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構造曲線による3次元自由曲面の記述

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3次元自由曲面の構造線は曲面の幾何学的な特徴を効果的に，かつ自然に抽出したものであり，曲面のトポロジカルな構造を記述する曲線群である．構造線は，その表面の本来の持つ特徴量を用いて，微分幾何学を基に定義されたものである．それらの定義は，視線の方向に依存しない．本報告では,3種類の構造点，ピーク,ピット，鞍部構造点と,5種類の構造線，凸領域輪郭線，凹領域輪郭線,鞍部分割線,凸稜線,凹稜線を定義し，各構造線の持つ特性を，”曲線を基にした自由曲面の分割と記述”の観点から分析する．

Surface Structure Curves: Toward a Smooth Surface Sketch

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We have been developing a framework for the visual representation of three-dimensional *free-form curved* surfaces based a special class of surface curves which we call the *surface structure curves.* By analyzing their properties, we attempt to construct the basis for describing the topological structures of curved surfaces that give a global description of the surface geometry.

Surface structure curves are a set of surface curves defined by using *viewpoint-invariant* features such as surface curvatures, and their gradients and asymptotes from differential geometry. From these surface structure curves, *surface sketches* of the surfaces by means of the topological structure of *ridges lines, valley lines,* and enclosing boundaries of *bumps,* and *dents* can be inferred.

In this paper, we define three types of *surface structure points* and five types of *surface structure curves* in terms of *zero-crossings, asymptotes* and *gradients* of the Gaussian and mean curvatures. We discuss their properties and usefullness in *edge-based* segmentation and description of a free-form curved surface. Some examples of a surface sketch by the surface structure curves are shown.

1 Introduction

Recent the development in range finding techniques have made it possible to measure 3-D coordinate data from object surface *directly,* and data aquired by techniques are refered to as range data and stored in the form (called *range image* or *depth map*) where each pixel contains the distance from a sensor to the surface of objects.

Recent work in computer vision and robotics has centerd on the problem of analyzing range images, i.e., describing and recognizing of various classes of 3D objects. The *surface shapes* are more easily and unambiguously computed in range image, than in conventional intensity (gray level) images.

One of the most significant problems is what *representation* should be chosen to describe and recognize objects effective. Our work address this problem of representation of objects, especially, those taht are consisted of *smooth curved surface.* Our main concern is what geometrical descriptions that should be chosen as bases for the purpose of range image segmentation, description, recognition and reconstruction.

We can categorize various representation schemes proposed [e.g.,1,2,3,5,6,7] into two classes, *region-based* and *edge(curve)-based* representations. In general, the geometrical bases for representation and recognition are either a set of classified surface patches in the region-based representation, or a set of surface feature lines(curves) in the edge(curve)-based representation.

Gaussian curvature, an *intrinsic* property of the surface, has been used primarily to classify surface points to a set of surface patches. Such sets are, for instance, planar patches for polyhedral surfaces; a set of spherical, cylindrical, and conic patches for simply curved surfaces; and a set of elliptic, saddle patches for smooth surfaces. Several authors[Besl,Jain,Sato] worked on curved surfaces with additional surface curvature,*mean curvature,* to obtain finer classification, up to eight types of surfaces patches for smooth surfaces. In the process of surface patches extraction, most methods adopt the *iteration scheme* such as *region-growing* techniqus[1] or using connectionist approach for the existence of certain types of surface patchs[2]. The description in terms of such surface patchs are sometime unstable when *Occlusion* are present.

The use of surface curves to describe the global structure of the surface has been proposed by [3]. The curves includeare lines of curvature, asymptotes, and parabolic curves, but confined to having the simple geometrical property of being planar. However, meaningful surface curves are not necessarily being planar. Enomoto.[7] proposed *structure lines* to capture and reconstruct a smooth surface. Normal curvature and principal directions together with gradients and Hessians were used to define structure lines. However such curves were not invariant to change in viewing direction. He did not discuss the view-dependency of structure lines.

In this paper, we propose three types of points, and five types of curves, namely *surface structure points* and *surface structure curves* as the geometrical bases for segmenting and describing a smooth curved surface. We define them in terms of *intrinsic* properties such as *zero-crossings, asymptotes, osculating paraboloid,* and the consept of taking *gradients* of Gaussian and mean curvatures. This descriptions will turn out to be *viewpoint independent.*

The three types of structure points are:

1) *peak,* 2) *pit,* and 3) *saddle structure points* which are the *flattest/steepest (a point of highest curvature)* point in convex/concave elliptic and saddle regions;

The five types of structure curves are:

1),2) a *peak/pit-bounding contour* which is a *parabolic curve* that enclose the boundary of a convex/concave elliptic region;

3) a *saddle-segmenting contour* which is an *asymptotic line* that divide a saddle region into convex and concave saddle regions;

4),5) a *ridge/valley contour* which is a *line of maximum/minimum curvature* which draws the ridge/valley lines. Ridge/valley contours pass thorough *all* of the structure points.

