Short Note

The Distribution of the Maximum Flow of a Stochastic Network

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A new method for deriving the distribution of the maximum flow from the source node to the terminal node of a stochastic network is presented. The stochastic network considered here consists of nodes and edges which can fail and have integer-valued capacity constraints. The new method is useful when one needs to estimate a stochastic network whose required flow from source to terminal is regarded as a random variable.

1. Introduction

A stochastic network considered here is the network whose edges and nodes can fail and have integer-valued capacity constraints. As the measure of performance of the network, reliability at level x, $R(x)$, which is the probability that the maximum flow from the source node to the terminal node is larger than or equal to x is defined in Ref. 1), 2). In Ref. 1), 2), algorithms for obtaining $R(x)$ are presented, where x is assumed to be a constant. When x is regarded as a random variable, it is useful to know the distribution of the maximum flow. The distribution function can be obtained by the algorithms in Ref. 1), 2) applying for all possible x. In this paper, we present a new algorithm which directly obtains the distribution of the maximum flow of a stochastic network. The basic idea and notations for the new algorithm are same to the ones in Ref. 2).

2. Assumptions and Notation

Assumptions

1) The components of the stochastic network are nodes and edges. The source node and the terminal node are failure-free and have infinite capacity. The other components can fail and have integer capacity constraints.
2) Each component that can fail is either operating or has failed.
3) Failures of the components are mutually statistically independent.
4) The network can contain directed as well as undirected edges.
5) The network has a monotone (coherent) structure.[3]

Notation and nomenclature

n : Number of the components in the network that can fail

s, t : Denote source node and terminal node

a_i : Component that can fail $(i=1, 2, \cdots, n)$

c_i : Flow capacity of a_i; If a_i is operating, a_i can transmit c_i units of flow, where c_i is an integer

p_i, $\bar{p}_i$: Reliability and unreliability of a_i

subnetwork at level x: Subnetwork which ensures that the $s-t$ flow is larger than or equal to x

$f(x)$: Probability density of maximum flow from s to t

$F(x)$: Distribution function of maximum flow from s to t

$E=\langle j_1, \cdots, j_p, \bar{k}_1, \cdots, \bar{k}_q\rangle$: search condition that $a_{j_1}, \cdots, a_{j_p}$ are operating and $a_{k_1}, \cdots, a_{k_q}$ are failed, and the others are operating or failed

Ω, ϕ : Universal set and empty set

S_E : Collection of search conditions

3. Algorithm

The basic idea of the algorithm

The basic procedure of the algorithm is an implicit enumeration method. Since the network has n unreliable components, there exist 2^n different states of the network. Let us term these states elementary states. There exists the following correspondence between the search conditions and the sets of elementary states.

$E=\langle\phi\rangle \Leftrightarrow \Omega$ (universal set which consists of all the elementary states)

$E=\langle 1\rangle \Leftrightarrow \{a_1$ is operating, $a_i (i=2, 3, \cdots, n)$ are operating or failed$\}$

$E=\langle \bar{1}\rangle \Leftrightarrow \{a_1$ is failed, $a_i (i=2, 3, \cdots, n)$ are

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operating or failed$\}$

$E=\langle 1, \bar{2}\rangle \Leftrightarrow \{a_1$ is operating, a_2 is failed, a_i $(i=3, 4, \cdots, n)$ are operating or failed$\}$

$\cdots$

$E=\langle 1, 2, \cdots, n\rangle \Leftrightarrow \{$all the elements are operating$\}$

The number of the elementary states which satisfy $E=\langle i_1, \cdots, i_p, \bar{j}_1, \cdots, \bar{j}_q\rangle$ is 2^{n-p-q}.

The new algorithm examines the maximum flows of elementary states which satisfy a given search condition. Let the maximum value of s-t flow be x under a given search condition. Then there exists at least one subnetwork at level x which satisfies the given condition. All the elementary states which satisfy both the given search condition and the condition that all the components in this subnetwork are operating attain the maximum flow x. All these elementary states are implicitly enumerated as their maximum flows being x. And sum of the probabilities of these elementary states are added to probability density at x. Once the maximum flow is calculated, the search condition is renewed as follows.

