高度交通システム　９－９
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都市道路ネットワークにおけるの最適料金の設定

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概要

道路料金の徴収は、都市交通ネットワークシステムの最適運営の重要な手段として、古くから交通経済学などの分野で議論されてきた。最適化の目標は、伝統的には、交通渋滞緩和、利用者全体のコストの最小化、経済的な効用の最大化などが典型的である。近年は特に環境問題の視点から、二酸化炭素排出の削減、エネルギーの有効利用なども最適化の目標になっている。ここでは、目標を関数として与えられたとき、それを最適化する道路料金パターンの設定のための計算法を提案する。料金パターンが決まると、利用者はそれをコストの一部として計算し、総合的なコストが最小となるように、車を利用するかしないか、利用する場合はどんな経路を選択するのかを決め、一種の均衡状態となる。目標関数は、道路交通量、旅客流量などを直接の変数とするが、これらの変数は均衡条件によって束縛される。料金パターンは均衡状態を変え、そしてそれらの変数の値を変え、最終的に目標関数の値を決める。このような最適化問題は”均衡制約下の数理計画 (Mathematical Program with Equilibrium Constraints, MPEC)”と呼ばれている。均衡状態変数が料金に対する感度（偏微分）が計算できれば、従来の非線形計画法で解ける。本論文は感度計算に基づく最適な料金設定の方法を提案する。

Optimal Congestion Pricing on Urban Road Networks

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ABSTRACT: Road pricing is an important economic measure for optimal management of transportation networks. The optimization objectives can be the total travel time or total cost incurred by all the travelers, or some other environmental objective such as minimum emission of dioxide, an so on. Suppose a certain toll is imposed on some link on the network, this will give an impact on flows over the whole network and brings about a new equilibrium state. An equilibrium state is a state of traffic network at which no traveler could improve her *perceived* travel cost by unilaterally changing her route. The goal of the toll setting is to effect the new state approach the objective in question. The problem can be formulated as a mathematical program with equilibrium constraints (MPEC). A key step for sovling such a MPEC problem is the sensitivity analysis of traffic flows with respect to the change of link characteristics such as toll prices. In this paper a sensitivity analysis based method is proposed for solving optimal road pricing problems.

Key words: transportation network, road pricing, sensitivity analysis, MPEC

1 Introduction

Road pricing is an important economic measure for optimal management of transportation networks. The optimization objectives can be the total travel time or total cost incurred by all the travelers, or some other environmental objective such as minimum emission of dioxide, an so on. Suppose a certain toll is posed on some link on the network, this will give an impact on flows over the whole network and brings about a new equilibrium state. An equilibrium state is a state of traffic network at which no traveler could improve her *perceived* travel cost by unilaterally changing her route. The goal of the toll setting is to effect the new state approach the objective in question. The problem can be formulated as a mathematical program with equilibrium constraints (MPEC). (See Luo, Pang and Ralph (1996) for a comprehensive treatment of this kind of mathematical programs.) A key step for sovling such a MPEC problem is the sensitivity analysis of traffic flows with respect to the change of link characteristics such as the toll prices. In a recent paper by Ying and Miyagi (2001) an efficient computational algorithm for sensitivity analysis of the stochastic user equilibrium (Sheffi, 1985) in traffic networks with fixed travel demand has been proposed. A stochastic user equilibrium (SUE) is an equilibrium state of the network where the travel costs perceived by the travelers have random errors. In this paper we extend that algorithm for dealing with traffic networks with elastic travel demand. Here by elasticity is meant that a traveler may choose to drive or to use public transit judging from the relative costs of the two choices. Based on the sensitivity analysis algorithm, we establish a computational method for solving some optimal road pricing problems.
The main advantage of our algorithm over the conventional sensitivity analysis methods (Tobin and Frisz, 1988, Qiu and Magnanty, 1989, Yang, 1997) is that our algorithm does not need to enumerate the huge number of paths on a practical large network. Another feature of our contribution is that our method is developed in the framework of stachostic user equilibrium (SUE), which is an extension of the Wordropian equilibrium treated in the above mentioned works. A Wardropian equilibrium is a state of traffic network at which no traveler could improve her *actual* travel cost by unilaterally changing her route, that is, a special case of SUE where the random error in perception of travel cost is zero. In the next section SUE conditions for mobile networks with fixed and elastic demands are formulated. In Section 3 the sensitivity analysis method are described. In Section 4 a typical optimal road pricing scheme is formulated. How to use the sensitivity analysis for solving such an optimization problem is briefly outlined. Some related problems are addressed in Section 5.

