Performance Limits of Parallel Server Systems based on Deterministic Optimal Routing

Kazumasa OIDA and Kazumasa SHINJO

ATR Adaptive Communications Research Laboratories

Abstract

This paper presents the performance limits of parallel server (PS) systems in which every server has its own queue. The average packet delays of the PS systems depend on the routing policy, which assigns each arriving packet to one of the parallel servers. The optimal routings for the PS systems are numerically calculated based on the condition that the input traffic is completely deterministic. The optimal routings show that under a heavy traffic load, PS systems outperform a single server (SS) system. When an infinite number of packets arrive simultaneously, the expected average delay of a PS system that includes 10 servers is 20% smaller than that of an SS system.

1 Deterministic Optimal Routing

1.1 Formulation of the Optimization Problem

Consider a parallel server (PS) system (Fig. 1) in which there are p (≥ 2) homogeneous parallel servers ($S_k, k = 1, \ldots, p$). A single server (SS) system corresponds to the case $p = 1$. Each server (S_k) has its own infinite-capacity queue (Q_k), and the transmission rates (C/p) of all servers are identical. Each arrival packet (e_i) joins one of the queues according to a routing policy and is transmitted on a first-come first-served (FCFS) basis. Once a packet joins a queue, it cannot change its queue (that is, no jockeying). The total number (n) of arrival packets ($e_i, i = 1, \ldots, n$) is finite. Let x_i and t_i be the size of packet e_i and its arrival time at the PS system, respectively. For the deterministic input assumption, we assume the values of the sizes and the arrival times ($x_i, t_i, i = 1, \ldots, n$) of all packets are given and $t_1 \leq t_2 \leq \cdots \leq t_n$. Let $W_k(t)$ denote the emptying time of server S_k at time t. Strictly speaking, $W_k(t)$ is equal to the sum of the remaining transmission time of the packet being transmitted by S_k at time t and the total transmission time of all packets waiting in Q_k at time t. If the assignment (u_i) of packet e_i to queue Q_k is given by $u_i = k$, then we have

$$W_k(t_{i+1}) = \max(W_k(t_i) + \frac{\theta_k^i x_i}{C/p} - (t_{i+1} - t_i), 0), \quad k = 1, \ldots, p, \tag{1}$$

where

$$\theta_k^i = \begin{cases} 1, & \text{if } u_i = k, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Then, the average packet delay becomes

$$D_{p,n} = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{p} \theta_k^i W_k(t_i) + \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{C/p}, \tag{3}$$

where $W_1(t_1) = W_2(t_1) = \cdots = W_p(t_1) = 0$. The first term and the second term on the right-hand side of Eq. (3) represent the average waiting time and the average transmission time, respectively. Accordingly, the optimization problem P to be solved in this section can be described as

$$P \left\{ \begin{array}{ll} \text{Minimize} & D_{p,n}(u_1, u_2, \ldots, u_n) \\ \text{Subject to} & u_i \in S, \\ & t_i \leq t_{i+1},\ i = 1, \ldots, n-1, \\ & x_i > 0,\ i = 1, \ldots, n, \\ & C > 0, \end{array} \right.$$

where $S = \{1, 2, \ldots, p\}$.

1.2 Numerical Results

We calculated the optimal numerical values $(D_{p,n}(\{u_i^*(R)\})$ of problem P when $p = 2, \ldots, 6$ and $n = 128$ and observed the following results. **Under light or moderate traffic, all PS systems are inferior to the SS system; in contrast, under heavy traffic, all PS systems are superior to the SS system.**

The input traffic intensity (I) assigned to the SS system and the PS systems can be described as $I = \frac{\lambda}{C\mu}$, where $1/\mu$ and $1/\lambda$ are mean values of distributions that generate $\{x_i\}$ and $\{t_{i+1} - t_i\}$, respectively. Fig. 2 compares the average packet delays ($D_{p,128}$) of the PS systems ($p = 2, 3, 4, 6$) with those of the SS system, when $1/\mu = 1/\lambda = 1$. From Fig. 2, if $I \leq 0.9$, the SS system scores the best performance. In contrast, if $I > 1.1$, all PS systems outperform the SS system. Note also that if $I \leq 0.9$, the average delay increases with an increase in p, while if $I \to \infty$, from the lines corresponding to the PS systems, the average delay seems to decrease with an increase in p. We will show in section 2 that this is true when $n = \infty$.