Fig.1 shows an example of *surface sketch* of an ellipsoid drawn by surface strucure point/curves (over lines of curvature of an ellipsoid). Three of the lines of curvature are chosen as ridge contours, which are aligned with the *locally* flattest/steepest direction and intersect at peak points, *globally* the flattest/steepest points. The surface of ellipsoid is effectively described by *geometrically meaningful* points and curves. This example show how surface structure point/curve can provide a *natural parameterization* of a smooth curved surface.

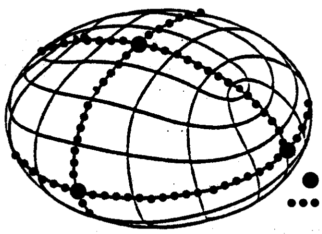

Fig.1 A sutface structure sketch drawn by surface structure points and curves, ,called *peak structure point* and *ridge contours, over the line of curvature of an ellipsoid.*

2 Surface Curvatures

2.1 surface curvature[3,7]

We recall some basic definitions from differential geometry. In the case, where a surface S is given in the form of $z = f(x, y)$, we have $\mathbf{X}(u, v) = (x(u, v), y(u, v), z(u, v)) = (u, v, f(u, v))$ as the parametrization of a surface S.

Two vectors, $\mathbf{X}_u$,$\mathbf{X}_v$, where the subscripts denote partial differentiation, form a basis $\{\mathbf{X}_u, \mathbf{X}_v\}$ for the tangent plane $\mathbf{T}$(u,v) at the point $\mathbf{X}(u, v)$.

The intersection of S with a plane Γ containing a tangent vector $\mathbf{t}(u, v) \in \mathbf{T}(u, v)$ and a surface normal $\mathbf{n}$ is called as a *normal section* of S at a point $\mathbf{X}(u, v)$ along $\mathbf{t}(u, v)$. The curvature of a normal section is called the *normal curvature*.The normal curvature is known as the ratio of the first and second fundamental forms, II/I and varies as a function of the direction (du, dv) of tangent vector $\mathbf{t}(u, v) = du\mathbf{X}_u(u, v) + dv\mathbf{X}_v(u, v)$. We refer the following as the normal curvature function $\lambda(du, dv)$.

$$\lambda(du, dv) = \frac{II}{I} = \frac{Ldudu + 2Mdudv + Ndvdv}{Edudu + 2Fdudv + Gdvdv} \quad (1)$$

where $E = \mathbf{X}_u \cdot \mathbf{X}_u, F = \mathbf{X}_u \cdot \mathbf{X}_v = \mathbf{X}_v \cdot \mathbf{X}_u, G = \mathbf{X}_v \cdot \mathbf{X}_v$ are the coefficients of I, and $L = \mathbf{X}_{uu} \cdot \mathbf{n}, M = \mathbf{X}_{uv} \cdot \mathbf{n} = \mathbf{X}_{vu} \cdot \mathbf{n}, N = \mathbf{X}_{vv} \cdot \mathbf{n}$ are the coefficient of II.

Two *extrema* of the normal curvature function, namely *the maximum/minimum normal curvatures,* are called the *principal curvatures* κ_1, κ_2, $(\kappa_1 \geq \kappa_2)$, and the corresponding directions, $(\xi_1, \eta_1), (\xi_2, \eta_2)$ given by the two unit tangent vectors, $\mathbf{w}_1, \mathbf{w}_2$, where $\mathbf{w}_i(u, v) = \xi_i\mathbf{X}_u(u, v) + \eta_i\mathbf{X}_v(u, v)$ and $i = 1, 2$, are called *principal directions* at $\mathbf{X}$(u, v). They are known to be orthogonal.

Let $\kappa_1 = \lambda(\xi_1, \eta_1), \kappa_2 = \lambda(\xi_2, \eta_2)$ be extrema of the normal curvature $\lambda(du, dv)$. Then we have,

$$\kappa_1 = \lambda(\xi_1, \eta_1) = \frac{L\xi_1^2 + 2M\xi_1\eta_1 + N\eta_1^2}{E\xi_1^2 + 2F\xi_1\eta_1 + G\eta_1^2} \quad (2)$$

$$\kappa_2 = \lambda(\xi_2, \eta_2) = \frac{L\xi_2^2 + 2M\xi_2\eta_2 + N\eta_2^2}{E\xi_2^2 + 2F\xi_2\eta_2 + G\eta_2^2} \quad (3)$$

The *Gaussian curvature* K and *mean curvature* $H$are defined in terms of these principal curvatures:

$$K = \kappa_1\kappa_2 = \frac{LN - M^2}{EG - F^2} \quad (4)$$

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{EN + GL - 2FM}{2(EG - F^2)} \quad (5)$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K}, \quad where\ \kappa_1 \geq \kappa_2 \quad (6)$$

2.2 Gradient of Surface Curvature

We introduce the concept of a *gradient* over *the space of surface curvatures* which is used to deal of surface structure curves in later sections.