Renewals of a search condition

Let $E_0=\langle i_1, i_2, \cdots, i_p, \bar{j}_1, \bar{j}_2, \cdots, \bar{j}_q\rangle$ and $\{a_{k_1}, a_{k_2}, \cdots, a_{k_r}\}$ be a search condition and the subnetwork which attains the maximum flow under the condition E_0.

Let $\{a_{h_1}, a_{h_2}, \cdots, a_{h_u}\}$

$=\{a_i | i \in (k_1, k_2, \cdots, k_r), i \notin (i_1, i_2, \cdots, i_p)\}$

$=\{a_{k_1}, a_{k_2}, \cdots, a_{k_r}\}-\{a_{i_1}, a_{i_2}, \cdots, a_{i_p}\}$.

Then all the elementary states which satisfy both condition $E'=\langle h_1, h_2, \cdots, h_u\rangle$ and E_0 are implicitly enumerated as their maximum flows being same to that of the subnetwork. The rest of the elementary states which satisfy E_0 satisfy both E_0 and $\bar{E}'$, where $\bar{E}'$ is negation of E'. The condition $E_0 \,\&\, \bar{E}'$ is partitioned into the following mutually disjoint conditions.

$E_0 \,\&\, \langle \bar{h}_1\rangle = \langle i_1, i_2, \cdots, i_p, \bar{j}_1, \bar{j}_2, \cdots, \bar{j}_q, \bar{h}_1\rangle$

$E_0 \,\&\, \langle h_1, \bar{h}_2\rangle$

$=\langle i_1, i_2, \cdots, i_p, \bar{j}_1, \bar{j}_2, \cdots, \bar{j}_q, h_1, \bar{h}_2\rangle$

$\cdots$

$E_0 \,\&\, \langle h_1, h_2, \cdots, h_{u-1}, \bar{h}_u\rangle$

$=\langle i_1, i_2, \cdots, i_p, \bar{j}_1, \bar{j}_2, \cdots, \bar{j}_q, h_1, h_2, \cdots, h_{u-1}, \bar{h}_u\rangle$.

Algorithm

S0: $S_E=\{\langle\phi\rangle\}$. Set $f(x)=0$ $(x=0, 1, \cdots, m)$, where m is a sufficiently large integer.

S1: If S_E is empty, go to S3. Otherwise, take a search condition E_0 from S_E.

S2: Let $E_0=\langle j_1, \cdots, j_p, \bar{k}_1, \cdots, \bar{k}_q\rangle$. Obtain the maximum flow x under the condition E_0 using the labeling method.[4] If $x=0$, $f(0)=f(0)+p_{j_1}\cdots p_{j_p}\cdot \bar{p}_{k_1}\cdots \bar{p}_{k_q}$ and go to S1. Otherwise, Let $L_0=\{a_{m_1}, \cdots, a_{m_r}\}$ be the subnetwork at level x. Set $E=E_0 \,\&\, \langle m_1, m_2, \cdots, m_r\rangle = \langle f_1, \cdots, f_c, \bar{k}_1, \cdots, \bar{k}_q\rangle$.

$f(x)=f(x)+p_{f_1}\cdots p_{f_c}\cdot \bar{p}_{k_1}\cdots \bar{p}_{k_q}$. Let $E-E_0=\langle h_1, \cdots, h_d\rangle$. Add the following search conditions into S_E.

$E_1=E_0 \,\&\, \langle \bar{h}_1\rangle = \langle j_1, \cdots, j_p, \bar{k}_1, \cdots, \bar{k}_q, \bar{h}_1\rangle$,

$E_2=E_0 \,\&\, \langle h_1, \bar{h}_2\rangle$

$=\langle j_1, \cdots, j_p, \bar{k}_1, \cdots, \bar{k}_q, h_1, \bar{h}_2\rangle$,

$\cdots$

$E_d=E_0 \,\&\, \langle h_1, \cdots, h_{d-1}, \bar{h}_d\rangle$

$=\langle j_1, \cdots, j_p, \bar{k}_1, \cdots, \bar{k}_q, h_1, \cdots, h_{d-1}, \bar{h}_d\rangle$

Go to S1.

S3: $F(x)=\sum_{y \le x} f(y)$. Stop.