2 Stochastic User Equilibrium of Traffic Networks with Fixed and Elastic Demands

A list of notation used in this paper follows.

Notations

- $N = \{i, j, \cdots\}$: set of nodes
- $A = \{ij, \cdots\}$: set of links
- $W = \{rs, \cdots\}$: set of O-D pairs
- $q_{rs} = D_{rs}(S_{rs})$: elastic O-D demand, $rs \in W$, where S_{rs} is the expected minimum cost for O-D rs, and D_{rs} is a strictly decreasing function

- $\boldsymbol{q}=(q_{rs})_{rs\in W}$ denotes the vector of all O-D demands
- $R_{rs} = \{k, p, \cdots\}$: set of paths connecting rs
- h_k^{rs}: flow on path k with origin r and destination s
- P_k^{rs}: probability that a traveler from r to s chooses path k
- P_{ij}^{rs}: probability that a traveler from r to s traces link ij
- x_{ij}: link flow, for $ij \in A$
- $t_{ij}(x_{ij}, \epsilon_{ij})$: differentiable cost function of link ij with respect to flow x_{ij}, and parameter ϵ_{ij}; ϵ_{ij} may represent link tolls or other link characteristics.
  It is assumed that t_{ij} is strongly monotone with respect to x_{ij}. For given ϵ_{ij}, the inverse of the cost function is denoted as $x_{ij}(t_{ij}, \epsilon_{ij})$, which is also strongly monotone in t_{ij}
- $(x_{ij})_{t_{ij}}$: partial derivative of x_{ij} with respect to t_{ij}
- $(x_{ij})_{\epsilon_{ij}}$: partial derivative of x_{ij} with respect to ϵ_{ij}
- $\boldsymbol{x}=(x_{ij})_{ij\in A}$, $\boldsymbol{t}=(t_{ij})_{ij\in A}$ and $\boldsymbol{\epsilon}=(\epsilon_{ij})_{ij\in A}$ denote the vectors of all link flows, link costs and uncertainty parameters, respectively
- $\delta_{ij,k}^{rs} = \begin{cases} 1 & \text{if } ij \text{ is a link on path } k; \\ 0 & \text{otherwise.} \end{cases}$
- $\delta_{ij,gh}^{rs} = \begin{cases} 1 & \text{if } ij = gh \in A; \\ 0 & \text{otherwise.} \end{cases}$
- $c_k^{rs} = \sum_{ij\in A} t_{ij}\delta_{ij,k}^{rs}$: the total cost of traveling on a path $k \in R_{rs}$
- θ: a dispersion parameter in SUE
- For simplicity, summation notations $\sum_{k\in R_{rs}}$, $\sum_{p\in R_{rs}}$, $\sum_{rs\in W}$ will be abbreviated as $\sum_k$, $\sum_p$, $\sum_{rs}$, respectively.

In a multinominal logit-based stochastic user equilibrium (SUE), the "expected utility" of traveling on path $k \in R_{rs}$ is given by $U_k^{rs} = -\theta c_k^{rs}$, where θ is a unit scaling parameter, see, e.g., Chapter 10 of Sheffi (1985). For a traveler on O-D pair $rs \in W$, the probability P_k^{rs} by which the path k is chosen is given by

$$P_k^{rs} = \frac{\exp(-\theta c_k^{rs})}{\sum_p \exp(-\theta c_p^{rs})}, \quad k \in R_{rs}. \tag{1}$$

