Short Note

The Divided Difference Table From A Matrix Viewpoint

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Given a complex function $w=f(z)$ defined in some region containing distinct points $z_1, \cdots, z_n$, we consider the divided difference table in the form of the divided difference matrix $f^+(z_1, \cdots, z_n)$ whose (i, j) component equals $f(z_i, \cdots, z_j)$ for $i \leqq j$ and 0 elsewhere. Two theorems are proved: the first asserts that $f \to f^+$ is an algebraic homomorphism; the second gives a Cauchy contour-integral representation of $f^+(z_1, \cdots, z_n)$, which also equals the Cauchy formula for $f(z^+)$, where z^+ denote the f^+-matrix corresponding to $f(z)=z$ and where $f(z)$ is assumed analytic in a region containing $z_1, \cdots, z_n$.

1. Introduction

Divided differences are a classical but important tool in numerical analysis. Let $w=f(z)$ be a function of a complex variable, analytic in some region D containing a set of distinct points $z_1, \cdots, z_n$. Given the functional values $f(z_1), \cdots, f(z_n)$, the standard method for computing the divided differences $f(z_i, z_{i+1}, \cdots, z_j)$, where $i<j$, is through the use of recursive relation

$$f(z_i, z_{i+1}, \cdots, z_j)=\frac{1}{z_i-z_j}\{f(z_i, \cdots, z_{j-1}) - f(z_{i+1}, \cdots, z_j)\}$$

Consider the divided difference matrix, denoted by $f^+(z_1, \cdots, z_n)$, whose $(i, j)$the entry equals $f(z_i, \cdots, z_j)$, where $i \leqq j$:

$$f^+(z_1, \cdots, z_n)=\begin{bmatrix} f(z_1) & f(z_1, z_2)\cdots & f(z_1, \cdots, z_n) \\ & f(z_2) & \cdots f(z_2, \cdots, z_n) \\ & & \vdots \\ 0 & & f(z_n) \end{bmatrix}.$$

This is a slight rearrangement of the conventional divided difference table such as the one below:

$$\begin{array}{llll} f(z_1) & & & \\ & f(z_1, z_2) & & \\ f(z_2) & & f(z_1, z_2, z_3)\cdots & \\ \cdot & f(z_2, z_3) & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ f(z_n) & \cdot & \cdot & \end{array}$$

In this paper, we will show that the divided difference matrix reveals a deeper mathematical structure existing among the divided differences. To be more specific, we will prove two theorems on $f^+(z_1, \cdots, z_n)$; THEOREM 1 gives a matrix representation of basic computational rules for divided differences and shows that the map $f \to f^+$ is an algebraic homomorphism; THEOREM 2 gives a Cauchy integral representation of the divided difference matrix $f^+(z_1, \cdots z_n)$.

Theorem 1. (Homomorphism Theorem)
In the below, f and g denote functions of a complex variable.

(a) $(\alpha f+\beta g)^+=\alpha f^+ +\beta g^+$, where α and β are constants;
(b) $(fg)^+=f^+g^+=g^+f^+$
(c) When $f(z)=1$, $f^+=1^+=I$ (identity matrix);
(d) When $f(z)=z$, $f^+=z^+=\begin{bmatrix} z_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & z_n \end{bmatrix}$;
(e) $(1/g)^+=(g^+)^{-1}$, where $g(z_1)\neq 0, \cdots, g(z_n)\neq 0$;
(f) $(f/g)^+=f^+(g^+)^{-1}=(g^+)^{-1}f^+$, where $g(z_1)\neq 0, \cdots, g(z_n)\neq 0$.

The proof of (a) through (f) is easy, where (b) is a matrix representation of Leibniz' formula [1, p. 5] (or, viewed differently, (b) is a good way to memorize Leibniz' formula).

Corollary 1. If $f(z)$ is a rational function of z, where the denominator does not vanish at each point z_i $(i=1, \cdots, n)$, or, a power series in z whose radius of

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convergence exceeds each of $|z_1|, \cdots, |z_n|$, then $f^+(z_1, \cdots, z_n) = f(z^+)$.

The corollary generalizes to:

Theorem 2. (Integral Representation Theorem)

Let $w=f(z)$ be analytic in a region D which contains the given set of distinct points $z_1, \cdots, z_n$ in its interior. Let C be a contour in D, enclosing $z_1, \cdots, z_n$ in its interior. Then

$$f^+(z_1, \cdots, z_n) = \frac{1}{2\pi i}\int_C f(t)(tI - z^+)^{-1}dt = f(z^+),$$

where z^+ is defined in Theorem 1 (d). This gives an integral representation of the divided difference matrix $f^+(z_1, \cdots, z_n)$.

Proof. By direct computation, the (i, j)th entry of

$$(tI - z^+)^{-1} = \begin{bmatrix} t-z_1 & -1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ 0 & & & t-z_n \end{bmatrix}^{-1}$$

equals $(t-z_i)^{-1}(t-z_{i+1})^{-1} \cdots (t-z_j)^{-1}$, where $i \leqq j$, and 0 elsewhere. Hence the proof reduces to that of

$$f(z_1, \cdots, z_n) = \frac{1}{2\pi i}\int_c \frac{f(t)}{(t-z_1)\cdots(t-z_n)}dt.$$

This is a known identity [3, p. 247, or 4, p. 171] whose proof is included here for self-containedness. Indeed, proof is immediate from Cauchy's formula

$$f(z_k) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z_k}dt, \quad k=1, \cdots, n,$$

the partial fraction decomposition

$$\frac{1}{(t-z_1)\cdots(t-z_n)} = \frac{1}{z_1-z_n}\left\{\frac{1}{(t-z_1)\cdots(t-z_{n-1})} - \frac{1}{(t-z_2)\cdots(t-z_n)}\right\},$$

where $n \geqq 2$, and the well-known recursive relation

$$f(z_1, \cdots, z_n) = \frac{1}{z_1-z_n}\left\{f(z_1, \cdots, z_{n-1}) - f(z_2, \cdots, z_n)\right\}.$$

The second equality in the theorem is a direct consequence of the Cauchy formula for a function of a matrix. See, for example, [2, Chapter VII] or [5], where, in the latter, a direct, polynomial-interpolation-free approach is given.

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