A *gradient* on a surface gives the direction to which *height* is increased *most.*

The gradient on a surface S given by $z = f(x, y)$, where $S : \mathbf{X}(u, v) = (u, v, f(u, v))$, is given in the following[3,pp.102]:

$$grad\ f = \frac{f_uG - f_vF}{EG - F^2}\mathbf{X}_u + \frac{f_vE - f_uF}{EG - F^2}\mathbf{X}_v \quad (7)$$

where $\{\mathbf{X}_u, \mathbf{X}_v\}$ is a basis for a tangent plane $T(u, v)$.

Using same analogy, in *surface of Gaussinan cuavature,* $K(x, y) = \kappa_1(x, y), \kappa_2(x, y)$, a *gradient* on a surface S_k given by $z = K(x, y) = \kappa_1(x, y,)\kappa_2(x, y)$, where $S_k : \mathbf{X}(u, v) = (u, v, K(u, v))$, can be similarly obtained.

The gradient of K gives the direction of *maximum variation of* K at each point. Thus a *gradK* gives the direction *closer* to a point of extremal curvature, a *flattest/steepest point.*

Since from the differential geometry; that is, a *gradient of a surface is always orthogonal to the contour lines of a surface,* it suggest that *grad K* can be obtained at each point along contour lines of *equi-curvature,* if a set of *equi-curvature contour lines are well defined over the space of surface curvature.*

Since the continuity of surface curvature is not guaranteed on a smooth surface which is differentiable up to order 2, points of particular curvature level, say zero, may not exist on the surface.

To accomodatethe use of gradient, which is defined over a smooth function, into our *surface curvature* space, where z may *notcontinuous,* we treat a surface of surface curvature $K(x, y) = \kappa_1(x, y)\kappa_2(x, y)$ as *stepwise continuous* rather than *discontinuous.*

Suppose that the surface curvature is discontinuous at a point where a level of curvature has a jump from K_a to K_b in its neighborhood. At such point in our *stepwise continuous* surface, we prefer to assign surface curvature a *real-range-value* K where $K_a \leq K \leq K_b$, rather than leaving curvature undefined at the point. For instance, when we refer to zero-crossings of curvature, we mean points at which a curvature is either zero or a sign change in curvature has occured.

Then over a *stepwise continuous* surface, we can associate *contour lines,* each of which is *continuous surface curve* and along which surface curvature is in a *equal level.* Note that several equi-curvature contours of curvature level K', where $K_a \leq K' \leq K_b$, pass through a point if a level of curvature has a jump from K_a to K_b in the neighborhood of the point. par In this manner, we can obtain a gradient of surface curvature as a vector normal to *equi-curvature contour lines.*

3 Geometric Interpretation of Gaussian and Mean Curvatures

Both Gaussian and mean curvatures are intrinsic surface propertis which posses invariance against to rotational and translational change. They describe *surface shape* at each individual point on a surface.

3.1 Surface Classification

Any point on a surface can be classified into one of eight distinct possiblities of surface types form the *sign* of the Gaussian and mean curvatures[1]. They are:

1. $K > 0, H < 0$: Peak Surface:
2. $K = 0, H < 0$: Convex-Flat Surface:
3. $K < 0, H < 0$: Saddle-Ridge Surface:
4. $K < 0, H = 0$: Minimal Surface:
5. $K < 0, H > 0$: Saddle-Valley Surface:
6. $K = 0, H > 0$: Concave-Flat Surface:
7. $K > 0, H > 0$: Pit Surface:
8. $K = 0, H = 0$: Flat Surface:

We use this surface classification for the porpose of range image segmentation. We will take each of surface types as a *region* type, that is, a peak region consists of a set points of peak type, a saddle-ridge region consists of points of saddle-ridge type, and so on.

From this classification, we observe that the *zero-crossings(=parabolic points)* of Gaussian and mean curvatures play important roles in segmenting a surface into distinct types of regions. For instance, a curve consists of zero-crossings of Gaussian curvature(= parabolic points) separate the surface into two distinct types, elliptic region and saddle region. Thus this class of curves (of *zero-crossings* of Gaussian and mean curvaturecan) can be used effectively to segment the surface according to the above classification. We will use parabolic points and minimal surface points to define our *surface structure curves,* in Sec.4.

The adjacency among regions is preserved between a peak and a convex-flat, a convex-flat and a saddle-ridge, a saddle-ridge and a saddle-valley, and so on. The above discussion is summarized in Fig. 2.

3.2 Surface Shape

For very point p of a surface S, these is a well-defined quadratic surface which approximate S up to second order at p. This quadratic surface is called a *osculating paraboloid*[8]. We will use it to derive an *analytic* equation for each of surface structure curves in Sec. 3.

