Regularization of Solutions of Nonlinear Equations with Singular Jacobian Matrices

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We consider a solution of a nonlinear equation with the singular Jacobian matrix at the solution. So far, to such a solution, it has been difficult to get an approximation with high accuracy since the Jacobian matrix is singular. We propose a method for overcoming the difficulty arising from the singularity of the Jacobian matrix. We call this process the "regularization of solutions of nonlinear equations". Since a system of nonlinear equations obtained from the regularization has a solution which makes the Jacobian matrix of the system non-singular and contains the above-mentioned solution, we can get an approximation of the system solution as accurately as we desire by the Newton method. Hence we can also obtain a desired approximation to the solution of the original equation.

1. Introduction

We consider a real nonlinear equation

$$F(x)=0, \tag{1.1}$$

where $x\in R^n$, $F(x)\in R^n$, and F is a continuously differentiable mapping from some region Ω in R^n into R^n. Here R^n denotes the real n-dimensional Euclidean space.

When we compute a solution $x=\hat{x}\in\Omega$ of the nonlinear eq. (1.1), it is difficult to get a highly accurate approximation to the solution $\hat{x}$ by applying the Newton method to the eq. (1.1) in the case where the Jacobian matrix $F_x(x)$ of the function $F(x)$ with respect to x is singular at $x=\hat{x}$.

However, introducing a parameter in the eq. (1.1) and making use of the singularity of $F_x(\hat{x})$, we can regularize the solution $\hat{x}$ of the eq. (1.1) and we can get an approximation to $\hat{x}$ as accurately as we desire by applying the Newton method to a system of equations obtained from the regularization, that is, (2.5) in Chapter 2. Such a system is called an "augmented system".

H. Weber and W. Werner [11] have proposed a regularization method similar to ours. When dim Ker $(F_x(\hat{x}))=1$ and Ker $(F_x(\hat{x}))\cap$Im $(F_x(\hat{x}))=\{0\}$, they have considered an augmented system similar to (2.5) and they have obtained a result similar to Theorem 1 in Chapter 2, where Ker $(F_x(\hat{x}))$ denotes the kernel of $F_x(\hat{x})$ and Im $(F_x(\hat{x}))$ denotes the image of $F_x(\hat{x})$. However, when the condition Ker $(F_x(\hat{x}))\cap$Im $(F_x(\hat{x}))=\{0\}$ is not satisfied, they have considered a complicated augmented system instead of the one similar to (2.5).

On the other hand, in our case, we consider only the system (2.5), whether the condition Ker $(F_x(\hat{x}))\cap$ Im $(F_x(\hat{x}))=\{0\}$ holds or not. Hence our method seems to be more useful and convenient than their method. For details, see Remark 4 in Chapter 2 and Example 1 in Chapter 3.

Further, when the Jacobian matrix has a high singularity at the solution, they did not describe anything. On the other hand, we can consider the solution with the Jacobian matrix which has such a high singularity and we give a condition for classifying solutions with singular Jacobian matrices.

In Chapter 3, in order to illustrate our theory and method, we present some examples of solutions of nonlinear equations with singular Jacobian matrices.

2. The Method for the Regularization of Solutions of Nonlinear Equations

We consider a solution $x=\hat{x}\in\Omega$ of a real n-dimensional nonlinear equation

$$F(x)=0 \tag{2.1}$$

such that the rank of the Jacobian matrix $F_x(x)$ is $n-1$ at $x=\hat{x}$, where the function $F(x)$ is defined in some region $\Omega(\subset R^n)$ and $F(x)$ is continuously differentiable with respect to x in Ω.

In order to simplify the following argument, we assume that

$$\text{rank } F_x(\hat{x})=\text{rank } F_0(\hat{x})=n-1, \tag{2.2}$$

where $F_0(\hat{x})$ is the $n\times(n-1)$ matrix obtained from $F_x(\hat{x})$ by deleting the first column vector.

Then there exists a positive integer $k(1\le k\le n)$ such that

$$\text{rank } (F_x(\hat{x}),\, e_k)=\text{rank } (F_0(\hat{x}),\, e_k)=n, \tag{2.3}$$

where e_k is the k-th unit vector, that is,

$$e_k=(0,\cdots,0,\underset{\substack{\wedge \\ (k\text{-th component of } e_k)}}{1},0,\cdots,0)^T. \tag{2.4}$$

Here $(\cdots)^T$ denotes the transposed vector of a vector $(\cdots)$.

