An Optimal Algorithm for Approximating a Piecewise Linear Function

Hiroshi Imai* and Masao Iri**

It is shown that the problem of approximating a given piecewise linear function with n nondifferentiable points by another piecewise linear function such that the absolute value of the difference between the two functions be bounded by a given constant w and that the number of nondifferentiable points of the latter function be minimum can be solved in $O(n)$ time.

1. Introduction

Problems of approximating a piecewise linear function by another such function have several applications in pattern recognition and data reduction. In this paper, we consider the problem of approximating a given piecewise linear function with n nondifferentiable points by another piecewise linear function such that the absolute value of the difference between the two functions be bounded by a positive constant w and that the number of nondifferentiable points of the latter function be minimum. For this problem, Tomek [6] gave heuristic algorithms, but no algorithm has been known that produces an optimal solution. We here show that an optimal solution of this problem can be found efficiently. In fact, our algorithm runs in $O(n)$ time, which is optimal with respect to the time complexity. The algorithm utilizes computational-geometric algorithms for the convex hull problem, is simple enough, and runs fast in practice.

2. An Outline of the Algorithm

A function $f:[x^-, x^+]\to\mathbf{R}$ is said to be "piecewise linear" if its graph $y=f(x)$ in the xy-plane is a polygonal line connecting points $p_1, p_2, \ldots, p_n$ in this order such that $x^-=x(p_1)<x(p_2)<\ldots<x(p_n)=x^+$, where $x(p)$ denotes the x-coordinate of point p. This polygonal line is denoted by $p_1p_2\ldots p_n$, and will often be identified with the function f. This polygonal line is strictly monotone with respect to the x-axis (henceforth, "monotone" will mean "strictly monotone with respect to the x-axis"). For the piecewise linear function f, another piecewise linear function $\tilde{f}$ is called an approximate piecewise linear function with error bound $w>0$ if the x-coordinates of two endpoints of

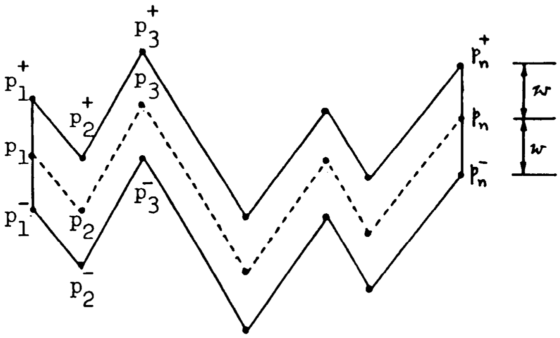

Fig. 1 Polygonal line $p_1p_2\ldots p_n$ (dotted line) and polygon $P(w)$.

polygonal line of $y=\tilde{f}(x)$ is $x(p_1)$ and $x(p_n)$, and if $|\tilde{f}(x)-f(x)|\le w$ for any x between $x(p_1)$ and $x(p_n)$. The problem is to find an approximate piecewise linear function with error bound w which has as few segments as possible.

For a polygonal line $p_1p_2\ldots p_n$, construct a polygon by sliding the polygonal line $p_1p_2\ldots p_n$ vertically by w both upwards and downwards. We shall denote this polygon by $P(w)$ (see Fig. 1). We denote vertices of the upper (lower) boundary of $P(w)$ by $p_1^+, p_2^+, \ldots, p_n^+$ ($p_1^-, p_2^-, \ldots, p_n^-$) from left to right. A polygonal line which connects a point on edge e and a point on edge e' and which is contained in $P(w)$ will be called simply a polygonal line connecting edges e and e'. Then the following is obvious.

Lemma 1. A polygonal line is an approximate polygonal line of $p_1p_2\ldots p_n$ with error bound w iff it is a monotone polygonal line connecting edge $\overline{p_1^+p_1^-}$ and edge $\overline{p_n^+p_n^-}$ of the polygon $P(w)$. □

In the sequel, we consider the problem of finding a monotone polygonal line having the minimum number of points connecting edge $\overline{p_1^+p_1^-}$ and edge $\overline{p_n^+p_n^-}$ of polygon $P(w)$. First, we show that this polygonal line can be found by repeatedly solving the edge-visibility problem. A point p of a polygon P is said to be visible

*Department of Computer Science and Communication Engineering, Kyushu University, Fukuoka 812, Japan.
**Department of Mathematical Engineering and Instrumentation Physics, University of Tokyo, Tokyo 113, Japan.

