two control points and the eq. (1) is of degree one with respect to $t$. We call it a segment of order two, or degree one. A single point is considered a segment of order one or degree zero. A curve segment given by eq. (2) is of order $n+1$ or degree $n$. surface segment by eq. (3) is of order $(m+1)\times(n+1)$ or degree $m\times n$. Just as a line segment, which is a curve of degree one, is generated from the linear interpolation of two shapes of adjacent curves of degree zero, a curve segment $\boldsymbol{R}_0(t\colon n)$ of degree $n$ is generated from the linear interpolation of two adjacent curves $\boldsymbol{R}_0(t\colon n-1)$ and $\boldsymbol{R}_1(t\colon n-1)$ of degree $n-1$, which is shown as

$$\boldsymbol{R}_0(t\colon n)=\boldsymbol{R}_0(t\colon n-1)\cdot(1-t)+\boldsymbol{R}_1(t\colon n-1)\cdot t. \tag{6}$$

As $R_0(t: n-1)$ and $R_1(t: n-1)$ are contained in the convex hull of the control point vector $\{P_i\}_0^n$, $R_0(t: n)$ defined by the interpolated point on the line inside the convex hull is also in the same convex hull.

In order to simplify the treatment of subscripts of control points, we introduce the shifting operators $E$ and $F$, which operate on the first and second subscripts of the control points in the following way.

$$P_{i+1}=EP_i, \quad P_{i+1,j}=EP_{ij}, \quad P_{i,j+1}=FP_{ij}. \tag{7}$$

As the shifting operators obey the integer exponent rule such as

$$E^i \cdot E^j = E^{i+j}, \quad E \cdot E^{-1}=1, \tag{8}$$

they can be treated as the algebraic constants in algebraic manipulation of expressions containing them. The shifting operators can also operate on curve and surface segment expressions. Then the eq. (1) can be written as

$$\begin{aligned} R_0(t: 1) &= (1-t+tE)P_0 \\ &= (1-t+tE)R_0(t: 0). \end{aligned} \tag{9}$$

Similarly,

$$R_0(t: n)=(1-t+tE)R_0(t: n-1). \tag{10}$$

From (9) and (10),

$$R_0(t: n)=(1-t+tE)^n P_0. \tag{11}$$

From this, $f_i(t)$ is obtained as

$$f_i(t)={}_nC_i \cdot t^i \cdot (1-t)^{n-i}. \tag{12}$$

This satisfies the condition (4). By using $P_n=E^n \cdot P_0$, eq. (11) is transformed into

$$R_0(t: n)=\{(1-t)E^{-1}+t\}^n P_n. \tag{13}$$

Equations (11) and (13) are the generating form of a curve segment expression.

From eq. (11) and $a_i=P_i-P_{i-1}$, we obtain

$$(1-t+Et)^n \cdot P_0 = P_0 - t \cdot \frac{\{1+(E-1)\cdot t\}^n - 1}{(E-1)\cdot t} \cdot a_1.$$

By putting $\tau=-(E-1)\cdot t$, and expanding $\{(1-\tau)^n-1\}/\tau$ around $\tau=0$, then we obtain the following expression.

$$R_0(t: n)=P_0+\sum_{i=1}^{n} \frac{(-t)^i}{(i-1)!} \cdot \frac{d^{i-1}}{dt^{i-1}} \left\{ \frac{(1-t)^n-1}{t} \right\} \cdot a_i. \tag{14}$$

This is the same as that given by P. Bézier. Comparing eqs. (11) and (14), (11) is far simpler, from which we can easily deduct various useful relations. Especially when surface segments are treated, the simple form is very useful.

A surface segment equation with a control point mesh $\{P_{ij}\}_{0,0}^{m,n}$ is obtained by a similar method, and the result is

$$S_{00}(u, v: m, n)=(1-u+uE)^m \cdot (1-v+vF)^n P_{00}. \tag{15}$$

In the above equation, expressions with the other end points are obtained as in (13).

A surface segment with three-sided lattice points $\{P_{ij}\}_{0,0}^{n,n-i}$ is given as follows,

$$T_{00}(u, v, w: n)=(u+vE+wF)^n P_{00}, \tag{16}$$

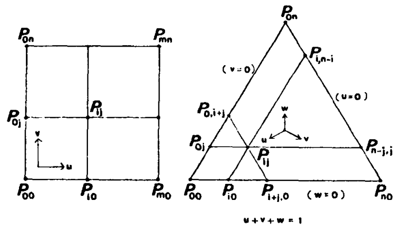


Fig. 1 Control points and parameter variables of rectangular and triangular patches.

where $u$, $v$, $w$ belong to an area coordinate system shown in Fig. 1, and have the relation $u+v+w=1$.

In the following sections, subscripts and parameters of $R$, $S$ and $T$ are omitted when they are clear from the context. The generating forms used in the shape expressions are called here $H$-form for simplicity. In $H$-form, the operand is called a control vector which may be shown by their elements written in { }.

## 3. General Characteristics and Control Points

By differentiating $R_0(t: n)$ with $t$, we obtain

$$\begin{aligned} \frac{1}{n} \cdot \frac{dR_0}{dt} &= (1-t+tE)^{n-1} \cdot (E-1)P_0 \\ &= R_1(t: n-1) - R_0(t: n-1). \end{aligned} \tag{17}$$

This is of $H$-form, and its control vector is $\{a_i\}_1^n$. This shows that the line segment connecting $R_1(t: n-1)$ and $R_0(t: n-1)$ is the tangent line of $R_0(t: n)$. A similar geometrical interpretation can be made to derivatives of $S$ or $T$. For example,

$$\begin{aligned} \frac{1}{m \cdot n} \cdot \frac{\partial^2 S}{\partial u \partial v} &= (E-1)\cdot(F-1)\cdot S_{00}(u, v: m-1, n-1) \\ &= 2 \cdot \left\{ \frac{1}{2} \cdot (S_{00}+S_{11}) - \frac{1}{2} \cdot (S_{01}+S_{10}) \right\}. \end{aligned} \tag{18}$$

Torsion $\partial^2 S/\partial u \partial v$ of a surface segment of degree $m \times n$ is determined from the difference between mid-points of two line segments made by connecting the corresponding points of surfaces of degree $(m-1) \times (n-1)$, $S_{00}$ and $S_{11}$, $S_{10}$ and $S_{01}$. A tangential plane of $T_{00}$ of degree $n$ is determined by the corresponding three points of $T_{00}$, $T_{10}$ and $T_{01}$ of degree $n-1$.

The $j$-th higher order derivative is given by

$$\frac{(n-j)!}{n!} \cdot \frac{d^j R_0}{dt^j} = (1-t+tE)^{n-j}(E-1)^j P_0. \tag{19}$$

This is also of $H$-form and its control vector elements are made from the $j$-th forward difference of the original control points $\{P_i\}_0^n$. Its value at each end point is expressed by $j+1$ control points begining from that point. A control point $P_i$ is determined by up to $i$-th derivatives of the curve at the end point,

$$P_i=(1+E-1)^i P_0 = \sum_{j=0}^{i} {}_iC_j (E-1)^j P_0. \tag{20}$$

Similar relations hold for surface segments $\boldsymbol{S}$ and $T$. The equations are easily obtained.