2 Performance Merit of Parallel Server System

2.1 Simultaneous Arrival Model

Consider a case in which all n packets arrive simultaneously at the PS systems. For simplification, the total transmission rate of all servers is one; i.e.,

$$\left\{ \begin{array}{l} t_1 = t_2 = \cdots = t_n, \\ C = 1. \end{array} \right. \tag{4}$$

In this case, we suppose that the decisions of routing $\{u_i\}$ are made in the following order: $u_1, u_2, \ldots, u_n$. By using constraints (4), the average packet delay $D_{p,n}$ in problem P can be rewritten in the following simple form.

$$J_{p,n} = \frac{p}{n} \sum_{k=1}^{p} u_k^t Q u_k, \tag{5}$$

where

$$u_k = \begin{pmatrix} \theta_k^1 \\ \theta_k^2 \\ \vdots \\ \theta_k^n \end{pmatrix}, Q = \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ x_1 & x_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}. \tag{6}$$

2.2 Expected Average Packet Delay for the FS policy

Here, we formulate the expected value of J_n according to the FS policy [1]. Let X_i denote a continuous random variable representing the size of packet e_i. Assume that $X_1, X_2, \ldots, X_n$ are all independent and have an identical probability distribution with a density function $p(x)$. Let $\tilde{J}_{p,n}(\alpha_1^p, \ldots, \alpha_{p-1}^p)$ be the expected value of $J_{p,n}$ according to the FS policy $\{\tilde{u}_i(\alpha_1^p, \ldots, \alpha_{p-1}^p)\}$.

Lemma 1 *For all $n \geq 1$ and $p \geq 2$,*

$$\tilde{J}_{p,n} = pm + \frac{p(n-1)}{2} F_p(\alpha_1^p, \ldots, \alpha_{p-1}^p), \tag{7}$$

where $m = \int_0^\infty x dP(x)$, $dP(x) = p(x)dx$, $F_p(\alpha_1^p, \ldots, \alpha_{p-1}^p) = \sum_{k=1}^p \int_{\alpha_{k-1}^p}^{\alpha_k^p} dP(x) \int_{\alpha_{k-1}^p}^{\alpha_k^p} x dP(x)$, $0 = \alpha_0^p \leq \alpha_1^p \leq \cdots \leq \alpha_{p-1}^p \leq \alpha_p^p = \infty$.

On the other hand, from (5), the average packet delay of the SS system is $J_{1,n} = \frac{1}{n}\{nx_1 + (n-1)x_2 + \cdots + 2x_{n-1} + x_n\}$, so that the expected average packet delay becomes

$$\tilde{J}_{1,n} = \frac{(n+1)}{2} m. \tag{8}$$

2.3 Single Server versus Parallel Server

We now compare the expected average delays of the PS systems and the SS system.