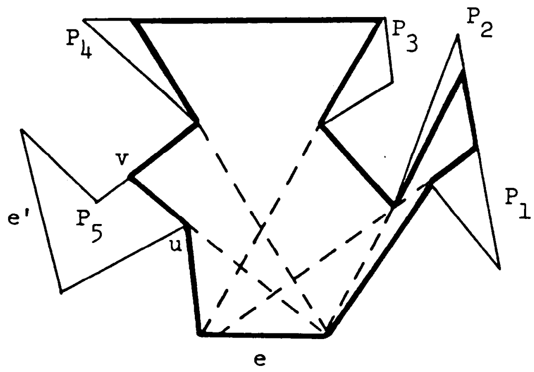

Fig. 2 Visibility from edge e.

from an edge e of P if there exists a point q on e such that the line segment $\overline{pq}$ is in P. An edge e' of P is said to be visible from edge e if there is a point on e' which is visible from e. The visibility polygon $VP(P, e)$ from an edge e of polygon P is then defined as the portion of boundary of P that is visible from e (see Fig. 2). (This visibility is the "weak visibility" in the sense of Toussaint and Avis [7].) By the visibility from e, the polygon P is partitioned into several connected regions, i.e., into $VP(P, e)$ and other polygons invisible from e, which latter will be referred to as invisible polygons. When e' is not visible from e, there is an invisible polygon P' the edges of which contain e'. The intersection of boundaries of $VP(P, e)$ and the invisible polygon P' is an edge of P', which is called the *window* (in P) from e to e' (In Fig. 2, the window from e to e' is $\overline{uv}$). Using these concepts, we can solve the problem as follows.

1. $P_1 := P(w)$; $e_1 := \overline{p_1^+ p_1^-}$; $i := 1$;
2. **while** $\overline{p_n^+ p_n^-}$ is not visible from e_i in polygon P_i **do**
   $e_{i+1} :=$ window from e_i to $\overline{p_n^+ p_n^-}$ in polygon P_i;
   $P_{i+1} :=$ invisible polygon of e_i containing $\overline{p_n^+ p_n^-}$ in polygon P_i;
   $q_i :=$ point of intersection of the line containing e_{i+1} and the edge e_i;
   $i := i+1$;
3. $m := i+1$; find a point q_{m-1} on edge e_i and a point q_m on edge $\overline{p_n^+ p_n^-}$ which are visible from each other in polygon P_{m-1}.

At the end of the algorithm, $q_1 q_2 \ldots q_m$ is an approximate polygonal line of the minimum number of points. In Fig. 3, we show how the algorithm proceeds. The validity of the algorithm can be shown as follows.

Lemma 2. Any polygonal line connecting $\overline{p_1^+ p_1^-}$ and $\overline{p_n^+ p_n^-}$ in $P(w)$ has at least m points.

Proof: It suffices to show the following claim: Any polygonal line in $P(w)$, with i points, starting from $\overline{p_1^+ p_1^-}$ cannot reach the interior of polygon P_i. The claim is true for $i=1$. Suppose the claim is true for $i-1$, and consider the claim for i. If there is a polygonal line in $P(w)$ with i points which starts from $\overline{p_1^+ p_1^-}$ and terminates in the interior of P_i, then, denoting the $(i-1)$-st point and the i-th of the polygonal line by v_{i-1} and v_i, v_i is in the interior of P_i and, from the induction hypothesis, v_{i-1} is not in the interior of P_{i-1}. Hence, line segment $\overline{v_{i-1} v_i}$ intersects e_{i-1} at some point q, and line segment $\overline{qv_i}$ is in P_{i-1}. But, this contradicts the definition of P_i that P_i is an invisible polygon from e_{i-1} in polygon P_{i-1}. □

