Short Note

Modular Term Rewriting Systems with Shared Constructors

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The idea of modular term rewriting systems (TRSs) is extended by allowing modules to share constructors, and it is proved that there is no infinite sequence of modular reduction; thus a modular TRS with shared constructors is terminating iff every module is terminating.

1. Introduction

It is known that termination of term rewriting systems (TRSs) is not "modular." In other words, termination of two systems R_0 and R_1 does not necessarily imply termination of $R_0 \cup R_1$, even when R_0 and R_1 share no function symbols. Modularity of termination is quite an important property for proving termination of complex systems, because if termination were modular, it would be possible to prove termination of the entire system by independently proving termination of the constituent systems. Unfortunately, however, termination is not modular [6]. To overcome the difficulty, Kurihara and Kaji [1] recently presented a novel approach to modularity, called a *modular TRS*, which is a family $\{R_1, \ldots, R_n\}$, rather than a union, of TRSs, where R_i, called a *module*, is a TRS on the set of terms over $\mathscr{F}_i$, a set of function symbols, and $\mathscr{V}$, a set of variables. (In the following, we set $R = R_1 \cup \ldots \cup R_n$ and $\mathscr{F} = \mathscr{F}_1 \cup \ldots \cup \mathscr{F}_n$.)

Computation in modular TRSs is defined by the following reduction relation called *modular reduction*:

$$s \Longrightarrow t \text{ iff } s \Longrightarrow_{R_i} t \quad \text{for some} \quad R_i \ (1 \leq i \leq n)$$

$$s \Longrightarrow_{R_i} t \text{ iff } s \rightarrow^{+}_{R_i} t \quad \text{and } t \in NF(R_i)$$

where $\rightarrow^{+}_{R_i}$ is the transitive closure of $\rightarrow_{R_i}$, the single-step reduction relation defined by R_i, and $NF(R_i)$ is the set of R_i-normal forms. In other words, $s \Longrightarrow_{R_i} t$ iff t can be obtained by repeatedly rewriting s by R_i until the term is in normal form with respect to R_i. Kurihara and Kaji [1] proved that if $\mathscr{F}_1, \ldots, \mathscr{F}_n$ *are pairwise disjoint*, then there in no infinite sequence $t_0 \Longrightarrow t_1 \Longrightarrow \cdots$ of modular reduction; thus the modular TRS is terminating iff all the modules $R_1, \ldots, R_n$ are terminating. However, because of the disjointness condition, the result restricts its full use.

In this paper, we extend the result by allowing modules to share *constructors*, which are function symbols that, by definition, cannot occur in the leftmost positions in the left-hand sides of rewriting rules; the other function symbols are *defined symbols*:

Theorem 1 *Let $\mathscr{C}$ be a set of constructors, and assume that $\mathscr{F}_1 - \mathscr{C}, \ldots, \mathscr{F}_n - \mathscr{C}$ are pairwise disjoint. Then there is no infinite sequence of modular reduction.*

2. Proof of the Theorem

The point of the proof is how we can extend the notions of aliens, alien trees, and ranks introduced by Kurihara and Kaji [1], while preserving the 'non-increasing' property of the ranks. Once we succeed in this extension, the proof will be almost the same as theirs [1].

We start with the preliminary definitions. In the following, we assume that the relation $\rightarrow_R$ is defined only on ground terms (containing no variables), and that every term in the following definitions is ground. This assumption is made solely to clarify the discussion, and our major results remain true for the more general case. We also assume that $\mathscr{F}_1 - \mathscr{C}, \ldots, \mathscr{F}_n - \mathscr{C}$ are pairwise disjoint.

Definition 1

- The *root* of a term t, notation *root*(t), is f if t is of the form $f(t_1, \ldots, t_n)$; otherwise, it is t itself.
- If $f(\ldots, t_i, \ldots)$ is a term, f is the *parent* of the occurrence *root*(t_i).
- Let $\square$ be an extra constant, which we assume is a constructor. A term $\mathscr{C}$ over $\mathscr{F} \cup \{\square\}$ and $\mathscr{V}$ is called a *context* on $\mathscr{F}$.

For mnemotechnical reasons, we provide $n+1$ distinct colors, which we assume include transparent as the $(n+1)$th color, and paint the defined symbols $\mathscr{F}_i - \mathscr{C}$ in the ith color $(1 \leq i \leq n)$. Each occurrence of the constructors $\mathscr{C}$ is painted according to the surrounding context: if the occurrence has no parent, it is transparent; otherwise, its color is the same as that of its parent. (The definition applies recursively if the parent is a constructor.)

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Definition 2 An *alien* in a term t is a nonvariable proper subterm u of t that is maximal with respect to the 'subterm' relation, such that $root(t)$ and $root(u)$ are in distinct colors.

We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if $t_1, \ldots, t_n$ are all the aliens in t (from left to right) and C is the context obtained by replacing each alien by the $\Box$.