Now, making use of the singularity of the Jacobian matrix $F_x(\hat{x})$, we consider an augmented system consist-

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ing of the eq. (2.1) and additional equations involving the Jacobian matrix. That is, since the equation $F_x(\hat{x})h=0$ has a nontrivial solution due to (2.2), we introduce a parameter B in the eq. (2.1) and we consider the system

$$G(\boldsymbol{x})=\begin{pmatrix}F(x)-Be_k\\ F_x(x)h\\ h_1-1\end{pmatrix}=0, \tag{2.5}$$

where $\boldsymbol{x}=(x, h, B)^T$, $x=(x_1,\cdots,x_n)^T$ and $h=(h_1,\cdots,h_n)^T$. Then, by (2.2), the system (2.5) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0)^T$ (where $\hat{h}$ is a solution of the equation $F_x(\hat{x})h=0$, $h_1-1=0$), and in particular, the x-component $\hat{x}$ of $\hat{\boldsymbol{x}}$ is the desired solution of (2.1). For the solution $\hat{\boldsymbol{x}}$ of (2.5), we have the following theorem.

Theorem 1. Assume that the function $F(x)$ is twice continuously differentiable with respect to x in Ω.

Then the matrix $G'(\hat{\boldsymbol{x}})$ is non-singular if and only if

$$\operatorname{rank}(F_0(\hat{x}), \hat{l})=n, \tag{2.6}$$

where $G'(\boldsymbol{x})$ denotes the Jacobian matrix of $G(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ and $\hat{l}=\{F_{xx}(\hat{x})\hat{h}\}\hat{h}$. Here $F_{xx}(x)$ denotes the second derivative of $F(x)$ with respect to x.

Proof. Since $F(x)$ is twice continuously differentiable with respect to x, the function $G(\boldsymbol{x})$ defined by the equality (2.5) is continuously differentiable with respect to $\boldsymbol{x}$ and we have

$$G'(\boldsymbol{x})=\begin{pmatrix}F_x(x) & 0 & -e_k\\ F_{xx}(x)h & F_x(x) & 0\\ 00\cdots0 & 10\cdots0 & 0\end{pmatrix}. \tag{2.7}$$

Then, for the solution $\hat{\boldsymbol{x}}$, we have

$$\det G'(\hat{\boldsymbol{x}})=\begin{vmatrix}F_x(\hat{x}) & 0 & -e_k\\ F_{xx}(\hat{x})\hat{h} & F_x(\hat{x}) & 0\\ 00\cdots0 & 10\cdots0 & 0\end{vmatrix} =\begin{vmatrix}0 & F_0(\hat{x}) & 0 & 0 & -e_k\\ \hat{l} & F_1(\hat{x},\hat{h}) & 0 & F_0(\hat{x}) & 0\\ 0 & 0\cdots0 & 1 & 0\cdots0 & 0\end{vmatrix}, \tag{2.8}$$

where $F_1(\hat{x}, \hat{h})$ is the $n\times(n-1)$ matrix obtained from $F_{xx}(\hat{x})\hat{h}$ by deleting the first column vector.

From (2.8) it follows that

$$\det G'(\hat{\boldsymbol{x}})\neq 0 \text{ is equivalent to (2.6).} \tag{2.9}$$

This completes the proof. Q.E.D.

Remark 1. We propose another way of introducing a parameter in the eq. (2.1). By Theorem 1, we may consider the system

$$\tilde{G}(\boldsymbol{x})=\begin{pmatrix}F(x)-B\{F_{xx}(x)h\}h\\ F_x(x)h\\ h_1-1\end{pmatrix}=0 \tag{2.10}$$

instead of the system (2.5) if the function $F(x)$ is three times continuously differentiable with respect to x in Ω and the condition (2.6) is satisfied. In this case, we must compute the second derivative $F_{xx}(x)$, but we need not look for the vector e_k satisfying the condition (2.4). Then the solution $\hat{\boldsymbol{x}}$ of (2.5) is also a solution of (2.10). For the solution $\hat{\boldsymbol{x}}$, we have a result similar to Theorem 1.