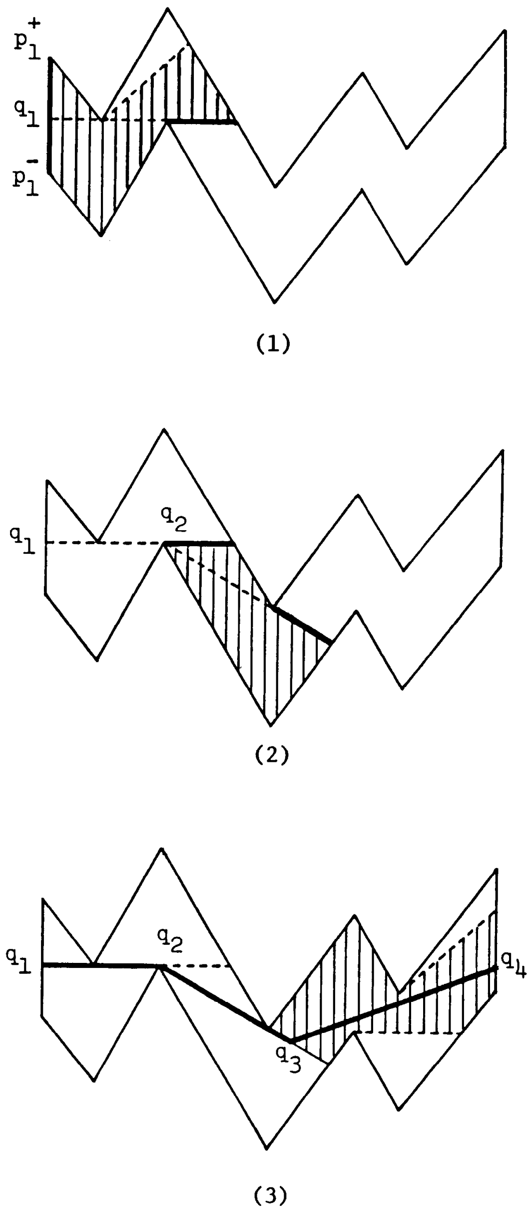

Fig. 3 How the algorithm proceeds.

Lemma 3. The polygonal line $q_1 q_2 \ldots q_m$ is in polygon $P(w)$.

Proof: From a property of the edge visibility, every point on e_{i+1} is visible from q_i, and hence $\overline{q_i q_{i+1}}$ is in

$P(w)$ for $i=1, \ldots, m-2$. □

From the monotonicity of $P(w)$, we have

Lemma 4. For each $i=1, \ldots, m-2$, $x(v_i)<x(v_{i+1})$ for any point v_i on e_i and any point v_{i+1} on e_{i+1}. □

From Lemmas 3 and 4, we see $\overline{q_1q_2 \ldots q_m}$ is a monotone polygonal line in $P(w)$ connecting $\overline{p_1^+p_1^-}$ and $\overline{p_n^+p_n^-}$. Combining this with Lemmas 1 and 2, we have.

Theorem 1. The polygonal line $q_1q_2 \ldots q_m$ obtained by the above algorithm is an approximate polygonal line of $p_1p_2 \ldots p_n$ with error bound w having the minimum number of points. □

From the algorithmic point of view, it is crucial to find windows efficiently. Of course, we can find a window by constructing an edge-visibility polygon completely. Since the visibility polygon from an edge in a polygon with n edges can be found in $O(n \log n)$ time, shown by ElGindy [2] and Lee and Lin [4] (as is reported in Lee and Preparata's survey [5]) and also by Chazelle and Guibas [1], the above algorithm can be executed in $O(mn \log n)$ time. However, we can do it much better by taking advantage of the monotonicity of polygon $P(w)$. In fact, we can solve the problem in $O(n)$ time, as will be shown in the following section.