To investigate how the osculating paraboloids are described in terms of principal curvatures k_1, k_2, and principal directions,$\mathbf{w_1}, \mathbf{w_2}$, we choose a particular local coordinate system $(\mathbf{w_1}, \mathbf{w_2}, \mathbf{n})$, where the origin is at p and **x, y, z** axes are along the direction of maximam, minimum curvatures and a surface normal, respectively.

We will refer coordinates in $\mathbf{w_1}, \mathbf{w_2}, \mathbf{n}$ axes by ξ, η, and θ respectively. In *our* local coordinate system, the coefficients of the first fundamental are defined as $E = G = 1, F = 0$[4].

Let us assume that a paraboloid is described in this local coordinate system $(\mathbf{w_1}, \mathbf{w_2}, \mathbf{n})$. Then the osculating paraboloid $\theta = h(\xi, \eta)$ which approximate the neighborhood of p up to second order is given as;

$$h(\xi,\eta) = \frac{1}{2}(\kappa_1\xi^2 + \kappa_2\eta^2) = \frac{\xi^2}{\sqrt{(2/\kappa_1)}} + \frac{\eta^2}{\sqrt{(2/\kappa_2)}} \quad (8)$$

The shape of this paraboloid depends on the signs of Gaussian curvature, $K = \kappa_1\kappa_2$ and its convexity is determined from the signs of mean curvature,$\frac{1}{2}(\kappa_1 + \kappa_2)$. The osculating paraboloid has the following four types.

1. Elliptic Paraboloid ($K = \kappa_1\kappa_2 > 0$): When $K > 0$,$\kappa_1\kappa_2 > 0$ have same sign, the surfaces is an *elliptic paraboloid;*(Fig.3)

$$h(\xi,\eta) = \frac{1}{2}(\kappa_1\xi^2 + \kappa_2\eta^2) \quad (9)$$

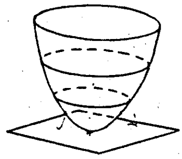

Fig.3 An elliptic paraboloid

An elliptic paraboloid intersects with planes parallel to the $(\mathbf{w_1}, \mathbf{w_2})$-plane in *ellipses*, while it intersects with the other coordinate planes in *parabolas* .

2. Hyperbolic Paraboloid ($K = \kappa_1\kappa_2 < 0$): When κ_1, κ_2 have opposite signs(i.e.$\kappa_1 > 0 > \kappa_2 > 0$, then the surfaces is a *hyperbolic paraboloid;*(Fig.4)

$$h(\xi,\eta) = \frac{\xi^2}{\sqrt{(2/\kappa_1)}} - \frac{\eta^2}{\sqrt{(2/\kappa_2}} \quad (10)$$

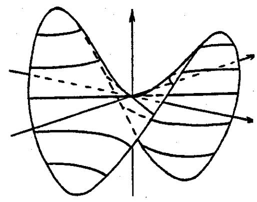

Fig.4 A hyperbolic paraboloid

An hyperbolic paraboloid intersects with planes parallel to the $(\mathbf{w_1}, \mathbf{w_2})$-plane in similar *hyperbolas,* while it intersects with the other coordinate planes in *parabolas.*

3. Parabolic Cylinder ($K = \kappa_1\kappa_2 = 0$,but not $\kappa_1 = \kappa_2 = 0$): When $K = 0$,$\kappa_1\kappa_2 = 0$, then the surfaces is a *parabolic cylinder;*(Fig.5)

If $\kappa_1 = 0$ and $\kappa_2 < 0$, we have

$$h(\xi,\eta) = \frac{1}{2}\kappa_2\eta^2 \quad (11)$$

If $\kappa_1 > 0$ and $\kappa_2 = 0$, we have

$$h(\xi,\eta) = \frac{1}{2}\kappa_1\xi^2 \quad (12)$$

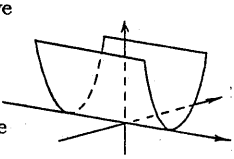

Fig.5 A parabolic Cylinder

A parabolic cylinder intersects with planes parallel to the $(\mathbf{w_1}, \mathbf{w_2})$-plane in *parallel lines,* while it intersects with the other coordinate planes in *parabolas.*

4. Plane: $\kappa_1 = \kappa_2 = 0$

When $\kappa_1\kappa_2 = 0$, then the surfaces is a *plane.*

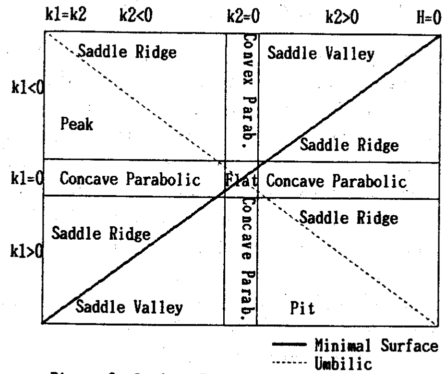

Figure 2. Surface Types determined by Signs of K(=k1k2), H(=(k1+k2)/2)

4 Surface Structure Curves

For the analysis of smooth surfaces, particular geometrical features such as the zero-crossings, and *gradient of surface curvature* play important roles both in segmenting and describing the topological structure of a smooth curved surface.