When rank $(F_0(\hat{x}), \hat{l})=n-1$, since the equation

$$\begin{cases}F_x(\hat{x})k+\hat{l}=0,\\ k_1=0\end{cases} \tag{2.11}$$

has a solution $\hat{k}=(\hat{k}_1,\cdots,\hat{k}_n)^T$, we introduce one more parameter and we consider the system

$$G_1(\boldsymbol{x}_1)=\begin{pmatrix}F(x)-B_1e_k\\ F_x(x)h_1-B_2e_k\\ F_x(x)h_2+l_1\\ h_1^1-1\\ h_2^1\end{pmatrix}=0, \tag{2.12}$$

where $\boldsymbol{x}_1=(x, h_1, h_2, B_1, B_2)^T$, $h_i=(h_i^1, h_i^2,\cdots,h_i^n)^T(i=1, 2)$ and $l_1=\{F_{xx}(x)h_1\}h_1$.

Evidently, the system (2.12) has a solution $\hat{\boldsymbol{x}}_1=(\hat{x}, \hat{h}_1, \hat{h}_2, 0, 0)^T$, where $\hat{h}_2$ is a solution of (2.11). For this solution $\hat{\boldsymbol{x}}_1$, we readily get the following theorem.

Theorem 2. Assume that the function $F(x)$ is three times continuously differentiable with respect to x in Ω.

Then the matrix $G_1'(\hat{\boldsymbol{x}}_1)$ is non-singular if and only if

$$\operatorname{rank}(F_0(\hat{x}), \hat{l}_2)=n, \tag{2.13}$$

where

$G_1'(\boldsymbol{x}_1)$ denotes the Jacobian matrix of $G_1(\boldsymbol{x}_1)$ with respect to $\boldsymbol{x}_1$;

$X^{(0)}=F_x(x)$, $X^{(1)}=X_x^{(0)}h_1$, $X^{(2)}=X_x^{(0)}h_2+X_x^{(1)}h_1$ and $\hat{l}_2=\hat{X}^{(2)}\hat{h}_1+2\hat{X}^{(1)}\hat{h}_2$, where $X_x^{(j)}(j=0, 1)$ denote the derivatives of $X^{(j)}(j=0, 1)$ with respect to x, respectively, and $\hat{X}^{(i)}(i=0, 1, 2)$ denote the values of $X^{(i)}(i=0, 1, 2)$ at $x=\hat{x}$, $h_1=\hat{h}_1$ and $h_2=\hat{h}_2$, respectively.

Remark 2. If the function $F(x)$ is four times continuously differentiable with respect to x in Ω and the condition (2.13) is satisfied, we may consider the system

$$\tilde{G}_1(\boldsymbol{x}_1)=\begin{pmatrix}F(x)-B_1l_2\\ F_x(x)h_1-B_2l_2\\ F_x(x)h_2+\{F_{xx}(x)h_1\}h_1\\ h_1^1-1\\ h_2^1\end{pmatrix}=0 \tag{2.14}$$

instead of the system (2.12), where $l_2=X^{(2)}h_1+2X^{(1)}h_2$.

More generally, let us suppose that the function $F(x)$ is $(d+2)$ times continuously differentiable with respect to x in $\Omega(d\geq 2)$.

Put

$$X^{(i+1)}=\sum_{k=0}^{i}{}_iC_kX_x^{(k)}h_{i+1-k} \quad (1\leq i\leq d) \tag{2.15}$$

and

$$l_i=\sum_{k=1}^{i}{}_iC_kX^{(k)}h_{i+1-k} \quad (1\leq i\leq d+1) \tag{2.16}$$

where $X_x^{(j)}(j=0, 1,\cdots,d)$ are the derivatives of $X^{(j)}(j=0, 1,\cdots,d)$ with respect to x, respectively, and $h_i(i=1, 2,\cdots,d+1)$ are n-dimensional vectors.

If there exists a $(d+1)n$-dimensional vector $\hat{y}_d=(\hat{x}, \hat{h}_1,\cdots,\hat{h}_d)^T$ such that the conditions

(i) $\hat{x}$ is a solution of (2.1) satisfying (2.2) and (2.3), (2.17)

(ii) $\hat{X}^{(0)}\hat{h}_1=0$, $\hat{h}_1^1-1=0$ and $\hat{X}^{(0)}\hat{h}_{j+1}+\hat{l}_j=0$, $\hat{h}_{j+1}^1=0$ $(j=1,2,\cdots,d-1)$, (2.18)