3. An $O(n)$-Time Algorithm

We first consider some geometric properties of windows. In polygon $P(w)$, let p^+ be a point on $\overline{p_{i-1}^+p_i^+}$ (distinct from p_i^+), and p^- be a point on $\overline{p_{j-1}^-p_j^-}$ (distinct from p_j^-) such that p^+ and p^- are visible from each other. Then, we can consider a simple monotone polygon P whose upper boundary is $p^+p_i^+p_{i+1}^+ \ldots p_n^+$ and lower boundary is $\underline{p^-p_j^-p_{j+1}^- \ldots p_n^-}$. We shall consider the window from edge $\overline{p^+p^-}$ to $\overline{p_n^+p_n^-}$ in this polygon P. For a polygonal line of $p_1'p_2' \ldots p_k'$ of a piecewise linear function $y=f'(x)$, denote by $\mathrm{CH}^+(p_1' \ldots p_k')$ the convex hull of a region $\{(x, y) \mid y \geq f'(x), x(p_1') \leq x \leq x(p_k')\}$, and denote by $\mathrm{CH}^-(p_1' \ldots p_k')$ the convex hull of a region $\{(x, y) \mid y \leq f'(x), x(p_1') \leq x \leq x(p_k')\}$. Then, we have the following.

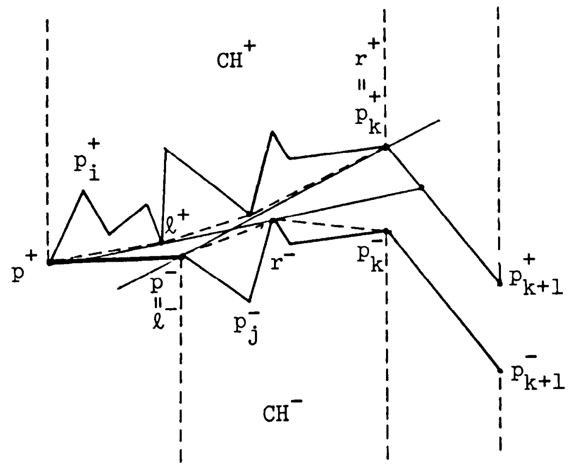

Fig. 4 CH^+, CH^-, two separating lines and a window.

Lemma 5. For $k \geq \max\{i, j\}$, the following three are equivalent.

(i) $\overline{p_k^+p_k^-}$ is visible from $\overline{p^+p^-}$.

(ii) $\mathrm{CH}^+(p^+p_i^+p_{i+1}^+ \ldots p_k^+)$ and $\mathrm{CH}^-(p^-p_j^-p_{j+1}^- \ldots p_k^-)$ have no common interior point (but they may touch each other).

(iii) There are two separating lines (which coincide with each other in case the two convex hulls touch each other at more than one point) of $\mathrm{CH}^+(p^+p_i^+ \ldots p_k^+)$ and $\mathrm{CH}^-(p^-p_j^- \ldots p_k^-)$ each of which supports the two convex hulls at points r^+ and l^- or at r^- and l^+, respectively, as in Fig. 4. □

For three points u, v and w, define angles $\angle^+uvw$ and $\angle^-uvw$ as in Fig. 5.

Lemma 6. For $k \geq \max\{i, j\}$, let us suppose that $\mathrm{CH}^+(p^+p_i^+p_{i+1}^+ \ldots p_k^+)$ and $\mathrm{CH}^-(p^-p_j^-p_{j+1}^- \ldots p_k^-)$ have no common interior point, and let r^+, r^-, l^+, l^- be the four supporting points as defined in Lemma 5. Then, we have

(i) if $\angle^+p_{k+1}^+l^+r^- \geq \pi$ and $\angle^-p_{k+1}^-l^-r^+ \geq \pi$, $\mathrm{CH}^+(p^+p_i^+ \ldots p_{k+1}^+)$ and $\mathrm{CH}^-(p^-p_i^- \ldots p_{k+1}^-)$ have no common interior point;

(ii) if $\angle^+p_{k+1}^+l^+r^- < \pi$, $\mathrm{CH}^+(p^+p_i^+ \ldots p_{k+1}^+)$ and $\mathrm{CH}^-(p^-p_j^- \ldots p_{k+1}^-)$ have a common interior point, and the window is $\overline{r^-v}$ where v is the point of intersection of line l^+r^- and $\overline{p_k^+p_{k+1}^+}$; and,

(iii) the proposition obtained by interchanging superscripts + and − in (ii).