Next, we discuss the relations between control edges and the geometric characteristics of shapes. An arbitrary point of a curve segment or a surface segment can be made as an end or a corner point of the shape segment, which is explained in the following section. Accordingly, any point of a shape segment has associated control points $\boldsymbol{P}_{00}$, $\boldsymbol{P}_{01}$, $\boldsymbol{P}_{02}$, $\boldsymbol{P}_{10}$, $\boldsymbol{P}_{20}$ and $\boldsymbol{P}_{11}$, or control edges $\boldsymbol{a}_{10}=\boldsymbol{P}_{10}-\boldsymbol{P}_{00}$, $\boldsymbol{a}_{20}=\boldsymbol{P}_{20}-\boldsymbol{P}_{10}$, $\boldsymbol{b}_{01}=\boldsymbol{P}_{01}-\boldsymbol{P}_{00}$, $\boldsymbol{b}_{02}=\boldsymbol{P}_{02}-\boldsymbol{P}_{01}$, $\boldsymbol{a}_{11}=\boldsymbol{P}_{11}-\boldsymbol{P}_{01}$ and $\boldsymbol{b}_{11}=\boldsymbol{P}_{11}-\boldsymbol{P}_{10}$. The geometrical characteristics such as position, tangent, tangential plane, osculating plane, curvature and torsion of a curve and a surface, are given with these values.

$$\text{Tangent:}\quad \hat{t}=\frac{\boldsymbol{a}_1}{a_1}, \tag{21}$$

$$\text{Binormal:}\quad \hat{\beta}=\frac{\boldsymbol{a}_1\times\boldsymbol{a}_2}{|\boldsymbol{a}_1\times\boldsymbol{a}_2|}, \tag{22}$$

$$\text{Principal normal:}\quad \hat{\nu}=\hat{t}\times\hat{\beta}. \tag{23}$$

Osculating plane: a plane made by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$,

$$\text{Curvature:}\quad \frac{1}{\rho}=\frac{n-1}{n}\cdot\frac{\boldsymbol{a}_2\cdot\hat{\nu}}{a_1^2}, \tag{24}$$

$$\text{Torsion:}\quad \frac{1}{\rho_t}=\frac{n-2}{n}\cdot\frac{\boldsymbol{a}_3\cdot\hat{\beta}}{(\boldsymbol{a}_2\cdot\hat{\nu})\cdot a_1}. \tag{25}$$

The curvature and the torsion of a space curve can be interpreted as proportional to the pseudo curvature and torsion of the associated control polygon.

For a surface segment, the geometric characteristics are given as follows.

$$\text{Normal:}\quad \hat{n}=\frac{\boldsymbol{a}_{10}\times\boldsymbol{b}_{01}}{|\boldsymbol{a}_{10}\times\boldsymbol{b}_{01}|}, \tag{26}$$

Tangential plane: a plane made by $\boldsymbol{a}_{10}$ and $\boldsymbol{b}_{01}$,

Gaussian curvature:

$$K=\left(\frac{1}{\rho_1}\cdot\frac{1}{\rho_2}-\frac{1}{\rho_{12}^2}\right)\operatorname{cosec}^2\theta, \tag{27}$$

Mean curvature:

$$H=\frac{1}{2}\cdot\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}-2\cos\theta\cdot\frac{1}{\rho_{12}}\right)\operatorname{cosec}^2\theta, \tag{28}$$

where $1/\rho_1$ and $1/\rho_2$ are the curvatures calculated from $\boldsymbol{a}_{10}$, $\hat{n}\cdot\boldsymbol{a}_{20}$ and $\boldsymbol{b}_{01}$, $\hat{n}\cdot\boldsymbol{b}_{02}$. $1/\rho_{12}$ is given by

$$\frac{1}{\rho_{12}}=\frac{\delta}{a_{10}\cdot b_{01}},$$

where $\delta$ is the distance between $\boldsymbol{P}_{11}$ and the tangential plane at $\boldsymbol{P}_{00}$. $\theta$ is an angle between $\boldsymbol{a}_{10}$ and $\boldsymbol{b}_{01}$. These geometric characteristic values can be easily obtained graphically.

## 4. Partitioning of Shape and Order Raising

### 4.1 Partitioning

A region of a surface segment bounded by two constant parameter curves or a portion of a curve segment bounded by a constant parameter point can be expressed as the new shape segment which has the same order as the original segment. The control points of the new segments are determined from the original control points.

A curve segment $\boldsymbol{R}(t)$ is divided at $t=t_0$ into two curve segments $\boldsymbol{R}^{\mathrm{I}}(t_1)$ and $\boldsymbol{R}^{\mathrm{II}}(t_2)$, where domain of $t_1$ and $t_2$ are both [0, 1]. Now let $E_1$ and $E_2$ be the shifting operators, $\{\boldsymbol{P}^{\mathrm{I}}\}_0^n$ and $\{\boldsymbol{P}^{\mathrm{II}}\}_0^n$ be the control points corresponding to each segment. Then the following equation holds for $0\le t\le t_0$.

$$(1-t_1+t_1E_1)^n\boldsymbol{P}_0^{\mathrm{I}}=(1-t+tE)^n\boldsymbol{P}_0.$$

As $t_1=t/t_0$, the right side of the above equation becomes

$$\{1-t_1+t_1\cdot(1-t_0+t_0E)\}^n\boldsymbol{P}_0.$$

Comparing the above two expressions, we obtain the following relation,

$$\boldsymbol{P}_i^{\mathrm{I}}=(1-t_0+t_0E)^i\boldsymbol{P}_0. \tag{29}$$

In the similar way, for $\boldsymbol{R}^{\mathrm{II}}(t_2)$

$$\boldsymbol{P}_{n-i}^{\mathrm{II}}=\{(1-t_0)E^{-1}+t_0\}^i\boldsymbol{P}_n, \tag{30}$$

is obtained. See Fig. 2. As they have simple geometrical interpretation, the points can be determined graphically. Thus, a curve segment with a control vector $\{\boldsymbol{P}_i\}_0^n$ has control vectors $\{\boldsymbol{P}_i^{\mathrm{I}}\}_0^n$ and $\{\boldsymbol{P}_i^{\mathrm{II}}\}_0^n$. In this way, control points of a segment can be increased indefinitely. At the junction point, the derivatives with respect to $t$ are continuous to the $n$-th order, but the derivatives with respect to $t_1$ and $t_2$ are discontinuous owing to the difference of scale. With increased control points, the shape of the curve segment can be finely controlled. An application example is shown in Fig. 6.

Just as the partitioning of a curve segment, a surface segment can be divided into four segments $\boldsymbol{S}^{\mathrm{I}}(u_1, v_1)$, $\boldsymbol{S}^{\mathrm{II}}(u_1, v_2)$, $\boldsymbol{S}^{\mathrm{III}}(u_2, v_1)$, and $\boldsymbol{S}^{\mathrm{IV}}(u_2, v_2)$ by two curves $\boldsymbol{S}(u_0, v)$ and $\boldsymbol{S}(u, v_0)$ as shown in Fig. 3. The expression of one segment is

$$\boldsymbol{S}^{\mathrm{I}}(u_1, v_1)=(1-u_1+u_1E_1)^m\cdot(1-v_1+v_1F_1)^n\boldsymbol{P}_{00}^{\mathrm{I}}, \tag{31}$$

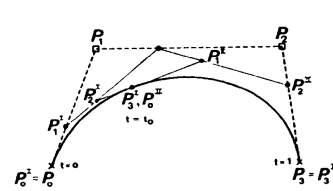


Fig. 2 Generation of new control points by partitioning.

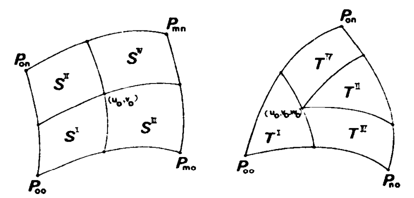


Fig. 3 Partitioning of rectangular and triangular patches.