(iii) $\operatorname{rank}(F_0(\hat{x}),\hat{l}_d)=n-1$ (2.19) are satisfied, then we introduce $(d+1)$ parameters $B_1, B_2, \cdots, B_{d+1}$ and we consider the system

$$G_d(x_d)=\begin{pmatrix} F(x)-B_1e_k \\ X^{(0)}h_1-B_2e_k \\ X^{(0)}h_2+X^{(1)}h_1-B_3e_k \\ \vdots \\ \sum_{i=0}^{d-1} {}_{d-1}C_iX^{(i)}h_{d-i}-B_{d+1}e_k \\ \sum_{i=0}^{d} {}_dC_iX^{(i)}h_{d+1-i} \\ \psi_d(x_d) \end{pmatrix} = \begin{pmatrix} F(x)-B_1e_k \\ X^{(0)}h_1-B_2e_k \\ X^{(0)}h_2+l_1-B_3e_k \\ \vdots \\ X^{(0)}h_d+l_{d-1}-B_{d+1}e_k \\ X^{(0)}h_{d+1}+l_d \\ \psi_d(x_d) \end{pmatrix}=0, \qquad (2.20)$$

where $x_d=(x, h_1, h_2, \cdots, h_{d+1}, B_1, B_2, \cdots, B_{d+1})^T$, $h_i=(h_i^1, h_i^2, \cdots, h_i^n)^T (i=1, 2, \cdots, d+1)$, $\psi_d(x_d)=(h_1^1-1, h_2^1, \cdots, h_{d+1}^1)^T$, and $\hat{X}^{(j)} (j=0, 1, \cdots, d)$ and $\hat{l}_i (i=1, 2, \cdots, d)$ denote the values of $X^{(j)} (j=0, 1, \cdots, d)$ and $l_i (i=1, 2, \cdots, d)$ at $x=\hat{x}$, $h_1=\hat{h}_1, \cdots, h_d=\hat{h}_d$, respectively. Then, by (2.17), (2.18) and (2.19), the system (2.20) has a solution $\hat{x}_d=(\hat{y}_d, \hat{h}_{d+1}, 0)^T$ (where $\hat{h}_{d+1}$ is a solution of the equation $\hat{X}^{(0)}h_{d+1}+\hat{l}_d=0$, $h_{d+1}^1=0$ and 0 is the $(d+1)$-dimensional zero vector) and for this solution $\hat{x}_d$, we have the following theorem.

Theorem 3. The matrix $G_d'(\hat{x}_d)$ is non-singular if and only if

$$\operatorname{rank}(F_0(\hat{x}), \hat{l}_{d+1})=n, \qquad (2.21)$$

where $G_d'(x_d)$ denotes the Jacobian matrix of $G_d(x_d)$ with respect to x_d and $\hat{l}_{d+1}$ denotes the values of l_{d+1} at $x=\hat{x}$, $h_1=\hat{h}_1, \cdots, h_{d+1}=\hat{h}_{d+1}$.

Proof. Since $F(x)$ is $(d+2)$ times continuously differentiable with respect to x in Ω, the function $G_d(x_d)$ defined by the equality (2.20) is continuously differentiable with respect to x_d. Then we have

$$G'(x_d)=\begin{pmatrix} {}_0C_0X^{(0)} & O & O & \cdots & O & O & -e_k & 0\cdots & 0 \\ {}_1C_0X^{(1)} & {}_1C_1X^{(0)} & O & \cdots & O & O & 0 & -e_k\cdots & 0 \\ {}_2C_0X^{(2)} & {}_2C_1X^{(1)} & {}_2C_2X^{(0)} & \cdots & O & O & 0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\ \ddots & \vdots \\ {}_dC_0X^{(d)} & {}_dC_1X^{(d-1)} & {}_dC_2X^{(d-2)} & \cdots & {}_dC_dX^{(0)} & O & 0 & 0\cdots & -e_k \\ {}_{d+1}C_0X^{(d+1)} & {}_{d+1}C_1X^{(d)} & {}_{d+1}C_2X^{(d-1)} & \cdots & {}_{d+1}C_dX^{(1)} & {}_{d+1}C_{d+1}X^{(0)} & 0 & 0\cdots & 0 \\ & 10\cdots0 & 00\cdots0 & & \boxed{O} & \boxed{O} & & & \\ \boxed{O} & 00\cdots0 & 10\cdots0 & \cdots & & & & \boxed{O} & \\ & \boxed{O} & \boxed{O} & & 10\cdots0 & 00\cdots0 & & & \\ & & & & 00\cdots0 & 10\cdots0 & & & \end{pmatrix}. \qquad (2.22)$$

From (2.22) it follows that

$$\det G_d'(\hat{x}_d)\neq 0 \text{ is equivalent to (2.21).} \qquad (2.23)$$

This completes the proof. Q.E.D.