A curve consisting of zero-crossings of Gaussian curvature(= parabolic points) separates the surface into two distinct types, elliptic region and saddle region, since the parabolic points are located intermediately between a elliptic region and a saddle region. Thus this class of surface curves can be used to uniquely describe the *global* structure of the surface geometry. We call such curves *surface structure curves.*

Since surface structure curves are defined by such intrinsic features, the definitions are *viewpoint-independent.* In other words, the definitions of surface structure curves are invariant to rotational and translational change of the coordinate system.

In the following section, we will define first three types of structure points, and then five types of structure curves.

4.1 Structure Points

We are interested in the special class of surface points, where the curvatures of both *lines* of (maximum/minimum) *curvature* reach their extrema(maxima/minima), or where more than two *lines of curvature* intersect. We call this class of points *structure points,* and define as follows.

Definition 1
A structure point of a smooth surface S is a point where *the gradient of Gaussian curvature* is zero.

We have three types of structure points according to the signs of Gaussian and mean curvature:

1. Peak structure points : $K > 0, H < 0$

   At a maximum/minimum peak structure point where both the maximum and minimum lines of curvature reach a negative (maxima/minima) extremal level of curvature, a peak surface (a convex elliptic region) becomes the steepest/flattest.

2. Pit structure points : $K > 0, H > 0$

   At a maximum/minimum pit structure point where both the maximum and minimum lines of curvature reach a positive (maxima/minima) extremal level of curvature, a pit surface(a concave elliptic region) becomes the steepest/flattest.

3. Saddle structure point: $K < 0, H = 0$

At a saddle structure point where the curvature of both maximum and minimum lines of curvature become zero, a saddle surface becomes the flattest.

Note that we will not define structure points on the surface of *constant* curvature. Such surfaces are, e.g., spheres, cylinders, cones, toruses, each of which has its own *analytic* form parameterized with a small set of parameters. Thus, we think that it is not useful to find extremal curvature points inside such (analytic) surfaces.

4.2 Peak/Pit Bounding Contour

Parabolic curves have not been considered to carry rich geometrical information, because they do not manifest in any particular simple forms. Since they are transitions from an elliptic region to a saddle region, they should be useful for region segmentation. So we investigate them more closely.

Individual analysis of each of principal curvature give two distinct types of parabolic curves. One draws the transition from a convex elliptic region to a saddle region and the other from a concave elliptic region to a saddle region.

When a smooth surface is crossed from inside a peak/pit region to a saddle-ridge/valley region, a point is passed where either the Gaussin curvature is zero, or the sign of k_1/k_2 curvature changes from negative/positive to posotive/negative, while k_2/k_1 curvature stays negative/positive in its neighborhood.

Traversing such zero-crossings,(call *convex/concave parabolic* points), along a boundary of a peak/pit region, we obtain a surface curve which encloses a peak/pit region and separate it from a saddle-ridge/valley region.

For convex parabolic points, we have the principal curvatures as $\kappa_1 = 0, \kappa_2 < 0$, since $G = \kappa_1\kappa_2 = 0, H = \frac{1}{2}(\kappa_1 + \kappa_2) < 0$, and $\kappa_2 \leq \kappa_1 \leq 0$. From Eq.2,3, the normal curvature at a convex parabolic point has a maximum extrema, $\kappa_1 = \lambda(\xi_1, \eta_1) = 0$, and a minimum extrema $\kappa_2 = \lambda(\xi_2, \eta_2) < 0$ in the direction of $\mathbf{w_1} = (\xi_1, \eta_1), \mathbf{w_2} = (\xi_2, \eta_2)$ respectively.

For a concave parabolic point, we have the principal curvatures as $\kappa_1 > 0, \kappa_2 = 0$, since $G = \kappa_1\kappa_2 = 0, H = \frac{1}{2}(\kappa_1 + \kappa_2) > 0$, and $\kappa_1 \geq \kappa_2 \geq 0$. The normal curvature at a concave parabolic point has a maximum extrema, $\kappa_1 = \lambda(\xi_1, \eta_1) > 0$, and a minimum extrema $\kappa_2 = \lambda(\xi_2, \eta_2) = 0$.