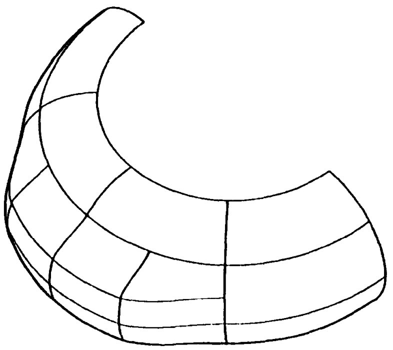


Fig. 4 An example of surface synthesis by partitioning and offset.

where

$$P^{\mathrm{I}}_{ij}=(1-u_0+u_0E)^i\cdot(1-v_0+v_0F)^jP_{00}. \qquad (32)$$

For the other surface segments, a similar expressions can be obtained. An example of shape control by the surface partition is shown in Fig. 4. For a three-sided surface $T$, expressions of surface segments $T^{\mathrm{I}}$ and $T^{\mathrm{II}}$ which are partitioned by curves passing through $(u_0, v_0, w_0)$, such as shown in Fig. 3, are

$$T^{\mathrm{I}}(u_1, v_1, w_1)=(u_1+v_1E_1+w_1F_1)^nP^{\mathrm{I}}_{00},$$
$$T^{\mathrm{II}}(u_2, v_2, w_2)=(u_2+v_2E_2+w_2F_2)^nP^{\mathrm{II}}_{00}, \qquad (33)$$

with their control points

$$P^{\mathrm{I}}_{ij}=\{u_0+(1-u_0)E\}^i\cdot\{u_0+(1-u_0)F\}^jP_{00},$$
$$P^{\mathrm{II}}_{ij}=\{(1-w_0)+w_0E^{-1}F\}^i\cdot\{(1-v_0)+v_0EF^{-1}\}^j$$
$$\times(u_0+v_0E+w_0F)^{n-i-j}P_{ij}. \qquad (34)$$

They are obtained by putting the independent variables as

$$u_1=\frac{u-u_0}{1-u_0},\quad v_1=\frac{v}{1-u_0},\quad w_1=\frac{w}{1-u_0},$$
$$u_2=\frac{u}{u_0},\quad v_2=\frac{v-v_0}{u_0},\quad w_2=\frac{w-w_0}{u_0},$$

and substituting them in $T$ with rearrangements. For remaining regions III and IV the expressions can be obtained by the similar method.

### 4.2 Order Raising

A curve segment of order $n+1$ can be expressed as a segment having order of more than $n+1$. The new control points can be determined in the following way. Let the initial control points be $\{P^{(n)}_i\}^n_0$ and the new control points with order increase of one be $\{P^{(n+1)}_i\}^{n+1}_0$, where a superscript means degree of a segment. Then as the shape is the same, the following equation holds.

$$(1-t+tE)^nP^{(n)}_0=(1-t+tE')^{n+1}P^{(n+1)}_0. \qquad (35)$$

By multiplying $(1-t)+t$ to the left side of eq. (35), the equation becomes

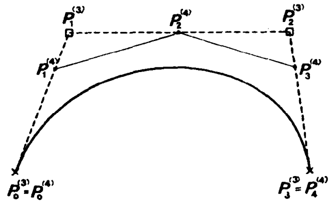



Fig. 5 Generation of new control points by order raising.

$$(1-t+tE)^n\cdot\{(1-t)+t\}P^{(n)}_0$$
$$=\sum_{i=0}^{n+1}\left\{\frac{n+1-i}{n+1}+\frac{i}{n+1}E^{-1}\right\}$$
$$\times{}_{n+1}C_i(tE)^i(1-t)^{n+1-i}P^{(n)}_0.$$

If $P^{(n+1)}_i$ is determined so as to satisfy the condition

$$P^{(n+1)}_i=\phi_i(n)\cdot P^{(n)}_i, \qquad (36)$$

where

$$\phi_i(n)=1-\frac{i}{n+1}+\frac{i}{n+1}\cdot E^{-1},$$

the above expression becomes $(1-t+tE')^{n+1}\cdot P^{(n+1)}_0$. See Fig. 5. The new control point $P^{(n+1)}_i$ is a point which divide the edge vector $\overline{P^{(n)}_{i-1}P^{(n)}_i}$ in the ratio $n+1-i$: $i$. Accordingly $P^{(n+1)}_i$'s are on the vector $a^{(n)}_i$'s, and $P^{(n+1)}_0$ and $P^{(n+1)}_{n+1}$ coincide with $P^{(n)}_0$ and $P^{(n)}_n$. The new control edge vectors are inside of the adjacent edge vectors. So the polygon comes indefinitely nearer to the original curve segment as the order increases indefinitely. With the increased control points, fine control of the shape becomes easy. Similarly the new control points with order increase of $j$ is given as

$$P^{(n+j)}_i=\left\{\prod_{k=n}^{n+j-1}\phi_i(k)\right\}\cdot P^{(n)}_i. \qquad (37)$$

Figure 6 shows an example of deformation from a single segment of degree three by the partitioning and the order raising.

For a surface segment $S$, the new control points with order increase of one are expressed as

$$P^{(m+1,n+1)}_{ij}=\phi_i(m)\cdot\phi'_j(n)\cdot P^{(m,n)}_{ij}, \qquad (38)$$

where

$$\phi'_j(n)=1-\frac{j}{n+1}+\frac{j}{n+1}\cdot F^{-1}.$$

Similarly for a surface segment $T$,

$$P^{(n+1)}_{ij}=\psi_{ij}(n)\cdot P^{(n)}_{ij}, \qquad (39)$$

where

$$\psi_{ij}(n)=1-\frac{i}{n+1}-\frac{j}{n+1}+\frac{i}{n+1}E^{-1}+\frac{j}{n+1}\cdot F^{-1}.$$

An example of surface shape control by the order raising is shown in Fig. 7.

## 5. Connection of Curve Segments

The shape modifying method with increased control points is powerful for local control of a shape. But for

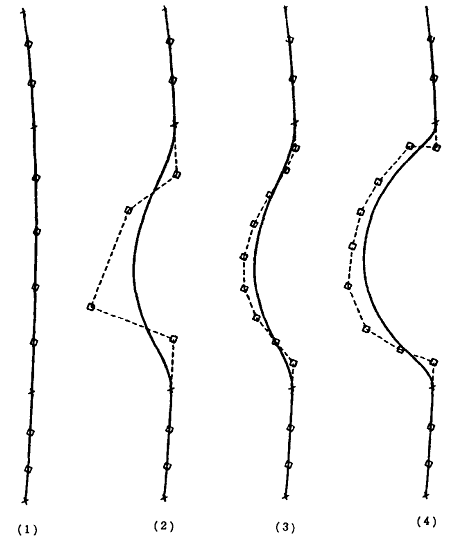



Fig. 6 An example of curve synthesis by partitioning and order raising.
×: Connection points.
□: Control points.
Initially a curve segment of degree three is generated.
(1) Partitioning into three segments and order raising of degree three to five.
(2) Displacement of control points.
(3) Order raising of degree five to degree ten.
(4) Displacement of control points.

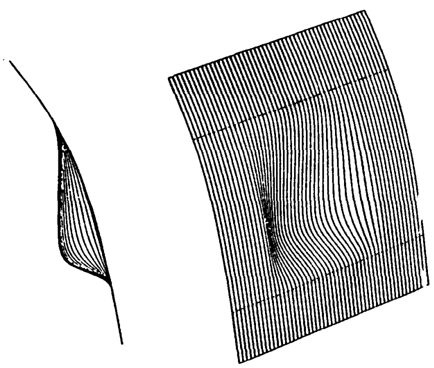


Fig. 7 An example of surface synthesis by partitioning and order raising.

construction of a large shape, it is necessary to connect shape segments with desired properties. In this section, conditions of connection of curve segments are investigated and equations of interpolating vector values are derived. As the degree of a curve segment can be increased without changing the shape, orders of two connecting curves can be assumed equal. The connection must be continuous to curvature at the junction. Naturally their osculating planes must be coincident. The geometrical relations of the control edges of both curve segments shown in Fig. 8 give the following conditions.