Thus, if the condition (2.21) is satisfied, then we can get an approximation to the solution $\hat{x}_d$ of (2.20) as accurately as we desire by the Newton method. Since the x-component $\hat{x}$ of $\hat{x}_d$ is a solution of (2.1), we can also obtain a desired approximation to $\hat{x}$.

Remark 3. If the function $F(x)$ is $(d+3)$ times continuously differentiable with respect to x in Ω and the condition (2.21) is satisfied, we may consider the system

$$\tilde{G}_d(x_d)=\begin{pmatrix} F(x)-B_1l_{d+1} \\ X^{(0)}h_1-B_2l_{d+1} \\ X^{(0)}h_2+l_1-B_3l_{d+1} \\ \vdots \\ X^{(0)}h_d+l_{d-1}-B_{d+1}l_{d+1} \\ X^{(0)}h_{d+1}+l_d \\ \psi_d(x_d) \end{pmatrix}=0 \qquad (2.24)$$

instead of the system (2.20).

Remark 4. When $\dim \operatorname{Ker}(F_x(\hat{x}))=1$ and $\operatorname{Ker}(F_x(\hat{x}))\cap \operatorname{Im}(F_x(\hat{x}))=\{0\}$, H. Weber and W. Werner [11] have considered the system

$$W(x)=\begin{pmatrix} F(x)+Bh \\ F_x(x)h \\ h^Th-1 \end{pmatrix}=0 \qquad (2.25)$$

instead of the system (2.5), where $x=(x, h, B)^T$, $x=(x_1, \cdots, x_n)^T$, $h=(h_1, \cdots, h_n)^T$ and B is a parameter. Evidently, the system (2.25) has a solution $\hat{x}=(\hat{x}, \hat{h}, 0)^T$ (where $\hat{h}$ is a solution of the equation $F_x(\hat{x})h=0$, $h^Th-1=0$) and for this solution $\hat{x}$, they have obtained a result similar to Theorem 1, that is, they have given a sufficient condition for guaranteeing that the Jacobian matrix $W'(x)$ of $W(x)$ with respect to x is non-singular at the solution $\hat{x}$.

But, when the condition $\operatorname{Ker}(F_x(\hat{x}))\cap\operatorname{Im}(F_x(\hat{x}))=\{0\}$ is not satisfied, the Jacobian matrix $W'(x)$ is singular at $\hat{x}$. Then they have considered the system

$$\tilde{W}(x)=\begin{pmatrix} F_x(x)^TF(x)+Bh \\ F_x(x)h \\ h^Th-1 \end{pmatrix}=0 \qquad (2.26)$$

instead of the system (2.25), where $F_x(x)^T$ denotes the transposed matrix of $F_x(x)$.

On the other hand, in our case, we consider only the system (2.5) whether the condition $\operatorname{Ker}(F_x(\hat{x}))\cap\operatorname{Im}(F_x(\hat{x}))=\{0\}$ holds or not. Hence our method seems to be simpler than their method. In particular, comparing the system (2.26) with the system (2.5), it seems that our method is more useful and convenient. Of course, we may adopt the condition $h_1-1=0$ instead of the condition $h^Th-1=0$ in the systems (2.25) and (2.26) when $\operatorname{rank} F_x(\hat{x})=\operatorname{rank} F_0(\hat{x})=n-1$.

Remark 5. When the condition (2.2) is satisfied, we have

$$\det F_x(\hat{x})=0.$$

Under the condition (2.3), we may consider the system

$$H(\boldsymbol{x})=\begin{pmatrix} F(x)-Be_k \\ g(x) \end{pmatrix}=0 \tag{2.27}$$

instead of the system (2.5), where $\boldsymbol{x}=(x, B)^T$ and $g(x)=\det F_x(x)$. Then $\hat{\boldsymbol{x}}=(\hat{x}, 0)^T$ is certainly a solution of (2.27) and for this solution $\hat{\boldsymbol{x}}$, under the same assumption as in Theorem 1, we have

$$\begin{cases} \text{the matrix } H'(\hat{\boldsymbol{x}}) \text{ is non-singular if and only if} \\ \text{the condition (2.6) is satisfied,} \end{cases} \tag{2.28}$$

where $H'(\boldsymbol{x})$ denotes the Jacobian matrix of $H(\boldsymbol{x})$ with respect to $\boldsymbol{x}$.