4. Numerical Examples

Example 1: Network of type A

Figure 1 shows one of the examples presented in Ref. 1), 2). This network consists of 17 components which can fail and have capacity constraints. The capacity constraints of the components are given in Fig. 1. Using this network, we generate 10 sample networks of type A by replacing the capacity constraints of all the edges, $c_6, c_7, \cdots, c_{17}$, with integer parts of random numbers distributed uniformly in (1, 16).

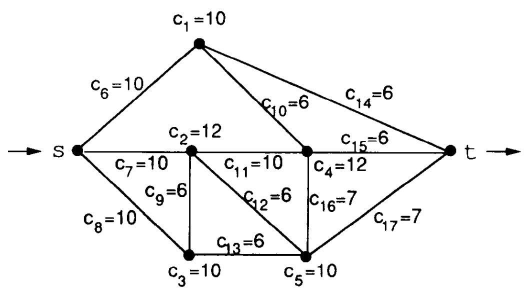

Fig. 1 A network of type A.

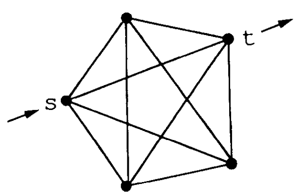

Fig. 2 A network of type B.

Example 2: Network of type B

Figure 2 shows the network of type B. The capacity constraints of all the nodes in this network are 30. Using this network, we generate 10 sample networks of type B by assigning integer parts of random numbers distributed uniformly in (1, 21) for the capacity constraints of all the edges.

R-Y method

The distribution function of the maximum flow can be obtained applying the method in Ref. 2). Let $R(x)$ be the probability that the maximum flow from s to t is larger than or equal to x. Then the procedure is as follows:

First, set $F(0)=0$ and $x=1$. Then obtain $R(x)$ using the method in Ref. 2).

Repeat "$x=x+1$ and obtain $R(x)$, then set $F(x-1)=1-R(x)$" until $R(x)=0$.

Let us refer to this procedure as R-Y method.

Computational results

Let us compare the new method and R-Y method using two quantities, maximum feasible max-flow and number of max-flow calculations, where the maximum feasible max-flow is the maximum flow when all the components are operating and the number of max-flow calculations is the number of the calls of the labeling method.

Figures 3 and **4** show the relations of these two quantities in Example 1 and Example 2, respectively. The number of the max-flow calculations by R-Y method has tendency to increase linearly as the maximum feasible max-flow increases. On the other hand, the number of max-flow calculations by the new method is nearly constant. Therefore, we may say that the new method is more effective than R-Y method unless the maximum feasible max-flow is small. The relations of the maximum feasible max-flow and the computation times are nearly same to the results shown in Figures 3 and 4. Finally, note that the reliabilities of the components are irrelevant to the computation times.

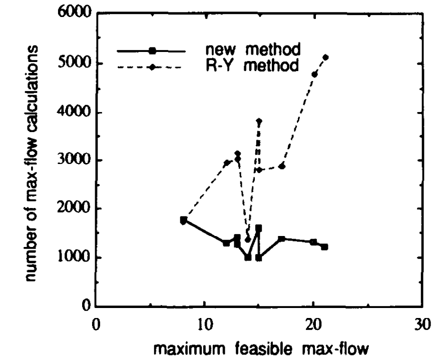

Fig. 3 The maximum feasible max-flow vs. the number of max-flow calculations in Example 1.

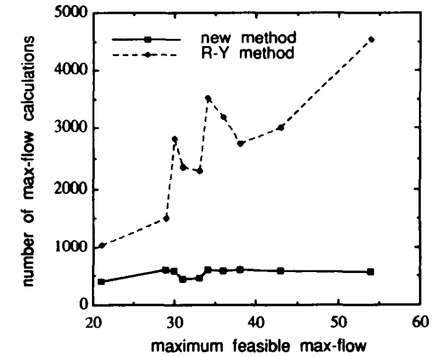

Fig. 4 The maximum feasible max-flow vs. the number of max-flow calculations in Example 2.

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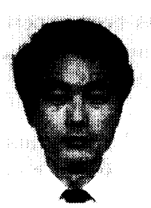

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