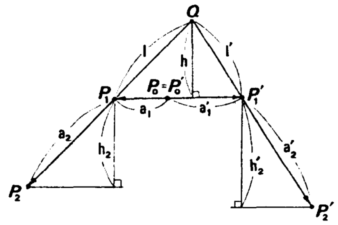



Fig. 8 Connection of space curves with tangent and curvature continuity.

$$\begin{aligned} &\boldsymbol{P}_0=\boldsymbol{P}_0' &&: \text{continuity of position.} \\ &\boldsymbol{a}_1=-k\cdot\boldsymbol{a}_1' &&: \text{continuity of tangent.} \\ &\frac{\boldsymbol{a}_2\cdot\boldsymbol{\vartheta}}{\boldsymbol{a}_1^2}=\frac{\boldsymbol{a}_2'\cdot\boldsymbol{\vartheta}}{\boldsymbol{a}_1'^2} &&: \text{continuity of curvature.} \end{aligned} \tag{40}$$

Extensions of $\boldsymbol{a}_2$ and $\boldsymbol{a}_2'$ must intersect at $\boldsymbol{Q}$ in the osculating plane. Now if $h$ be the length of the perpendicular from $\boldsymbol{Q}$ to $\boldsymbol{a}_1$, from the following relations

$$k=\frac{a_1}{a_1'},\quad k_a\equiv\frac{a_2}{l}=\frac{h_2}{h},\quad k_b\equiv\frac{l'}{a_2'}=\frac{h}{h_2'},\tag{41}$$

and eq. (40),

$$k=\frac{a_1}{a_1'}=\sqrt{\frac{h_2}{h_2'}}=\sqrt{k_a\cdot k_b}\tag{42}$$

is obtained. This is the condition imposed among the arbitrary constants $k$, $k_a$ and $k_b$. The curvature at the junction is $(2/3)\cdot(h_2/a_1^2)$.

When each segment of a synthesized curve by connection is of degree three, points $\boldsymbol{Q}_i$, which are made by the same process as the point $\boldsymbol{Q}$ stated above, are considered as another kind of control points. If $k_a=k_b$ for simplicity, they are equal to $k$. With $k_i$'s, we descriminate $k$ corresponding to each control point $\boldsymbol{Q}_i$. Now we divide each edge of the polygon made by $\{\boldsymbol{Q}_i\}$ into three sections in the ratios as shown in Fig. 9. That is,

$$\overline{Q_{i-1}Q_i'}:\overline{Q_i'Q_i''}=k_{i-1}:1,$$
$$\overline{Q_i'Q_i''}:\overline{Q_i''Q_i}=k_i:1,$$

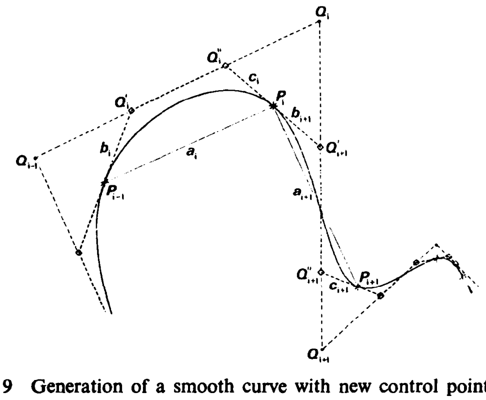


Fig. 9 Generation of a smooth curve with new control points $Q_i$'s.

$$\overline{Q_iQ'_{i+1}}:\overline{Q'_{i+1}Q''_{i+1}}=k_i:1,$$
$$\overline{Q'_{i+1}Q''_{i+1}}:\overline{Q''_{i+1}Q_{i+1}}=k_{i+1}:1.$$

A connecting point $\boldsymbol{P}_i$ of the curve segments is determined by dividing $\overline{Q''_iQ'_{i+1}}$ in the ratio $k_i$: 1. Then $\boldsymbol{P}_{i-1}$, $\boldsymbol{Q}'_i$, $\boldsymbol{Q}''_i$, $\boldsymbol{P}_i$ and $\boldsymbol{P}_i$, $\boldsymbol{Q}'_{i+1}$, $\boldsymbol{Q}''_{i+1}$, $\boldsymbol{P}_{i+1}$ are the control points of the previous type. Curve segments are connected at $\boldsymbol{P}_i$'s with continuity to curvature. The relation between $\boldsymbol{P}_i$'s and $\boldsymbol{Q}_i$'s is given by

$$\frac{\boldsymbol{Q}_{i-1}+k_i\cdot(1+k_{i-1})\cdot\boldsymbol{Q}_i}{1+k_i\cdot(1+k_{i-1})}+\frac{k_i\cdot(1+k_{i+1})\cdot\boldsymbol{Q}_i+k_i^2\cdot k_{i+1}\cdot\boldsymbol{Q}_{i+1}}{1+k_{i+1}\cdot(1+k_i)}=(1+k_i)\cdot\boldsymbol{P}_i. \tag{43}$$

When $k_i=1$, eq. (43) becomes simply

$$\boldsymbol{Q}_{i-1}+4\boldsymbol{Q}_i+\boldsymbol{Q}_{i+1}=6\boldsymbol{P}_i. \tag{44}$$

On of the uses of eqs. (43) or (44) is to control the synthesized curve by $\boldsymbol{Q}_i$'s. Displacement of $\boldsymbol{Q}_i$ has an influence only on $\boldsymbol{P}_{i-1}$, $\boldsymbol{P}_i$ and $\boldsymbol{P}_{i+1}$. That is, its effect is limited to the neighbouring four segments without changing the continuity conditions. This property is favourable to local control of shape. The so called parametric *B*-spline corresponds to the case $k_i=1$. Another use of eq. (43) or (44) is to obtain $\{\boldsymbol{Q}_i\}$ given $\{\boldsymbol{P}_i\}$. Instead of the variables in (43) or (44), the equivalent control edge vectors are to be used for simplicity. Here the new notation of control edge vectors $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{a}$ are used, which were denoted previously as $\boldsymbol{a}_1$, $\boldsymbol{a}_3$ and $\boldsymbol{a}_1+\boldsymbol{a}_2+\boldsymbol{a}_3$ according to the definition of a single control edge vector. And a subscript $i$ attached to the new notation vectors indicates that they belong to the $i$-th segment. Then, for eq. (43),

$$\begin{aligned}&\boldsymbol{b}_i+2k_i\cdot(1+k_i)\cdot\boldsymbol{b}_{i+1}+k_i^2\cdot\boldsymbol{c}_{i+1}=\boldsymbol{a}_i+k_i^2\cdot\boldsymbol{a}_{i+1},\\&\boldsymbol{c}_i=k_i\cdot\boldsymbol{b}_{i+1},\end{aligned} \tag{45}$$

and for eq. (44),

$$\boldsymbol{b}_i+4\boldsymbol{b}_{i+1}+\boldsymbol{b}_{i+2}=\boldsymbol{a}_i+\boldsymbol{a}_{i+1} \tag{46}$$

are obtained. The boundary conditions for the synthesized curve are curvatures at end points, which are given by eq. (24). When the curvatures at both ends are equal to zero, then the following equations are the conditions.

$$2\boldsymbol{b}_1+\boldsymbol{c}_1=\boldsymbol{a}_1,\quad \boldsymbol{b}_n+2\boldsymbol{c}_n=\boldsymbol{a}_n. \tag{47}$$

Given $\boldsymbol{a}_i$'s and $k_i$'s, eqs. (45) and (47) are linear simultaneous vector equations with unknown $\boldsymbol{b}_i$'s. Even when $\boldsymbol{a}_i$'s are not geometrical chord vectors but differences of adjacent sampled vectors, the solution of eq. (45) can be used for vector interpolation between the sampled vectors.

This interpolation method was introduced by one of the authors more than ten years ago with the different derivation [3] and has been used in Japanese car industries as the best feasible interpolation formula. In the interpolation, a method of determining the values of $k_i$'s is important, which have much influence on the shapes of polygons made by $\{\boldsymbol{Q}_i\}$ and $\{\boldsymbol{P}_i\}$. When adjacent chord lengthes $\boldsymbol{a}_i$ and $\boldsymbol{a}_{i+1}$ are quite different, if all $k_i$'s are set to 1, then the shapes of the two polygons $\{\boldsymbol{Q}_i\}$ and $\{\boldsymbol{P}_i\}$ are also greatly different. Natural relation of the shapes of the two polygons seems to be kept when $k_i$'s are proportional to $a_i/a_{i+1}$. So the following relation is assumed.

$$k_i=\left(\frac{a_i}{a_{i+1}}\right)^{\omega}=r_i^{\omega}, \tag{48}$$

where $\omega>0$ and $r_i=a_i/a_{i+1}$.