The proof of (2.28) is as follows: By the definition of the function $H(\boldsymbol{x})$, we have

$$H'(\boldsymbol{x})=\begin{pmatrix} F_x(x) & -e_k \\ \dfrac{\partial g(x)}{\partial x} & 0 \end{pmatrix}, \tag{2.29}$$

where $\partial g(x)/\partial x=(\partial g(x)/\partial x_1,\cdots,\partial g(x)/\partial x_n)$. Then, for the solution $\hat{\boldsymbol{x}}=(\hat{x}, 0)^T$ of (2.27), we have

$$\det H'(\hat{\boldsymbol{x}})=\begin{vmatrix} F_x(\hat{x}) & -e_k \\ \dfrac{\partial g(\hat{x})}{\partial x} & 0 \end{vmatrix}=\begin{vmatrix} 0 & F_0(\hat{x}) & -e_k \\ \hat{\eta} & \dfrac{\partial g}{\partial x_2}\cdots\dfrac{\partial g}{\partial x_n} & 0 \end{vmatrix}, \tag{2.30}$$

where $\hat{\eta}=[\det(\hat{l}, F_0(\hat{x}))]/\hat{h}_1$. Here $\hat{h}=(\hat{h}_1,\cdots,\hat{h}_n)^T$ is the h-component of the solution $(\hat{x}, \hat{h}, 0)^T$ of (2.5). By (2.30) we easily get

$$\det H'(\hat{\boldsymbol{x}})\neq 0 \text{ is equivalent to } \hat{\eta}\neq 0. \tag{2.31}$$

Thus (2.28) follows from (2.31).

Next, we consider the case where $\operatorname{rank} F_x(\hat{x})=n-d$ $(1<d\leq n)$. For the sake of simplicity, we assume that

$$n-d=\operatorname{rank} F_x(\hat{x})=\operatorname{rank} F_d(\hat{x}), \tag{2.32}$$

where $F_d(\hat{x})$ is the $n\times(n-d)$ matrix obtained from $F_x(\hat{x})$ by deleting the first column vector through the d-th column vector.

Then there exist d positive integers $k_1, k_2,\cdots,k_d$ $(1\leq k_1, k_2,\cdots,k_d\leq n)$ such that

$$\operatorname{rank}(F_d(\hat{x}), e_{k_1}, e_{k_2},\cdots,e_{k_d})=n, \tag{2.33}$$

where $e_{k_i}=(0,\cdots,0,\underset{(k_i\text{-th component of } e_{k_i})}{\overset{\wedge}{1}},0,\cdots,0)^T (i=1,2,\cdots,d)$.

Then we introduce d parameters $B_1, B_2,\cdots,B_d$ in the eq. (2.1) and we consider the system

$$G(\boldsymbol{x})=\begin{pmatrix} F(x)-B_1e_{k_1}-B_2e_{k_2}-\cdots-B_de_{k_d} \\ F_x(x)h \\ \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_d \end{pmatrix}-\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \end{pmatrix}=0, \tag{2.34}$$

where $\boldsymbol{x}=(x, h, B_1, B_2,\cdots,B_d)^T$, $x=(x_1,\cdots,x_n)^T$, $h=(h_1,\cdots,h_n)^T$ and $a=(a_1,\cdots,a_d)^T$ is a d-dimensional non-zero constant vector. The condition (2.32) implies that the system (2.34) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0, 0,\cdots,0)^T$. For this solution $\hat{\boldsymbol{x}}$, we have the following theorem.

Theorem 4. Assume that the function $F(x)$ is twice continuously differentiable with respect to x in Ω.