Hence we define a peak/pit-bounding contour as follows:

Definition 2
A *peak-bounding* contour is a set of convex parabolic points whose principal curvatures are $k_1 = 0, k_2 < 0$, given by the following conditions:

$$L\xi_1^2 + 2M\xi_1\eta_1 + N\eta_1^2 = 0 \tag{13}$$

$$\frac{L\xi_2^2 + 2M\xi_2\eta_2 + N\eta_2^2}{E\xi_2^2 + 2F\xi_2\eta_2 + G\eta_2^2} = \kappa_2 < 0 \tag{14}$$

where κ_1, κ_2 are principal curvatures and $\mathbf{w_1} = (\xi_1, \eta_1)$, $\mathbf{w_2} = (\xi_2, \eta_2)$ are principal directions.

Definition 3
A *pit-bounding contour* is a set of *concave parabolic* points whose curvature property are $k_1 > 0, k_2 = 0$, given by the following conditions:

$$\frac{L\xi_1^2 + 2M\xi_1\eta_1 + N\eta_1^2}{E\xi_1^2 + 2F\xi_1\eta_1 + G\eta_1^2} = \kappa_1 > 0 \tag{15}$$

$$L\xi_2^2 + 2M\xi_2\eta_2 + N\eta_2^2 = 0 \tag{16}$$

where κ_1, κ_2 are principal curvatures and $\mathbf{w_1} = (\xi_1, \eta_1)$; $\mathbf{w_2} = (\xi_2, \eta_2)$ are principal directions.

To get a geometrical interpretation of above conditions, we describe them in our local coordinate system $(\mathbf{w_1}, \mathbf{w_2}, \mathbf{n})$. As we mentioned in Sec 3.2, we have known that the shape of the osculating paraboloid at a parabolic point is a *parabolic cylinder.* Further, we also know that convex/concave parabolic points are connected in the direction of *non-zero* curvature.

For a peak-bounding contour we obtain the following *analytic* equation, i.e. *(convex) parabola,* from Eq.12.

$$\theta = h(0, \eta) = \frac{1}{2}\kappa_2\eta^2 \tag{17}$$

This concave parabola passes through the origin and lie on the $(\mathbf{w_2}, \mathbf{n})$ coordinate plane. A tangent line of this parabola agrees with the minimal principal direction $\mathbf{w_2} = (\xi_2, \eta_2)$, and its curvature is $\kappa_2 < 0$.

For a pit-bounding contour we obtain the following *analytic* equation, i.e.,*(concave) parabola,* from Eq.11. .

$$\theta = h(\xi, \eta) = \frac{1}{2}\kappa_1\xi^2 \tag{18}$$

This convex parabola pass through the origin and lie on the $(\mathbf{w_1}, \mathbf{n})$ coordinate plane. A tangent line of the parabola is with the maximum principal direction $\mathbf{w_1}$, and its curvature is $\kappa_1 > 0$.

A peak/pit-bounding contour does not pass through any of peak/pit structure points.

4.3 Saddle-Segmenting Contour

In a saddle region, the total convexity of a surface is exchanged along a curve of the zero-crossings of mean curvature(=minimal surface points). The amount of convexity and concavity at a minimal surface point are equal, i.e., $\kappa_1 = -\kappa_2 > 0$. We may perceive that a saddle surface of one side of the curve is more convex and the other side is more concave. We call this surface curve a saddle-segmenting contour and adopt it to be the third surface structure curve.

For a minimal surface point, we have the principal curvatures as $\kappa_1 = -\kappa_2 > 0$, since $G = \kappa_1\kappa_2 < 0, H = \frac{1}{2}(\kappa_1 + \kappa_2) = 0$. From this curvature property we have the special geometry of minimal surface point such that[4],

Gaussian curvature is *zero* along the direction of *asymptotic lines.*

At a minimal surface point, from Eq.2,3, the normal curvature has extrema, $\kappa_1 = \lambda(\xi_1, \eta_1) = -\kappa_2 = -\lambda(\xi_2, \eta_2) > 0$, and $\lambda((\xi_1 \pm \xi_2)/\sqrt{2}, (\eta_1 \pm \eta_2)/\sqrt{2}) = 0$ in the direction of $\mathbf{w_1} = (\xi_1, \eta_1)$, $\mathbf{w_2} = (\xi_2, \eta_2)$, $(\mathbf{w_1} + \mathbf{w_2})/\sqrt{2} = ((\xi_1 \pm \xi_2)/\sqrt{2}, (\eta_1 \pm \eta_2)/\sqrt{2})$ respectively. We define the saddle-segmenting contour as follows:

Definition 4
A *saddle-segmenting* contour is a set of minimal surface points whose curvature properties are $k_1 = -k_2 > 0$ and $k_1k_2 = 0$ along the direction of the *asymptotic lines*$((\xi_1 \pm \xi_2)/\sqrt{2}, (\eta_1 \pm \eta_2)/\sqrt{2})$ given by the following conditions:

$$L(\xi_1 \pm \xi_2)^2 + 2M(\xi_1 \pm \xi_2)(\eta_1 \pm \eta_2) + N(\eta_1 \pm \eta_2)^2 = 0 \tag{19}$$

where κ_1, κ_2 are principal curvatures and $\mathbf{w_1} = (\xi_1, \eta_1)$, $\mathbf{w_2} = (\xi_2, \eta_2)$ are principal directions.