Proof: (i) In this case, there is a point p on $\overline{p_{k+1}^+p_{k+1}^-}$ such that $\angle^+pl^-r^+<\pi$ and $\angle^-pl^+r^-<\pi$. Then, a line connecting p and the point of intersection of $\overline{r^+l^-}$ and $\overline{r^-l^+}$ is a separating line.

(ii) In this case, $\overline{p_{k+1}^+l^+}$ and $\mathrm{CH}^-(p^-p_j^- \ldots p_k^-)$ intersect, and hence $\mathrm{CH}^+(p^+p_i^+ \ldots p_{k+1}^+)$ and $\mathrm{CH}^-(p^-p_j^- \ldots p_{k+1}^-)$ intersect. Therefore, any point p in P with either "$x(p) \geq x(p_{k+1}^+)$" or "$x(r^-) \leq x(p) < x(p_{k+1}^+)$ and $\angle^+pl^+r^- \leq \pi$" is invisible from $\overline{p^+p^-}$. $\overline{r^-v}$ is visible from $\overline{p^+p^-}$, so that $\overline{r^-v}$ is the window from $\overline{p^+p^-}$ to $\overline{p_n^+p_n^-}$.

(iii) Similar. □

Based on these lemmas, we can develop the following algorithm: Starting with $p^+=p_1^+$, $p^-=p_1^-$ and $k=3$ (the case of $k=2$ is trivial), check whether $\mathrm{CH}^+(p^+ \ldots p_k^+)$

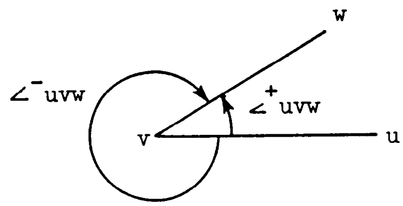

Fig. 5 Angles $\angle^+uvw$ and $\angle^-uvw$.

and $CH^-(p^- \dots p_k^-)$ intersect for $k=3, 4, 5, \dots$; if they have a common interior point, find the window from $\overline{p^+p^-}$ to $\overline{p_n^+p_n^-}$ by using Lemma 6, and updating $\overline{p^+p^-}$ to be this window, repeat this procedure until $\overline{p_n^+p_n^-}$ becomes visible from $\overline{p^+p^-}$. Below, we describe this algorithm more precisely, where $s^+(p)$, $s^-(p)$ are the right-adjacent points of p in CH^+ and CH^-, respectively, and $t^+(p)$ and $t^-(p)$ are the left-adjacent points of p in CH^+ and CH^-, respectively $(x(t^{\pm}(p))<x(p)<x(s^{\pm}(p)))$.

1. (Initialization)
   $p^+:=p_1^+$; $l^+:=p_1^+$; $r^+:=p_2^+$; $s^+(p_1^+):=p_2^+$; $t^+(p_2^+):=p_1^+$; $p^-:=p_1^-$; $l^-:=p_1^-$; $r^-:=p_2^-$; $s^-(p_1^-):=p_2^-$; $t^-(p_2^-):=p_1^-$; $i:=3$; $j:=1$.
2. (Updating CH^+ and CH^-)

(2.1) $p:=p_{i-1}^+$;
  while $p\neq p^+$ **and** $\angle^+ p_i^+ p t^+(p)>\pi$ **do**
    $p:=t^+(p)$;
  $s^+(p):=p_i^+$; $t^+(p_i^+):=p$;

(2.2) Interchanging superscripts + and − with each other, execute (2.1).

3. (Checking whether CH^+ and CH^- intersect or not)

(3.1) **if** $\angle^+ p_i^+ l^+ r^- < \pi$ **then**
  | q_j: = point of intersection of line l^+r^- and $\overline{p^+p^-}$;
  | $j:=j+1$;
  | $p^-:=r^-$; $p^+:=$ point of intersection of line l^+r^- and $\overline{p_{i-1}^+p_i^+}$;
  | $s^+(p^+):=p_i^+$; $t^+(p_i^+):=p^+$;
  | $r^+:=p_i^+$; $r^-:=p_i^-$; $l^+:=p^+$; $l^-:=p^-$;
  | **while** $\angle^- l^- r^+ s^-(l^-)<\pi$ **do** $l^-:=s^-(l^-)$;
  | go to 5.