Then the matrix $G'(\hat{\boldsymbol{x}})$ is non-singular if and only if

$$\operatorname{rank}(F_d(\hat{x}), \hat{m}_1, \hat{m}_2,\cdots,\hat{m}_d)=n, \tag{2.35}$$

where

$G'(\boldsymbol{x})$ denotes the Jacobian matrix of $G(\boldsymbol{x})$ with respect to $\boldsymbol{x}$; $\hat{m}_i=\{F_{xx}(\hat{x})\hat{h}\}\hat{h}^{(i)}(1\leq i\leq d)$, where $\hat{h}^{(i)}(1\leq i\leq d)$ are solutions of the equations

$$\begin{cases} F_x(\hat{x})h^{(i)}=0 \\ h_1^{(i)}=0, \\ \vdots \\ h_i^{(i)}=1, \\ \vdots \\ h_d^{(i)}=0 \end{cases} \qquad (1\leq i\leq d) \tag{2.36}$$

respectively. Here $h^{(i)}=(h_1^{(i)}, h_2^{(i)},\cdots,h_n^{(i)})^T(1\leq i\leq d)$.

We must choose a d-dimensional vector a $(\neq 0)$ so that the condition (2.35) is satisfied.

But, if there is no d-dimensional vector satisfying (2.35), that is,

$$\operatorname{rank}(F_d(\hat{x}), \hat{m}_1, \hat{m}_2,\cdots,\hat{m}_d)<n \tag{2.37}$$

for any d-dimensional vector a $(\neq 0)$, then we regard the system $G(\boldsymbol{x})=0$ as the original equation $F(x)=0$ and we repeat the above-mentioned process for the system $G(\boldsymbol{x})=0$.

3. Examples

In this section, in order to illustrate our theory and method mentioned in Chapter 2, we present some examples of solutions of nonlinear equations with singular Jacobian matrices.

First, we consider the case where $\operatorname{rank} F_x(\hat{x})=n-1$.

Example 1([11]). We consider the equation

$$F(x)=\begin{pmatrix} x_1^2-2x_1+\dfrac{1}{3}x_2^3+\dfrac{2}{3} \\ x_1^3-x_1x_2-2x_1+\dfrac{1}{2}x_2^2+\dfrac{3}{2} \end{pmatrix}=0, \tag{3.1}$$

where $x=(x_1, x_2)^T$. The eq. (3.1) has a solution $\hat{x}=(\hat{x}_1, \hat{x}_2)^T=(1, 1)^T$ and for this solution $\hat{x}$, we have

$$\operatorname{rank} F_x(\hat{x})=\operatorname{rank}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=1.$$

Since $\operatorname{rank}(F_x(\hat{x}), e_2)=2$, we introduce a parameter B in (3.1) and we consider the system

$$G(\boldsymbol{x})=\begin{pmatrix} F(x)-Be_2 \\ F_x(x)h \\ h_1-1 \end{pmatrix}=0 \tag{3.2}$$

where $\boldsymbol{x}=(x, h, B)^T$, $x=(x_1, x_2)^T$, $h=(h_1, h_2)^T$ and $e_2=(0, 1)^T$. The system (3.2) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0)^T$ (where $\hat{x}=(1, 1)^T$ and $\hat{h}=(1, 0)^T$) and for this solution $\hat{\boldsymbol{x}}$, we have

$$\det G'(\hat{\boldsymbol{x}}) \neq 0$$

since

$$G'(\hat{\boldsymbol{x}}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 1 & 0 \\ 6 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

H. Weber and W. Werner [11] also considered this example. As is seen from Remark 4 in Chapter 2, their method for introducing a parameter in (3.1) is different from our method. In fact, in this example, they considered the system

$$\tilde{W}(\boldsymbol{x}) = \begin{pmatrix} F_x(x)^T F(x) + Bh \\ F_x(x)h \\ h^T h - 1 \end{pmatrix} = 0, \tag{3.3}$$

where $\boldsymbol{x} = (x, h, B)^T$, $x = (x_1, x_2)^T$, $h = (h_1, h_2)^T$ and $F_x(x)^T$ denotes the transposed matrix of $F_x(x)$ and B is a parameter. Comparing the system (3.3) with the system (3.2), our method seems to be more useful and convenient. Of course, we may take the condition $h_1 - 1 = 0$ instead of the condition $h^T h - 1 = 0$ in the system (3.3).