From experiments, $\omega=0$ gives unacceptable results and $\omega=1$ or $\omega=1/2$ is good. Difference of the shapes by $\omega=1$ or 1/2 is not so great, but the selection of this value depends on man's subjective criterion on the shape. For convenience in surface connection, the following compromised formulas for $k_i$'s are usable.

$$\begin{cases} r_i<\dfrac{2}{3} & k_i=\dfrac{3}{2}r_i,\\[2mm] \dfrac{2}{3}\le r_i\le\dfrac{3}{2} & k_i=1,\\[2mm] r_i>\dfrac{3}{2} & k_i=\dfrac{2}{3}r_i.\end{cases} \tag{49}$$

An example of curve generation, in which there are $k_i$'s quite different from 1, is shown in Fig. 10 with the distribution of radius of curvature.

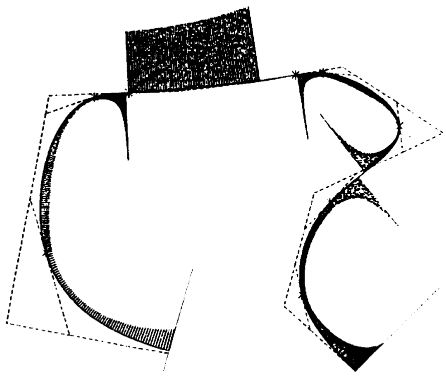


Fig. 10 An example of a smooth curve passing through arbitrarily specified points with the distribution of radius of curvature.

## 6. Connection of Surface Segments

In eqs. (43) and (45), if $\boldsymbol{Q}_i$'s, $\boldsymbol{P}_i$'s, $k_i$'s or $\boldsymbol{b}_i$'s, $\boldsymbol{a}_i$'s are functions of another parameter, they can supply data for surface generation and there arise no problem of connection of surface segments. When these vector values can be expressed by their associated control points, the latter control points can be considered as those of the surface. In this case, there arise no problem of surface segment connection, but we cannot directly specify the various conditions given to the surface such

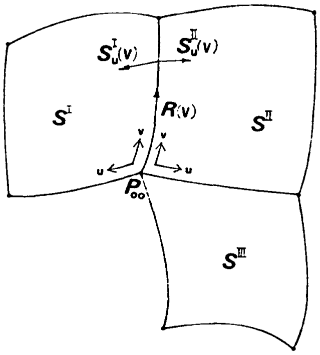


Fig. 11 Smooth connection of surface patches.

as passing points, boundary curves, etc. When a surface is to be composed by connecting surface segments, smooth connection on their boundary curves and continuous surface curvature condition at corner points of segments have to be satisfied.

As shown in Fig. 11, a surface segment $S^{II}$ connects surface segments $S^{I}$ and $S^{III}$ with a common corner point $\boldsymbol{P}_{00}$. $\boldsymbol{R}(v)$ is a boundary curve between $S^{I}$ and $S^{II}$. The $u$-direction tangent vectors of $S^{I}$ and $S^{II}$ on $\boldsymbol{R}(v)$, which we call tangent vectors of the first kind, and the $v$-direction tangent vector of $\boldsymbol{R}(v)$, which is a tangent vector of the second kind, must be coplanar on each point of the boundary curve. From the above conditions, the following equation must hold

$$\lambda(v)\cdot \boldsymbol{S}^{I}_{u}(v)+\mu(v)\cdot \boldsymbol{S}^{II}_{u}(v)=\nu(v)\cdot \dot{\boldsymbol{R}}(v), \tag{50}$$

where $\lambda$, $\mu$, $\nu$ are scalar functions of $v$. As each term of the above equation can be transformed in $H$-form of the same degree, relations between control edge vectors $(\boldsymbol{a}_{10}, \boldsymbol{a}_{11}, \cdots, \boldsymbol{a}_{1n})$ of both surfaces can be determined. In the similar way, relations between control edge vectors $(\boldsymbol{b}_{01}, \boldsymbol{b}_{11}, \cdots, \boldsymbol{b}_{m1})$ can be obtained. As $u$ and $v$ direction control edges of $S^{II}$ have $\boldsymbol{P}_{11}$ in common, the following relation must hold.

$$\boldsymbol{b}_{11}-\boldsymbol{a}_{11}=\boldsymbol{b}_{01}-\boldsymbol{a}_{10}. \tag{51}$$

All surface segments meeting at $\boldsymbol{P}_{00}$ must have a common tangent plane and the relation (51).

One of the easiest way of shape definition is to define a space curve network which covers the shape to be designed. When the network has a desirable shape, then meshes made by the network have to be fitted by surface patches connecting smoothly with adjacent patches. If a mesh is too large to be fitted by a single patch, then the mesh should be divided into fine meshes by added space curves. Degree of a curve segment between adjacent nodes of network should be made three in order not to increase degree of a surface patch more than $5\times 5$. Shape division with the network usually generates four-sided surface patches, but sometimes generates three-, five-, six-sided patches. Accordingly, methods of connection of surface patches of various shapes have to be developed. Even in the connection of four-sided patches, when magnitude ratio of the tangents of the first kind along the boundary of the patches is not constant, a new connection method has to be established. They are described in this section.

### 6.1 Connection of Four-Sided Surface Patches

#### 6.1.1 Case for constant ratio of the tangents of the first kind

For this case, the surface segments $S^{I}$, $S^{II}$, $S^{III}$ and $S^{IV}$ with their coordinates are shown in Fig. 12. Let ratios of tangent vectors in $u$(or $v$) direction at a junction be $k_1$(or $k_2$), and these values are constant at each junction on the $v$ (or $u$) direction curve passing through $\boldsymbol{P}_{00}$. Surface segments are to be of degree $3\times 3$. Then the tangent vectors of the first kind of the boundaries $u=0$ or $v=0$ are

$$\boldsymbol{S}_u(v)=3\cdot(1-v+vF)^3\cdot \boldsymbol{a}_{10}, \tag{52}$$

$$\boldsymbol{S}_v(u)=3\cdot(1-u+uE)^3\cdot \boldsymbol{b}_{01}. \tag{53}$$

For smooth connection of the surface segments $S^{II}$ and $S^{IV}$ with the segments $S^{I}$ and $S^{III}$, the following conditions on the control vectors $\{\boldsymbol{a}_{1i}\}$, $\{\boldsymbol{b}_{i1}\}$ of each segment must hold.

$$\begin{cases} k_1\cdot \boldsymbol{a}^{I}_{1i}=-\boldsymbol{a}^{II}_{1i}, \\ k_2\cdot \boldsymbol{b}^{III}_{i1}=-\boldsymbol{b}^{II}_{i1}, \end{cases} \quad (i=0,\cdots,3) \tag{54}$$

$$\begin{cases} k_1\cdot \boldsymbol{a}^{IV}_{1i}=-\boldsymbol{a}^{III}_{1i}, \\ k_2\cdot \boldsymbol{b}^{IV}_{i1}=-\boldsymbol{b}^{I}_{i1}. \end{cases} \quad (i=0,\cdots,3) \tag{55}$$

These relations are derived from eq. (50), by putting $\lambda=\mu=1$ and $\nu=0$. If the condition (51) at the node $\boldsymbol{P}_{00}$ is to hold for segments $S^{I}$ and $S^{III}$, that is

$$\begin{aligned} \boldsymbol{b}^{I}_{11}-\boldsymbol{a}^{I}_{11}&=\boldsymbol{b}^{I}_{01}-\boldsymbol{a}^{I}_{10}, \\ \boldsymbol{b}^{III}_{11}-\boldsymbol{a}^{III}_{11}&=\boldsymbol{b}^{III}_{01}-\boldsymbol{a}^{III}_{10}, \end{aligned} \tag{56}$$

the same condition for the segments $S^{II}$ and $S^{IV}$ becomes, using eqs. (54) and (55),

$$\begin{aligned} k_2\cdot \boldsymbol{b}^{III}_{11}-k_1\cdot \boldsymbol{a}^{I}_{11}&=k_2\cdot \boldsymbol{b}^{III}_{01}-k_1\cdot \boldsymbol{a}^{I}_{10}, \\ \frac{1}{k_2}\cdot \boldsymbol{b}^{I}_{11}-\frac{1}{k_1}\cdot \boldsymbol{a}^{III}_{11}&=\frac{1}{k_2}\cdot \boldsymbol{b}^{I}_{01}-\frac{1}{k_1}\cdot \boldsymbol{a}^{III}_{10}. \end{aligned} \tag{57}$$

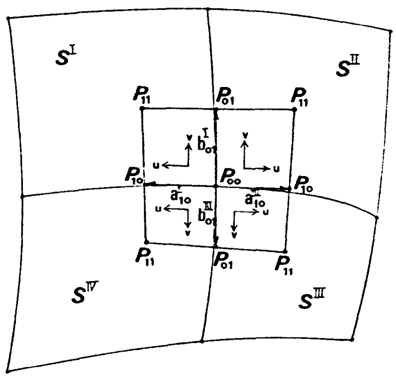


Fig. 12 Smooth connection of surface patches. (Case for constant ratio of the tangents of the first kind).