The osculating paraboloid constrained by the above conditions is a *hyperbolic paraboloid*from Eq.10. We also know that minimal surface points are connected in the direction of *asymptotic lines*$((\xi_1 + \xi_2)/\sqrt{2}, (\eta_1 + \eta_2)/\sqrt{2})$, i.e., in the direction of *zero* curvature. Thus we obtain an analytic form, *two straght lines,* for a saddle-segmenting contour.

$$\theta = h(\xi, \eta) = \xi \pm \eta = 0 \tag{20}$$

The two straight lines intersect at the origin at right angle and lie on the $(\mathbf{w_1}, \mathbf{w_2})$ coordinate plane.

Two saddle-segmenting contour intersect at saddle structure point along asymptotic lines.

4.4 Ridge/Valley contour

Ridge/valley contours are surface curves of ridge/valley points which describe the global shape of regions. They pass through *peak/pit structure points,* points of extremal curvature. Thus ridge/valley contours are characterized as bases for the description of surface shape, while the last three structure curves are useful for surface segmentation.

Ridge/valley lines are a set of points of whose principal direction are coincident with the *gradient* of Gaussian curvature K. We call such points ridge/valley points. The gradient of K gives the direction *pointing* to a *peak/pit-structure point.*

In a peak/pit region, the *ridge/valley contour* is a surface curve along which a direction of a gradient of Gaussian curvature K is equivalent to one of principal directions, $\mathbf{w}_1 = (\xi_1, \eta_1)$ or $\mathbf{w}_2 = (\xi_2, \eta_2)$.

In a saddle-ridge/valley region, a point is on a the *ridge/vallye contour* if a direction of a grad K is equivalent to maximum principal directions, $\mathbf{w}_1/\mathbf{w}_2 = (\xi_1/\xi_2, \eta_1/\eta_2)$, not to $\mathbf{w}_2/\mathbf{w}_1 = (\xi_2/\xi_1, \eta_2/\eta_1)$. Note that there is no ridge/valley contour either in the pit/peak region nor in the saddle-valley/ridge region.

Let $\mathbf{w_g} = (\xi_\mathbf{g}, \eta_\mathbf{g})$ be the direction of a grad K that is obtained as the unit vector orthogonal to the *equi-curvature contour* lines. For ridge/valley points, the normal curvature in the direction $\mathbf{w_g} = (\xi_\mathbf{g}, \eta_\mathbf{g})$ of $gradK$ is equivalent to maximum/minimum extrema, $\kappa_1 = \lambda(\xi_g, \eta_1 g)$, or $\kappa_2 = \lambda(\xi_g, \eta_g)$.

If $\mathbf{w}_g = \mathbf{w}_1$, the normal curvature in the direction $\mathbf{w_g}$) is equivalent to the maximum extrema κ_1, from Eq.2,

$$\frac{L\xi_g^2 + 2M\xi_g\eta_g + N\eta_g^2}{E\xi_g^2 + 2F\xi_g\eta_g + G\eta_g^2} = \kappa_1 \qquad (21)$$

If $\mathbf{w}_g = \mathbf{w}_2$, the normal curvature in the direction $\mathbf{w_g}$) is equivalent to the minimum extrema κ_2, from Eq.3,

$$\frac{L\xi_g^2 + 2M\xi_g\eta_g + N\eta_g^2}{E\xi_g^2 + 2F\xi_g\eta_g + G\eta_g^2} = \kappa_2 \qquad (22)$$

Hence we define a ridge/valley contour as follows:

Definition 5

A *ridge* contour is a set of ridge points of which one principal directions is coincident with the direction of*a gradient* of Gaussian curvature K and $H < 0$, given by the following conditions:

$$\frac{L\xi_g^2 + 2M\xi_g\eta_g + N\eta_g^2}{E\xi_g^2 + 2F\xi_g\eta_g + G\eta_g^2} = \kappa_i \qquad (23)$$

where $\kappa_i, i = 1, 2$ are principal curvatures. If $G > 0$ then $i = 1$ or $i = 2$, otherwise $i = 1$. $\mathbf{w_g} = (\xi_\mathbf{g}, \eta_\mathbf{g})$ is the direcion of $gradK (= \kappa_1\kappa_2)$.