Definition 3 The *alien tree* $AT(t)$ of a term t with $root(t) \in \mathcal{F} - \mathcal{C}$ is the tree defined below. Note that each node is a context on some $\mathcal{F}_i$:

1. If t has no alien, then $AT(t)$ consists of a single node t, the root of the tree.
2. If $t \equiv C[\![t_1, \ldots, t_n]\!]$ $(n>0)$, then $AT(t)$ consists of the root C and the subtrees $AT(t_i)$, $1 \leq i \leq n$.

Definition 4 The *alien forest* $AF(t)$ of a term t is the singleton set $\{AT(t)\}$, if $root(t) \in \mathcal{F} - \mathcal{C}$; otherwise (if $root(t) \in \mathcal{C}$), it is the set $\{AT(t_i) \mid 1 \leq i \leq n\}$ of the alien trees of the aliens $t_1, \ldots, t_n$ in t.

Definition 5 The *rank* of a term t, notation $rank(t)$, is $1+$the height of the highest alien tree in $AF(t)$. If $AF(t) = \emptyset$, then we define $rank(t)=0$.

For example, let $\mathcal{F}_1 = \{F, \mathrm{c}, \mathrm{H}\}$, $\mathcal{F}_2 = \{g, b, \mathrm{c}, \mathrm{H}\}$, and $\mathcal{C} = \{\mathrm{c}, \mathrm{H}\}$; then the alien forest of the term $t \equiv \mathrm{H}(F(b, b), g(F(b, \mathrm{c}), \mathrm{c}))$ with rank 3 is depicted below:

$$AF(t)=\left\{\begin{array}{ccccc} F(& \Box, & \Box), & & g(\Box, \mathrm{c})\} \\ & | & | & & | \\ & b & b & & F(\Box, \mathrm{c}) \\ & & & & | \\ & & & & b \end{array}\right.$$

The rank of a term is nerver increased by rewriting:

Lemma 1 *If* $s \rightarrow_R t$ *then* $rank(s) \geq rank(t)$.

Proof. Routine, using the induction on $rank(s)$. $\Box$

Since we have established the important non-increasing property of the ranks, we can now reuse the proof in Kurihara and Kaji [1] to prove our extended result. For readers' convenience, we will briefly review the proof. The proof is by induction on n, the number of modules. The case $n=1$ is trivial.

Induction step: Assume that there is an infinite sequence $t_0 \Longrightarrow t_1 \Longrightarrow \cdots$. We can divide the sequence into infinitely many pieces of finite subsequences $t_{p_i} \Longrightarrow \cdots \Longrightarrow t_{p_{i+1}}$ $(i \geq 0,\ p_0 = 0)$ such that every piece uses all the modules, because every module must be used an infinite number of times according to the inductive hypothesis. Let $r = rank(t_0)$, and consider all the subterms of t_0 that are at the bottom-most (rth) layer of $AT(t_0)$. After the first piece of rewriting, the layer contains only normal forms, and from that point on will always be in normal form. In general, if all layers below the ith are in normal form, then the next piece brings the ith layer into normal form, while preserving that property for lower layers. Thus, after at most r pieces, the rewriting must terminate. Note that this discussion is valid only if the ranks are non-increasing, as established in the previous lemma.

3. Conclusion

We have extended the idea of modular TRSs by allowing modules to share constructors. The result is novel, and we believe that it provides new insight and direction for future research on modularity, because none of the related work done so far [1–8] has allowed sharing of function symbols. Typically, constructors represent primitive functions or constants for constructing structural data objects. Thus sharing of constructors means sharing of data, and greatly enhances the utility of modular TRSs.

Now an obvious question arises: under what suitable restrictions could we allow sharing of defined symbols while preserving our modularity theorem? This is left as an open question.

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References

1. KURIHARA, M. and KAJI, I. Modular term rewriting systems and the termination, *Inf. Process. Lett.*, **34** (1990), 1-4.
2. KURIHARA, M. and OHUCHI, A. Termination of the direct sum of simply-terminating term rewriting systems. *Proc. Info Japan '90* (1990), 11-18.
3. MIDDELDORP, A. Modular aspects of properties of term rewriting systems related to normal forms, *Proc. RTA '89, Lect. Notes in Comp. Sci.*, **355** (1989), 263-278.
4. MIDDELDORP, A. A sufficient condition for the termination of the direct sum of term rewriting systems, *Proc. 4th IEEE symp. Logic in Comp. Sci.* (1989), 396-401.
5. RUSINOWITCH, M. On termination of the direct sum of term-rewriting systems, *Inf. Process. Lett.*, **26** (1987), 65-70.
6. TOYAMA, Y. Counterexamples to termination for the direct sum of term rewriting systems, *Inf. Process. Lett.* **25** (1987), 141-143.
7. TOYAMA, Y. On the Church-Rosser property for the direct sum of term rewriting systems. *J. ACM*, **34** (1987), 128-143.
8. TOYAMA, Y., KLOP, J. W. and BARENDREGT, H. P. Termination for the direct sum of left-linear term rewriting systems (preliminary draft), *Proc. RTA '89, Lect. Notes in Comp. Sci.*, **355** (1989), 477-491.

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