From these relations, the following theorem is deduced. Theorem: If eqs. (54) to (57) hold, then $(P^{\mathrm{I}}_{11}, P^{\mathrm{I}}_{01}, P^{\mathrm{II}}_{11})$, $(P^{\mathrm{II}}_{11}, P^{\mathrm{II}}_{10}, P^{\mathrm{III}}_{11})$, $(P^{\mathrm{III}}_{11}, P^{\mathrm{IV}}_{01}, P^{\mathrm{IV}}_{11})$ and $(P^{\mathrm{IV}}_{11}, P^{\mathrm{IV}}_{10}, P^{\mathrm{I}}_{11})$ are colinear respectively, and $P^{\mathrm{I}}_{01}$ (or $P^{\mathrm{III}}_{01}$) divides the line segment $\overline{P^{\mathrm{I}}_{11}P^{\mathrm{II}}_{11}}$ (or $\overline{P^{\mathrm{IV}}_{11}P^{\mathrm{III}}_{11}}$) in the ratio $1:k_1$, and $P^{\mathrm{II}}_{10}$ (or $P^{\mathrm{IV}}_{10}$) divides the line segment $\overline{P^{\mathrm{III}}_{11}P^{\mathrm{II}}_{11}}$ (or $\overline{P^{\mathrm{IV}}_{11}P^{\mathrm{I}}_{11}}$ in the ratio $1:k_2$.

If $P_{11}$ of any one of the four segments can be determined, the other $P_{11}$'s around the $P_{00}$ are determined with this theorem. $P_{11}$ of a segment has the following relation with $S_{uv}$ at the $P_{00}$.

$$\begin{aligned}P_{11}&=EFP_{00}\\&=P_{00}+(E-1)P_{00}+(F-1)P_{00}+\frac{1}{3^2}\cdot S_{uv}.\end{aligned}\quad (58)$$

$S_{uv}$ can be assumed to be the mean value of $(\partial/\partial v)S_u$ and $(\partial/\partial u)S_v$ at $P_{00}$. They are obtained from values of $S_u$'s or $S_v$'s at neighbouring nodes along $v$ or $u$ directions by using the interpolation eq. (45), in which $b$ becomes $S_{uv}$ or $S_{vu}$ and $a$ becomes difference of $S_u$ or $S_v$. When the surface shape is simple, $k_i$'s are set equal to 1, and surface synthesis by patch connection becomes simple.

### 6.1.2 Case for non-constant ratio of the tangents of the first kind

Ratios of magnitude of tangents of boundary curves at each node are assumed not to be constant. In order to satisfy the smooth connection conditions, surface segments of degree $3\times3$ and degree $5\times5$ should be diagonally distributed as shown in Fig. 13. Let surface $S^{\mathrm{V}}$ be of degree $5\times5$ and $S^{\mathrm{I}}$, $S^{\mathrm{II}}$, $S^{\mathrm{III}}$, $S^{\mathrm{IV}}$ be of degree $3\times3$, and boundaries of the surface $S^{\mathrm{V}}$ be numbered as 1, 2, 3, 4, and coordinates of each segment be as shown in the figure. Ratios of tangent magnitude at each node in $u$ direction are $k_1$, $k'_1$, $k_3$, $k'_3$ and in $v$ direction $k_2$, $k'_2$, $k_4$, $k'_4$, where a subscript indicates the boundary curve, and a prime indicates a distant node from the origin. For the boundaries 1 and 3, the conditions of smooth connection are

$$\begin{aligned}\lambda_1(v)\cdot S^{\mathrm{I}}_u(1, v)+\nu_1(v)\cdot S^{\mathrm{I}}_v(1, v)&=S^{\mathrm{V}}_u(0, v),\\ \lambda_3(v)\cdot S^{\mathrm{III}}_u(0, v)+\nu_3(v)\cdot S^{\mathrm{III}}_v(0, v)&=S^{\mathrm{V}}_u(1, v).\end{aligned}\quad (59)$$

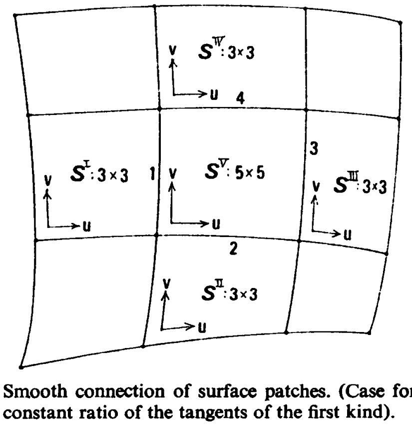



Fig. 13 Smooth connection of surface patches. (Case for non-constant ratio of the tangents of the first kind).

For the boundaries 2 and 4,

$$\begin{aligned}\lambda_2(u)\cdot S^{\mathrm{II}}_v(u, 1)+\nu_2(u)\cdot S^{\mathrm{II}}_u(u, 1)&=S^{\mathrm{V}}_v(u, 0),\\ \lambda_4(u)\cdot S^{\mathrm{IV}}_v(u, 0)+\nu_4(u)\cdot S^{\mathrm{IV}}_u(u, 0)&=S^{\mathrm{V}}_v(u, 1).\end{aligned}\quad (60)$$

For $\lambda_i$'s and $\nu_i$'s, the following equations are assumed

$$\lambda_i(t)=k_i\cdot(1-t)^2+2\sqrt{k_i\cdot k'_i}\cdot t\cdot(1-t)+k'_i\cdot t^2,\quad (61)$$

$$\nu_i(t)=\eta_i\cdot t\cdot(1-t)^2+\eta'_i\cdot t^2\cdot(1-t),\quad (62)$$

where

$$t=v\quad\text{for}\quad i=1, 3,$$
$$t=u\quad\text{for}\quad i=2, 4.$$

And $\eta_i$'s and $\eta'_i$'s are determined by the eqs. (59), (60) and (61) to take the following values.

$$\begin{aligned}\eta_1&=6\cdot\left(\sqrt{\frac{k'_2}{k_2}}-1\right),\quad \eta'_1=6\cdot\left(1-\sqrt{\frac{k_4}{k'_4}}\right),\\ \eta_2&=6\cdot\left(\sqrt{\frac{k'_1}{k_1}}-1\right),\quad \eta'_2=6\cdot\left(1-\sqrt{\frac{k_3}{k'_3}}\right),\\ \eta_3&=6\cdot\left(1-\sqrt{\frac{k_2}{k'_2}}\right),\quad \eta'_3=6\cdot\left(\sqrt{\frac{k'_4}{k_4}}-1\right),\\ \eta_4&=6\cdot\left(1-\sqrt{\frac{k_1}{k'_1}}\right),\quad \eta'_4=6\cdot\left(\sqrt{\frac{k'_3}{k_3}}-1\right),\end{aligned}\quad (63)$$