Definition 6

A *valley* contour is a set of valley points of which one principal directions is coincident with the direction of*a gradient* of Gaussian curvature K and $H > 0$, given by the following conditions:

$$\frac{L\xi_g^2 + 2M\xi_g\eta_g + N\eta_g^2}{E\xi_g^2 + 2F\xi_g\eta_g + G\eta_g^2} = \kappa_i \qquad (24)$$

where $\kappa_i, i = 1, 2$ are principal curvatures. If $G > 0$ then $i = 1$ or $i = 2$, otherwise $i = 2$. $\mathbf{w_g} = (\xi_\mathbf{g}, \eta_\mathbf{g})$ is the direcion of $gradK$.

In our local coordinate system $(\mathbf{w_1}, \mathbf{w_2}, \mathbf{n})$, the shape of the osculating paraboloid constrained by the above conditions is the elliptic paraboloid(from Eq.9) in a peak/pit region and the hyperbolic paraboloid(from Eq.10) in a saddle-ridge/valley region. In either case an analytic form of ridge/valley curve become a parabola.

If $\mathbf{w_1} = \mathbf{w_g}$,

$$\theta = h(\xi, 0) = \frac{1}{2}\kappa_1\eta^1 \qquad (25)$$

The parabola lie on the $(\mathbf{w_1}, \mathbf{n})$ coordinate plane. A tangent line agrees with the maximal principal direction $\mathbf{w_1} = (\xi_1, \eta_1)$, and its curvature is κ_1.

If $\mathbf{w_2} = \mathbf{w_g}$,

$$\theta = h(0, \eta) = \frac{1}{2}\kappa_2\eta^2 \qquad (26)$$

The parabola lie on the $(\mathbf{w_2}, \mathbf{n})$ coordinate plane. A tangent line agrees with the maximal principal direction $\mathbf{w_2} = (\xi_2, \eta_2)$, and its curvature is κ_2.

At a peak/pit structure point, two ridge/valley coutour are intersect along maximum/minimum line of curvature. A ridge contour and a valley contour meet at a saddle structure point.

An example of the surface structure curves is illustrated in Fig.6 . We illustrate second example of a surface sketch o ver the surface of a double sinusoid given by $z = sin2\pi x * sin\pi y; 0 \le x \le 1, 0 \le y \le 1$.

5 conclusion

The surfce structure points and curves are defined on a three-dimensional *smooth* surface which possesses up to second order.

A set of *structure points* which are critical points on *smooth* surfaces and locate a *globally* flattest/steepest points, points of maximum/minimum curvature are then defined.

Five types of surface structure curves have been proposed. The structure curves are the collection of *parabolic lines, a line of minimal surface points,* and lines of *maximum/minimum curvatures* . The five surface structure curves are: *peak* and *pit-bounding* contours, a *saddle-segmenting contour, ridge* and *valley contours.* They are all geometrically meaningful curves and possess global properties that enclose dents and bumps, and pass thorough *globally* the flattest/steepest(i.e.,highest curvature) points.

At each point on the surface structure curves, an *analytic* equation, either a line or a parabola are defined. We have derived the formulation by analyzing their behaviors of *maximun and minimum* curvatures κ_1, κ_2 *individually.*

Surface structure points and curves can provide the *natural* parameterization of a smooth surface. Examples of a *surface sketch* shows that surface structure curves can capture the global topological structure of surfaces and are potentially very powerful and stringent discriptors for three-dimensional free-form curved surfaces.

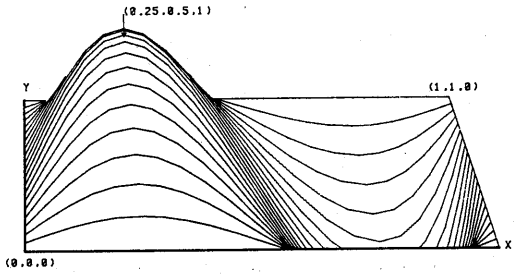

Fig.6(a)

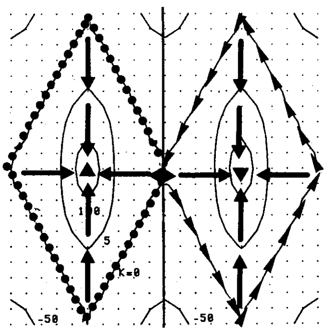

Fig.6(b)

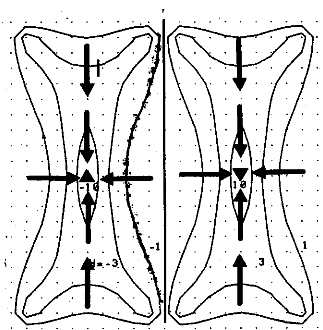

Fig.6(c)

Fig.6(a) Line drawing of the surface of a double sinusoid; Surface Sketch over four contours of Gaussian curvature K (Fig.6(b) and over six contours of mean curvature H

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▲ Peak point
▼ Pit point
◆ Saddle point
••••••• Peak-bounding contour
◄ ◄ ◄ Pit-bounding contour
——— Saddle-Segmenting contour
→▲ Ridge contour
→▼ Valley contour