Let α_p^* be an optimal $(p-1)$-vector $(\alpha_1^p, \ldots, \alpha_{p-1}^p)$ that minimizes $F_p(\alpha_1^p, \ldots, \alpha_{p-1}^p)$. when the packet size distribution is given. From (7), α_p^* also minimizes $\tilde{J}_{p,n}(\alpha_1^p, \ldots, \alpha_{p-1}^p)$ for any $n (\geq 2)$. Let $\tilde{J}_{p,n}^*$ and F_p^* be the minimum values of $\tilde{J}_{p,n}$ and F_p, respectively. From (7) and (8),

$$\frac{\tilde{J}_{p,n}^*}{\tilde{J}_{1,n}} = p\left(\frac{2}{n+1} + \frac{n-1}{n+1}\frac{F_p^*}{m}\right). \tag{9}$$

Lemma 2 *If the packet sizes have a negative exponential distribution, then for any $m > 0$, F_p^*/m is constant.*

Lemma 2 indicates that if packet sizes have a negative exponential distribution, then the ratio $\tilde{J}_{p,n}^*/\tilde{J}_{1,n}$ does not depend on the mean value m of the distribution.

Fig. 3 plots $\tilde{J}_{p,n}^*/\tilde{J}_{1,n}$, when the packet sizes have a negative exponential distribution. The figure shows that when n (the number of arrival packets) is small, the SS system is better than the PS systems ($\tilde{J}_{p,n}^*/\tilde{J}_{1,n} > 1$). Conversely, when n is large, the PS systems surpass the SS system. When $n = \infty$, the minimum average delay of the PS system that includes 10 servers is approximately 20% smaller than the average delay of the SS system. Note that if $n = \infty$, the ratio $\tilde{J}_{p,n}^*/\tilde{J}_{1,n}$ decreases when p increases.

References

[1] K. Oida and K. Shinjo, "Characteristics of Deterministic Optimal Routing for a Simple Traffic Control Problem," *Proc. of IEEE Int'l Performance, Computing, and Communications Conf.*, pp. 386-392, 1999.

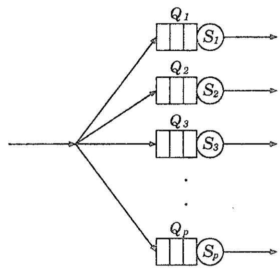

Figure 1: Parallel server system including p servers.

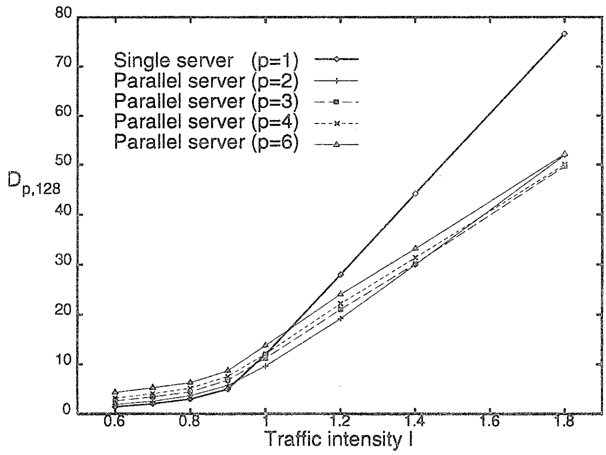

Figure 2: Comparison between the minimum average delays of four PS systems ($p = 2, 3, 4, 6$) and the average delays of the SS system.

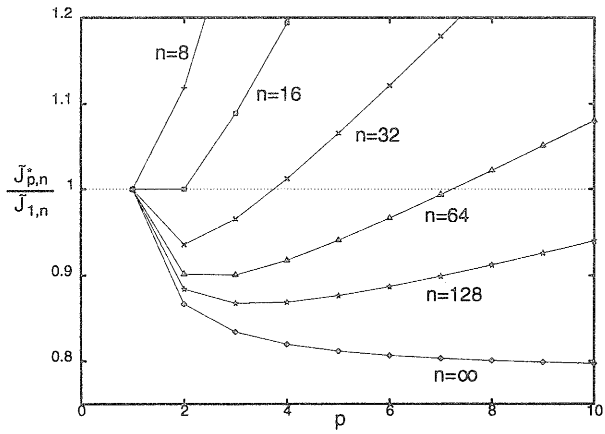

Figure 3: A comparison between the expected average delays in PS systems and a SS system, when packet sizes have a negative exponential distribution.