Also, the control vectors of the surface segment $S^{\mathrm{V}}$ are determined as

$$\begin{aligned}&5\cdot a^{\mathrm{V}}_{10}=3\cdot k_1\cdot a^{\mathrm{I}}_{30},\\ &25\cdot a^{\mathrm{V}}_{11}=3\cdot(3\cdot k_1\cdot a^{\mathrm{I}}_{31}+2\cdot\sqrt{k_1k'_1}\cdot a^{\mathrm{I}}_{30})+\eta_1\cdot b^{\mathrm{I}}_{31},\\ &50\cdot a^{\mathrm{V}}_{12}=3\cdot(3\cdot k_1a^{\mathrm{I}}_{32}+6\cdot\sqrt{k_1k'_1}\cdot a^{\mathrm{I}}_{31}+k'_1\cdot a^{\mathrm{I}}_{30})\\ &\qquad+2\cdot\eta_1\cdot b^{\mathrm{I}}_{32}+\eta'_1\cdot b^{\mathrm{I}}_{31},\\ &50\cdot a^{\mathrm{V}}_{13}=3\cdot(k_1\cdot a^{\mathrm{I}}_{33}+6\cdot\sqrt{k_1k'_1}\cdot a^{\mathrm{I}}_{32}+3\cdot k'_1a^{\mathrm{I}}_{31})\\ &\qquad+\eta_1\cdot b^{\mathrm{I}}_{33}+2\cdot\eta'_1\cdot b^{\mathrm{I}}_{32},\\ &25\cdot a^{\mathrm{V}}_{14}=3\cdot(2\cdot\sqrt{k_1k'_1}\cdot a^{\mathrm{I}}_{33}+3\cdot k'_1\cdot a^{\mathrm{I}}_{32})+\eta'_1\cdot b^{\mathrm{I}}_{33},\\ &5\cdot a^{\mathrm{V}}_{15}=3\cdot k'_1\cdot a^{\mathrm{I}}_{33}.\end{aligned}\quad (64)$$

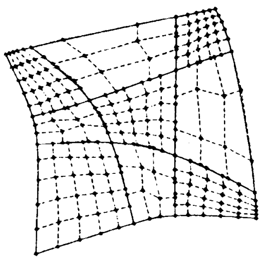
The other control vectors $a^{\mathrm{V}}_{5i}$, $b^{\mathrm{V}}_{i1}$ and $b^{\mathrm{V}}_{i5}$ $(i=0,\cdots,5)$ are given by similar expressions. From these equations, the control points $P_{ij}$'s are determined except for $P_{22}$, $P_{23}$, $P_{32}$ and $P_{33}$ which are arbitrary, but may well take values such as

$$\begin{aligned}P_{22}&=P_{11}+a_{21}+b_{12},\\ P_{23}&=P_{14}+a_{24}-b_{14},\\ P_{32}&=P_{41}-a_{41}+b_{42},\\ P_{33}&=P_{44}-a_{44}-b_{44}.\end{aligned}$$

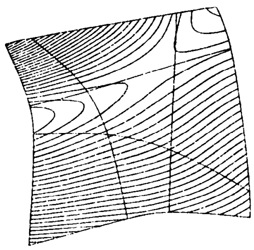
An example of surface generation of this case is shown in Fig. 14.

## 6.2 Generation of Non-Four-Sided Surface Patches

By connecting four-sided or three-sided surface segments on each neighbouring boundary around a common corner point of the segments, a non-four-sided surface patch can be constructed. Such surface patches are needed at convex or concave corner regions of an object made from synthesized surfaces.



(1) control polygons.



(2) contour lines.

Fig. 14 An example of a surface made of segments of degree $3 \times 3$ and degree $5 \times 5$.

### 6.2.1 Connection of four-sided surface segments

As shown in Fig. 15, $n$ four-sided surface segments of degree $3 \times 3$ are connected around the center point $\boldsymbol{P}$. All the second control points $\boldsymbol{P}_i$'s of boundary curves which start from the center point $\boldsymbol{P}$ must be coplanar, and in the tangential plane of the synthesized surface at $\boldsymbol{P}$. For simplicity, these second control points are made to coincide with the vertices of a regular polygon of $n$ sides. Let the second control point of the boundary curve $\boldsymbol{R}_i(t)$ be denoted as $\boldsymbol{P}_i$, and the control edge vectors of $\boldsymbol{R}_i(t)$ be $\boldsymbol{b}_{i1}$, $\boldsymbol{b}_{i2}$, $\boldsymbol{b}_{i3}$. The control vector of the tangent vector $\boldsymbol{R}_i(t)$ multiplied by $\nu \cdot (1-t)$ is given by

$$\left\{\nu \cdot \boldsymbol{b}_{i1}, \frac{2}{3} \cdot \nu \cdot \boldsymbol{b}_{i2}, \frac{1}{3} \cdot \nu \cdot \boldsymbol{b}_{i3}, 0\right\}. \tag{65}$$

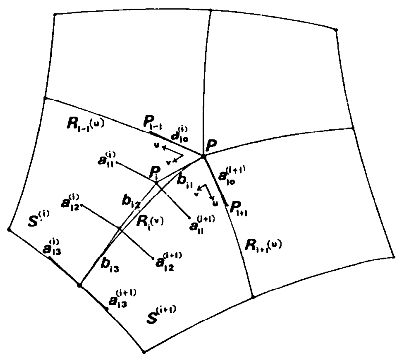



Fig. 15 Connection of rectangular surface segments.

Let the boundary curve of the surface segments $\boldsymbol{S}^{(i)}(u, v)$ and $\boldsymbol{S}^{(i+1)}(u, v)$ be $\boldsymbol{R}_i(v)$ and $u=0$, and the boundary curves $\boldsymbol{R}_{i-1}$ and $\boldsymbol{R}_{i+1}$ be $\boldsymbol{R}_{i-1}(u)$, $\boldsymbol{R}_{i+1}(u)$ and $v=0$. On the boundary curve $\boldsymbol{R}_i(v)$, the following conditions are to hold according to eq. (50).

$$\boldsymbol{S}_u^{(i)} + \boldsymbol{S}_u^{(i+1)} = \nu \cdot (1-v) \cdot \dot{\boldsymbol{R}}_i(v). \tag{66}$$

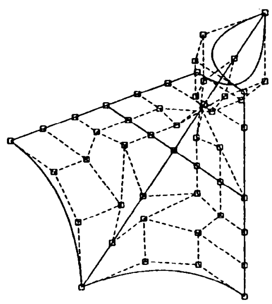
With the expression (65), the following relations between control vector elements are obtained.

$$\begin{aligned} \boldsymbol{a}_{10}^{(i)} + \boldsymbol{a}_{10}^{(i+1)} &= \nu \cdot \boldsymbol{b}_{i1}, \\ \boldsymbol{a}_{11}^{(i)} + \boldsymbol{a}_{11}^{(i+1)} &= \frac{2}{3} \cdot \nu \cdot \boldsymbol{b}_{i2}, \\ \boldsymbol{a}_{12}^{(i)} + \boldsymbol{a}_{12}^{(i+1)} &= \frac{1}{3} \cdot \nu \cdot \boldsymbol{b}_{i3}, \\ \boldsymbol{a}_{13}^{(i)} + \boldsymbol{a}_{13}^{(i+1)} &= 0. \end{aligned} \tag{67}$$

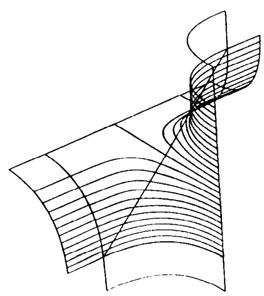
And from the definition of the notation,

$$\boldsymbol{a}_{10}^{(i)} = \boldsymbol{b}_{i-1,1}, \quad \boldsymbol{a}_{10}^{(i+1)} = \boldsymbol{b}_{i+1,1}. \tag{68}$$

With these conditions and the first relation of (67), $\nu = 2 \cdot \cos(2\pi/n)$ is determined. The second equation of (67) gives $\boldsymbol{P}_{11}$'s of surface segments. $\boldsymbol{P}_{11}$'s of surface $\boldsymbol{S}^{(i)}$ and $\boldsymbol{S}^{(i+1)}$ are symmetric with respect to the vector $\boldsymbol{b}_{i2}$. When $\boldsymbol{b}_{i2}$'s are at the symmetrical locations around the normal axis vector at $\boldsymbol{P}$, all $\boldsymbol{P}_{11}$'s are easily determined. Otherwise the second equation of (67) has to be solved. The third and fourth equations of (67) determine $\boldsymbol{P}_{21}$, $\boldsymbol{P}_{31}$, $\boldsymbol{P}_{12}$, $\boldsymbol{P}_{13}$. The remaining control points can be arbitray. If the surface segments are of degree $2 \times 2$, only $\boldsymbol{P}_{22}$ is arbitrary. An example of this connection is shown in Fig. 16, with the distribution of control points, where a



(1) control polygons.