(3.2) Interchanging + and − superscripts with each other, execute (3.1);

4. (Updating the two separating and supporting lines)

(4.1) **if** $\angle^+ p_i^+ l^- r^+ < \pi$ **then**
  | $r^+:=p_i^+$;
  | **while** $\angle^+ p_i^+ l^- s^-(l^-)<\pi$ **do** $l^-:=s^-(l^-)$.

(4.2) Interchanging + and − superscripts with each other, execute (4.1);

5. **if** $i=n$ **then** | $m:=j+1$;
   | find q_{m-1} and q_m and halt
   else $[i:=i+1;$ return to 2$]$.

Since the polygons constructed in the course of the algorithm are monotone, the simple algorithm in step 2 correctly updates the convex hulls (Toussaint and Avis [7]). When, in (3.1), the condition of "**if**" holds with $r^-=p_k^-$, point p_k^- is an extreme point of CH^-, and $CH^-(p_k^-p_{k+1}^- \dots p_i^-)$ has already been formed. The above algorithm obviously runs in $O(n)$ time (note that step 3 can be executed in $O(n)$ time in total due to the property of Lemma 4). Thus, we obtain the following.

Theorem 2. An approximate polygonal line of $p_1p_2 \dots p_n$ with error bound w with the minimum number of points can be found in $O(n)$ time. □

4. Concluding Remarks

The above algorithm is so simple that it runs fast in practice. It produces an approximate polygonal line for which the maximum absolute error attains the upper bound w (if $m>2$). Hence, the next problem to challenge would be to find an approximate polygonal line with at most m points such that the maximum error is minimum. It would also be interesting to apply the approach taken in this paper to more general polygonal approximation problems such as that raised in [3]. It would be useful to generalize the approach to the case of a function of many variables, but seems quite difficult since this approach relies substantially upon efficient algorithms in computational geometry, which are valid only in the one-dimensional case, such as two-dimensional convex hull algorithms.

The algorithm presented in this paper can readily be extended to an algorithm for finding the visibility polygon from an edge in a monotone polygon with n edges in $O(n)$ time. Applying this extended algorithm, we can find a polygonal path, which connects two points or two edges in a general polygon and which has the minimum number of points, in $O(n \log n)$ time by first slicing the polygon with vertical lines passing the vertices, next removing the subpolygons redundant with respect to the shortest paths and decomposing the polygon into monotone subpolygons, and then applying the extended algorithm to the respective monotone polygons in $O(n)$ time in total.

Acknowledgment

The research by the first author was supported in part by the Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture of Japan.

References

1. CHAZELLE, B. and GUIBAS, L. J. Visibility and Intersection Problems in Plane Geometry, *Proc. ACM Symp. on Computational Geometry*, Baltimore, Maryland, 1985, 135-146.
2. ELGINDY, H. An Efficient Algorithm for Computing the Weak Visibility Polygon from an Edge in Simple Polygons, Unpublished Note, McGill University (1984).
3. IMAI, H. and IRI, M. Computational-Geometric Methods for Polygonal Approximations of a Curve, *Research Memorandum RMI 85-01*, Department of Mathematical Engineering and Instrumentation Physics, University of Tokyo, 1985; *Comput. Gr. Image Process.*, **36** (1986), 31-41.
4. LEE, D. T. and LIN, A. Computing the Visibility Polygon from an Edge, *Comput. Gr. Image Process*, **34** (1986), 1-19.
5. LEE, D. T. and Preparata, F. P. Computational Geometry—A Survey, *IEEE Trans. Comput.*, C-33, 12 (1984), 1072-1101.
6. TOMEK, I. Two Algorithms for Piecewise-Linear Continuous Approximation of Functions of One Variable, *IEEE Trans. Comput.*, C-**23** (1974), 445-448.
7. TOUSSAINT, G. T. and AVIS, D. On a Convex Hull Algorithm for Polygons and Its Applications to Triangulation Problems, *Pattern Recogn.*, **15**, 1 (1982), 23-29.

(Received March 20, 1986; revised September 8, 1986)