At stochastic user equilibrium, the path flows are

$$h_k^{rs} = q_{rs}\frac{\exp(-\theta c_k^{rs})}{\sum_p \exp(-\theta c_p^{rs})}, \quad k \in R_{rs}. \tag{2}$$

θ can be understood as a dispersion parameter indicating how precisely a driver can correctly choose the shortest routes; the larger the θ, the higher the probability that a driver chooses the shortest routes. From (2) it can be derived that the Wardropian equilibrium is a special case of SUE when we take $\theta \to \infty$.
The link flows are

$$x_{ij} = \sum_{rs}\sum_k h_k^{rs}\delta_{ij,k}^{rs} = \sum_{rs} q_{rs}\frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})}, \quad ij \in A. \tag{3}$$

If in some region mobile road network is the only available transport means, then a traveler has to accomplish her travel by car. Such a situation is referred to as "fixed demand" traffic network, on which the travel demands q_{rs} remains invariable indifferent of variation the travel costs on the network. On the other hand, if there is some alternative travel means such as railway transit, then the travel demand on the mobile road network q_{rs} is a function $D_{rs}(S_{rs})$ of the *expected minimum cost* S_{rs} (Sheffi, 1985) on the network,

$$S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t})) = -\frac{1}{\theta}\ln\sum_k \exp(-\theta c_k^{rs}).$$

When S_{rs} increases due to, e.g., a toll posed on some link, then some travelers from r to s may give up driving and use the public transit and causes a decrease in the demand q_{rs}. Such a case is referred to as a "elastic demand" traffic network. For a clear presentation of the new contribution in this work, we will firstly review a convenient mathematical formulation of the SUE for networks with fixed demand, and then treat the case of elastic demand. The positive definiteness of certain matrix that plays a critical role in the sensitivity computation is also established.

2.1 Dual Mathematical Programming Formulation of SUE with Fixed Demand

As was shown by Daganzo (1982) (p. 346, the Extremal Equivalence Theorem), the stochastic user equilibrium of a traffic network with fixed travel demand is achieved if and only if $\boldsymbol{t} = (t_{ij})_{ij\in A}$ is a minimizing point of a function Z,

$$Z(\boldsymbol{t}, \boldsymbol{\epsilon}) = \sum_{ij}\int_{t_{ij}(0,\epsilon_{ij})}^{t_{ij}} x_{ij}(\nu, \epsilon_{ij})d\nu - \sum_{rs} q_{rs}S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t})), \tag{4}$$

where

$$S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t})) = -\frac{1}{\theta}\ln\sum_k \exp(-\theta c_k^{rs}) \tag{5}$$

is the expected minimum cost perceived by a traveler from r to s. Note that q_{rs} are assumed to be fixed constants here. In this case, the minimizing condition for this unconstrained program is as follows

$$\begin{aligned}
&\frac{\partial Z}{\partial t_{ij}} \\
&= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs}\sum_k \frac{\partial S_{rs}}{\partial c_k^{rs}}\frac{\partial c_k^{rs}}{\partial t_{ij}} \\
&= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs}\sum_k \frac{\partial S_{rs}}{\partial c_k^{rs}}\delta_{ij,k}^{rs} \\
&= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs}\frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})} \\
&= 0, \quad ij \in A.
\end{aligned} \tag{6}$$

This implies that

$$x_{ij}(t_{ij}, \epsilon_{ij}) = \sum_{rs} q_{rs}\frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})}, \quad ij \in A,$$

which are exactly the link flows at the stochastic user equilibrium.
Rewriting (6) in a compact vector expression, we have

$$\nabla_{\boldsymbol{t}} Z = \boldsymbol{0}. \tag{7}$$

It can be shown that the Hessian of Z

$$\nabla_{\boldsymbol{t}}^2 Z = \left(\frac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}}\right)_{ij,gh} \tag{8}$$

is a positive definite matrix. This implies that Z is convex and hence the minimum point is unique. In fact, it is trivial to show that the Hessian of the first term of Z