(2) contour lines.

Fig. 16 An example of a hexagonal patch made of rectangular surface segments.

six-sided patch is synthesized from connection of four four-sided segments. Intersection lines with the equidistant planes are also shown in the figure.

### 6.2.2 Connection of three-sided surface segments

Three-sided surface segments are connected around the center point $\boldsymbol{P}_{00}$ as shown in Fig. 17, to make a many-sided surface patch. The expression of a segment is

$$T=(u+vE+wF)^3\boldsymbol{P}_{00}. \qquad (69)$$

Let a common boundary curve be at $w=0$, and its control edges be $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$. Tangent of the surface segment in $w$ direction can be obtained from the following operation.

$$\frac{dT}{dw}=\frac{\partial T}{\partial w}-\frac{1}{2}\cdot\left(\frac{\partial T}{\partial u}+\frac{\partial T}{\partial v}\right). \qquad (70)$$

Now new vectors $\boldsymbol{a}_i$'s are defined as

$$\boldsymbol{a}_i=\boldsymbol{P}_{i-1,1}-\frac{1}{2}\cdot(\boldsymbol{P}_{i-1,0}+\boldsymbol{P}_{i0}), \quad (i=1, 2, 3). \qquad (71)$$

For smooth connection of surface segments $T^{\mathrm{I}}$ and $T^{\mathrm{II}}$, $\boldsymbol{P}_{i-1,1}$'s are to be selected to satisfy the following conditions.

$$\lambda\cdot\boldsymbol{a}_i^{\mathrm{I}}+\mu\cdot\boldsymbol{a}_i^{\mathrm{II}}=\nu\cdot\boldsymbol{b}_i, \quad (i=1, 2, 3). \qquad (72)$$

For each segment, $\boldsymbol{P}_{00}$, $\boldsymbol{P}_{01}$ and $\boldsymbol{P}_{10}$ are coplanar, when $\boldsymbol{P}_{01}$ and $\boldsymbol{P}_{10}$ of all segments are vertices of the regular polygon of $n$ sides. Equation (72) are satisfied by $\lambda=\mu=1$ and $\nu=2\cdot\cos\theta$, where $\theta$ is an angle between $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$. $\boldsymbol{P}_{11}$'s of each segment can easily be determined if $\boldsymbol{b}_2$'s are in the symmetrical positions, otherwise their determination becomes complicated. For $i=3$, $\boldsymbol{P}_{21}$'s and $\boldsymbol{P}_{12}$'s are easily determined as they do not interfere with others. An example of a five-sided patch made by this method is shown in Fig. 18, with equiheight contour lines.

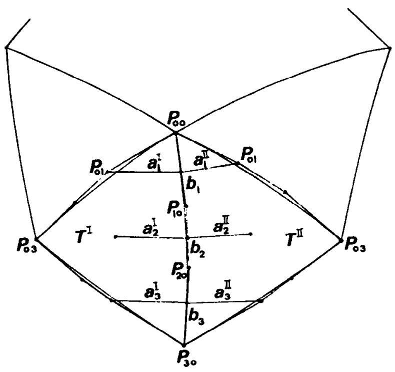
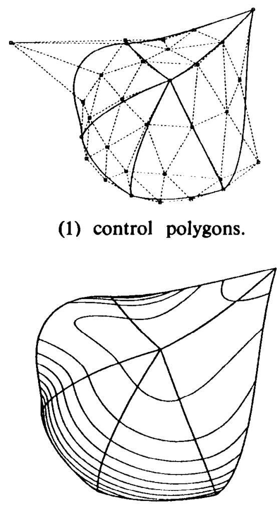



Fig. 17 Connection of triangular surface segments.

### 6.2.3 Connection of three-sided and four-sided surface segments

When three-sided surface segments are used, frequently they have to be connected to four-sided segments. So the connection method needs to be established. As shown



(1) control polygons.

(2) contour lines.

Fig. 18 An example of a pentagonal patch made of triangular surface segments.

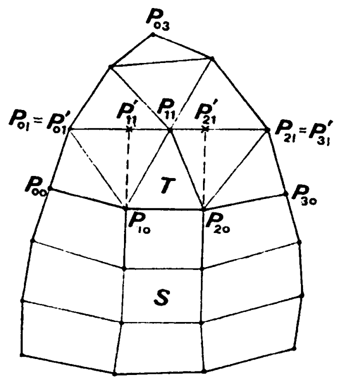



Fig. 19 Connection of triangular and rectangular surface segments.

in Fig. 19, two boundary curves $T(u, v, 0)$ and $S(u, 0)$ are to be smoothly connected. Both surface segments are of degree $3\times3$ and have smooth tangent connection along side boundary curves at $\boldsymbol{P}_{00}$ and $\boldsymbol{P}_{30}$. The first tangent vector of $T$ along the boundary $w=0$ can be determined by the following operation.

$$\frac{d}{dw}=\frac{\partial}{\partial w}-(1-v)\cdot\frac{\partial}{\partial u}-v\cdot\frac{\partial}{\partial v}. \qquad (73)$$

By using the above operation, $dT/dw$ can be expressed by the control vector $\{\boldsymbol{P}'_{0i}\}_0^3$.

$$\{\boldsymbol{P}'_{0i}\}=\left\{\boldsymbol{P}_{01}, \boldsymbol{P}_{01}+\frac{2}{3}(\boldsymbol{P}_{11}-\boldsymbol{P}_{01}), \boldsymbol{P}_{21}+\frac{2}{3}(\boldsymbol{P}_{11}-\boldsymbol{P}_{21}), \boldsymbol{P}_{21}\right\}. \qquad (74)$$

With this control vector, the three-sided surface segment can be considered as a four-sided surface segment whose first tangent vector along the connecting boundary is given by

$$3\cdot(1-v+vE)^3\cdot(P'_{01}-P_{00}). \tag{75}$$

The positions of the control points $\{P'_{0i}\}_0^3$ are geometrically easily determined from the original control points of $T$. Connection of two four-sided segments is not difficult, if their tangent vectors of the first kind are parallel on the boundary.

## 7. Conclusion

In order to include free form shapes as the element of our geometric modelling and processing system (GEOMAP) [8], and to respond to strong demand on CAD and CAM of die making and other shape design in industries, we have developed a new theory for shape expressions and various application methods of the theory. The features of our work are summarized in the Sec. 1 of this paper. Though there may be further development of the theory or application methods in future, we think the basic ones have been completed. Accordingly, we hope we could have feedback information and evaluation of applying them in practical cases.

### References

1. Barnhill, R. E. and Riesenfeld, R. F. (ed), Computer Aided Geometric Design, Academic Press, New York (1974).
2. Coons, S. A., Surfaces for Computer Aided Design of Space Forms, MIT Project MAC TR-41 (1967).
3. Hosaka, M., Theory of Curve and Surface Synthesis and their Smooth Fitting, Information Processing in Japan, 9 (1969) 60–68.
4. Bézier, P. E., Numerical Control-Mathematics and Applications, John Wiley and Sons, London (1972).
5. Riesenfeld, R. F., Applications of B-spline Approximation of Geometric Problems of Computer Aided Design, Univ. of Utah, UTEC-CSc-73-126 (1973).
6. Forrest, A. R., Computational Geometry-Achievements and Problems, in (1), 17–44.
7. Forrest, A. R., Recent Trends in Computer Aided Geometric Design, Proc. Conf. Interactive Techniques in Computer Aided Design, 78CH1289-8C, IEEE, New York (1978) 141–146,.
8. Hosaka, M. and Kimura, F., An Interactive Geometrical Design System with Handwriting Input, Information Processing 77, North-Holland, Amsterdam (1977)167–172.