Example 2([8]). Let us consider the equation

$$F(x) = \begin{pmatrix} x_1^3 + x_1 x_2 \\ x_2 + x_2^2 \end{pmatrix} = 0, \tag{3.4}$$

where $x = (x_1, x_2)^T$. The eq. (3.4) has a solution $\hat{x} = (\hat{x}_1, \hat{x}_2)^T = (0, 0)^T$ and for this solution $\hat{x}$, we have

$$\operatorname{rank} F_x(\hat{x}) = \operatorname{rank} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

In this case, since $\hat{l} = \{F_{xx}(\hat{x})\hat{h}\}\hat{h} = (0, 0)^T$, we have

$$\operatorname{rank}(F_x(\hat{x}), \hat{l}) = \operatorname{rank}(F_0(\hat{x}), \hat{l}) = 1,$$

where $F_0(\hat{x}) = (0, 1)^T$ and $\hat{h} = (1, 0)^T$.

Therefore we introduce two parameters B_1, B_2 and we consider the system

$$G_1(\boldsymbol{x}_1) = \begin{pmatrix} F(x) - B_1 e_1 \\ F_x(x)h_1 - B_2 e_1 \\ F_x(x)h_2 + \{F_{xx}(x)h_1\}h_1 \\ h_1^1 - 1 \\ h_2^1 \end{pmatrix} = 0, \tag{3.5}$$

where $\boldsymbol{x}_1 = (x, h_1, h_2, B_1, B_2)^T$, $x = (x_1, x_2)^T$, $h_i = (h_i^1, h_i^2)^T (i = 1, 2)$ and $e_1 = (1, 0)^T$. The system (3.5) has a solution $\hat{\boldsymbol{x}}_1 = (\hat{x}, \hat{h}_1, \hat{h}_2, 0, 0)^T$ (where $\hat{x} = (0, 0)^T$, $\hat{h}_1 = (1, 0)^T$ and $\hat{h}_2 = (0, 0)^T$) and for this solution $\hat{\boldsymbol{x}}_1$, we have

$$\det G_1'(\hat{\boldsymbol{x}}_1) \neq 0$$

since

$$G_1'(\hat{\boldsymbol{x}}_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Secondly, we consider the case where rank $F_x(\hat{x}) = n - 2$.

Example 3([4]). We consider the equation

$$F(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = 0, \tag{3.6}$$

where $x = (x_1, x_2)^T$ and

$$\begin{aligned} f_1(x_1, x_2) &= x_1^5 - 10x_1^3 x_2^2 + 5x_1 x_2^4 - 3x_1^4 + 18x_1^2 x_2^2 - 3x_2^4 - 2x_1^3 \\ &\quad + 6x_1 x_2^2 + 3x_1^2 x_2 - x_2^3 + 12x_1^2 - 12x_2^2 - 10x_1 x_2 \\ &\quad - 8x_1 + 8x_2, \\ f_2(x_1, x_2) &= 5x_1^4 x_2 - 10x_1^2 x_2^3 + x_2^5 - 12x_1^3 x_2 + 12x_1 x_2^3 - x_1^3 + 3x_1 x_2^2 \\ &\quad - 6x_1^2 x_2 + 2x_2^3 + 5x_1^2 - 5x_2^2 + 24x_1 x_2 - 8x_1 - 8x_2 + 4. \end{aligned}$$

The eq. (3.6) has a solution $\hat{x} = (\hat{x}_1, \hat{x}_2)^T = (2, 0)^T$ and for this solution $\hat{x}$, we have

$$\operatorname{rank} F_x(\hat{x}) = \operatorname{rank} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Hence we introduce two parameters B_1, B_2 in (3.6) and we consider the system

$$G(\boldsymbol{x}) = \begin{pmatrix} F(x) - B_1 e_1 - B_2 e_2 \\ F_x(x)h \\ h_1 - a_1 \\ h_2 - a_2 \end{pmatrix} = 0, \tag{3.7}$$

where $\boldsymbol{x} = (x, h, B_1, B_2)^T$, $x = (x_1, x_2)^T$, $h = (h_1, h_2)^T$, $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$. In this example, we take $a_1 = 1$ and $a_2 = 0$ in (3.7). Evidently, $\hat{\boldsymbol{x}} = (\hat{x}, \hat{h}, 0, 0)^T$ (where $\hat{x} = (2, 0)^T$ and $\hat{h} = (1, 0)^T$) is a solution of (3.7) and for this solution $\hat{\boldsymbol{x}}$, we have

$$\det G'(\hat{\boldsymbol{x}}) \neq 0$$

since

$$G'(\hat{\boldsymbol{x}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 16 & 2 & 0 & 0 & 0 & 0 \\ -2 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

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