$$\nabla_{\boldsymbol{t}}^2 (\sum_{ij} \int_{t_{ij}(0,\epsilon_{ij})}^{t_{ij}} x_{ij}(\nu, \epsilon_{ij}) d\nu) = \text{diag}((x_{ij})_{t_{ij}})_{ij} \tag{9}$$

is positive definite, since each diagonal entry is positive from the assumption that x_{ij} is strongly monotone in t_{ij}. In (9) "diag" denotes a diagonal matrix with corresponding diagonal entries. It is well known (see, e.g., p. 278, Sheffi, 1985) that $S_{rs}(\boldsymbol{c}^{rs})$ is concave with respect to $\boldsymbol{c}^{rs}$. As $\boldsymbol{c}^{rs}$ is a vector with components which are linear conbinations of t_{ij}, it is thus shown that the function $-\sum_{rs} q_{rs} S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t}))$ is convex with respect to t_{ij}, or equivalently, its Hessian with respect to t_{ij} is positive semi-definite. It then follows that the Hessian of Z

$$\nabla_{\boldsymbol{t}}^2 Z = \text{diag}((x_{ij})_{t_{ij}})_{ij} + \nabla_{\boldsymbol{t}}^2 (-\sum_{rs} q_{rs} S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t}))) \tag{10}$$

is a positive definite matrix.

2.2 SUE Conditions of Networks with Elastic Demand

Let us define the following functions in the variables $t_{ij},\ \epsilon_{ij},\ ij \in A$:

$$\begin{aligned}
&F_{ij}(\boldsymbol{t}, \boldsymbol{\epsilon}) \\
&= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs} \sum_k \frac{\partial S_{rs}}{\partial c_k^{rs}} \frac{\partial c_k^{rs}}{\partial t_{ij}} \\
&= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs} \sum_k \frac{\partial S_{rs}}{\partial c_k^{rs}} \delta_{ij,k}^{rs} \\
&= x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs} \frac{\sum_k \exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})}, \quad ij \in A,
\end{aligned} \tag{11}$$

where $q_{ij} = D_{rs}(S_{rs})$ are now elastic demands varying with S_{rs} which are functions of the link costs $\boldsymbol{t}$. The following equations define the conditions of a stochastic user equilibrium of the network with elastic demand:

$$F_{ij}(\boldsymbol{t}, \boldsymbol{\epsilon}) = 0, \quad ij \in A \tag{12}$$

These conditions define the link costs $\boldsymbol{t}$ as implicit functions in $\boldsymbol{\epsilon}$.
From (11) we have

$$\begin{aligned}
&\frac{\partial F_{ij}}{\partial t_{gh}} \\
&= (x_{ij})_{t_{ij}} \delta_{ij,gh} - \sum_{rs} q_{rs} \left[\frac{-\theta \sum_k \exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs} \delta_{gh,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})} \right. \\
&\qquad \left. - \frac{-\theta (\sum_k \exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs})(\sum_l \exp(-\theta c_l^{rs}) \delta_{gh,l}^{rs})}{(\sum_p \exp(-\theta c_p^{rs}))^2} \right] \\
&\qquad - \sum_{rs} \frac{dD_{rs}}{dS_{rs}} \frac{\sum_k \exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})} \frac{\sum_k \exp(-\theta c_k^{rs}) \delta_{gh,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})}, \\
&ij,\ gh \in A.
\end{aligned} \tag{13}$$

Let

$$F(t, \epsilon) = (F_{ij}(t, \epsilon))_{ij \in A}$$

be a vector of functions. Formula (13) can be summarized as

$$\nabla_t F = \mathrm{diag}((x_{ij})_{t_{ij}})_{ij} - \sum_{rs} q_{rs} \nabla_t^2 S_{rs}(c^{rs}(t))$$

$$-\sum_{rs} \frac{dD_{rs}}{dS_{rs}} \left(\frac{\sum_k \exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})} \frac{\sum_k \exp(-\theta c_k^{rs}) \delta_{gh,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})} \right)_{ij,gh} . \tag{14}$$

As is addressed above, the first term is a positive definite matrix, the second term is positive semi-definite. The third term is also a positive semi-definite matrix from the assumption that $\frac{dD_{rs}}{dS_{rs}} < 0$ and the fact that the matrix in bracket is the product of a column vector with its transpose. Therefore $\nabla_t F$ is a positive definite matrix. The system of equations (12) actually define t as an implicit function in ϵ, which is a single valued function due to the positive definiteness of $\nabla_t F$.

3 Sensitivity Analysis of Stochastic User Equilibrium of Traffic Networks with Elastic Demands

For a given ϵ, the link cost vector t is uniquely determined as shown above, so are uniquely determined the link flows x, the demands q and the path flows h_k^{rs}. Thus the equilibrium states of the traffic network are parametrized by ϵ. The problem of sensitivity analysis addressed in this paper is the computation of the derivatives of t, x and q with respect to ϵ. In Ying and Miyagi (2001), it has been shown that

$$\text{(i)} \ \frac{\sum_k \exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs}}{\sum_k \exp(-\theta c_k^{rs})}$$

and

$$\text{(ii)} \ \frac{\sum_k \exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs} \delta_{gh,k}^{rs}}{\sum_k \exp(-\theta c_k^{rs})}$$

can be computed by using Dial's algorithm in a link-based manner in the sense tat the paths do not have to be enumerated. The other terms in $\frac{\partial F_{ij}}{\partial t_{gh}}$ are routinely computable. Therefore $\nabla_t F$ can be efficiently computed. In the following we establish the formulae for computing the derivatives of t with respect to ϵ. Rewrite (12) as

$$F(t, \epsilon) = \mathbf{0}.$$

Looking x as mediate (apparent) variables in F, the following formulae are derived.

$$(F)_x (x)_\epsilon + \nabla_t F (\frac{\partial t}{\partial \epsilon}) = \mathbf{0},$$

$$(\frac{\partial t}{\partial \epsilon}) = \left(\frac{\partial t_{ij}}{\partial \epsilon_{gh}} \right)_{ij,gh} = -(\nabla_t F)^{-1} (F)_x (x)_\epsilon$$

$$= -(\nabla_t F)^{-1} \mathrm{diag}((x_{ij})_{\epsilon_{ij}})_{ij}, \tag{15}$$

and

$$(\frac{\partial x}{\partial \epsilon}) = \left(\frac{\partial x_{ij}}{\partial \epsilon_{gh}} \right)_{ij,gh} = \left((x_{ij})_{\epsilon_{ij}} \delta_{ij,gh} + (x_{ij})_{t_{ij}} \frac{\partial t_{ij}}{\partial \epsilon_{gh}} \right)_{ij,gh} . \tag{16}$$

The terms $(x_{ij})_{t_{ij}}$, $(x_{ij})_{\epsilon_{ij}}$ are directly computable from the explicit cost functions $t_{ij}(x_{ij}, \epsilon_{ij})$. Therefore all these derivatives can be easily obtained, once $\nabla_{\boldsymbol{t}}\boldsymbol{F}$ has been computed, by the algorithm developed in Ying and Miyagi (2001).

In applications where the change of demands caused by the change of link parameters are also of interest, we want to know the derivatives of q_{rs} with respect to ϵ_{ij}, which can be efficiently computed as follows

$$\frac{\partial q_{rs}}{\partial \epsilon_{ij}} = \frac{dD_{rs}}{dS_{rs}} \sum_{gh} \frac{\partial S_{rs}}{\partial t_{gh}} \frac{\partial t_{gh}}{\partial \epsilon_{ij}}, \tag{17}$$

as

$$\frac{\partial S_{rs}}{\partial t_{gh}} = \frac{\sum_k \exp(-\theta c_k^{rs})\delta_{gh,k}^{rs}}{\sum_k \exp(-\theta c_k^{rs})}$$

can be efficiently computed.

4 Optimal Road Pricing Problems

Road pricing can be used as a measure for achieving various economic objectives. Here we treat only one of such objectives for illustrating how the sensitivity analysis method developed in the previous section can be applied to solve optimal network management problems. The problem addressed here is to find an optimal road pricing system $\boldsymbol{p} = \{p_{ij}, ij \in A\}$ that effect the traffic network to achieve an equilibrium state at which the following cost function is minimum

$$C(\boldsymbol{p}) = T(\boldsymbol{p}) + V^{tran}(\boldsymbol{p}), \tag{18}$$

where

$$T(\boldsymbol{p}) = \sum_{ij \in A} x_{ij} \cdot (t_{ij}(x_{ij}, p_{ij}) - p_{ij}), \tag{19}$$

is the total travel time of all travelers on the road network and

$$V^{tran}(\boldsymbol{p}) = \sum_{rs} V_{rs}(\bar{q}_{rs} - q_{rs}(\boldsymbol{p})), \tag{20}$$

is the total variable operational cost of the transit system depending on the flow of passengers. $\bar{q}_{rs}$ is the total travel demand for using either the road network or the transit system. As was stated earlier, once $\boldsymbol{p} = \boldsymbol{\epsilon}$ is given, the link cost and flow vectors $\boldsymbol{t}$, $\boldsymbol{x}$ and demands $\boldsymbol{q} = (q_{rs},\ rs \in W)$ are determined at equilibrium, thus the cost function $C(\boldsymbol{p})$ is well defined.

Minimization of such an objective function may have several environmental and social implications. In the term $T(\boldsymbol{p})$, $t_{ij}(x_{ij}, p_{ij}) - p_{ij}) = t'_{ij}(x_{ij})$ is purely the travel time on link ij, which depends on the link flow x_{ij}. A small T implies that the overall congestion level on the network could not be high, and that the total amount of gasoline spent must also be small. It is clear that the travelers who change to use public transit due to expensive tolls may increase the cost of public transit. Such an effect is taken into account in the term $V^{tran}(\boldsymbol{p})$. In real world, a same amount of increase of travelers in a public transit system usually brings about a lower environmental burden (such as energy consumption) than that in a mobile road network. This part of variable cost on transit routes for OD pair rs is denoted by the function $V_{rs}(\bar{q}_{rs} - q_{rs}(\boldsymbol{p}))$, which is assumed to be monotonely increasing and differentiable. The road toll policy aims at realizing an optimal split of travelers into public transit and mobile network with some restriction.

It is impractical to collect a negative toll or to collect a positve amount beyond a certain limit. Therefore we assume that the tolls satisfy the following inequality constraints:

$$p_{ij} \geq 0, \quad p_{ij} \leq p, \quad ij \in A; \tag{21}$$

for some constants $p > 0$.
Corresponding to the notations used in previous sections, we have

$$p_{ij} = \epsilon_{ij} \text{ and } t_{ij} = t'_{ij}(x_{ij}) + p_{ij}.$$

By our sensitivity analysis algorithm,

$$(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}}),\ (\frac{\partial \boldsymbol{t}}{\partial \boldsymbol{p}}) \text{ and } (\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{p}})$$

can be efficiently computed. Using these derivative data, one can use some standard mathematical programming techniques (Ref., e.g., Luenberger, 1984, Luo, Pang and Ralph, 1996) to find a solution minimizing $C(\boldsymbol{p})$ with inequality constraints on $\boldsymbol{p}$. Note that by viewing $C(\boldsymbol{p})$ as a function in $\boldsymbol{p}$, the original mathematical program with equilibrium constraints is actually reduced to a mathematical program with linear constraints.

5 Discussion

We have implemented the algorithms proposed in this paper on a network studied in Ying and Miyagi (2001) and verified their correctness. We used a modified gradient projection method for finding the minimum point. The main difficulty encountered in implementation lies in the determination of the iteration step size, which requires repeated compuation of the equilibrium. This is a common difficulty in solving such kind of optimization problems. An important theme for future research is the resolution of this difficulty by exploring the particular structure of this particular optimization problem, and by applying suitable advanced MPEC techniques.

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